Contents

Chapter 1

Cyclic Inequalities

1.1 Applications

1.1. If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then

$$ab^2 + bc^2 + ca^2 \leq 4.$$

1.2. If a, b, c are positive real numbers such that $a + b + c = 3$, then

$$(ab + bc + ca)(ab^2 + bc^2 + ca^2) \leq 9.$$

1.3. If a, b, c are nonnegative real numbers such that $a^2 + b^2 + c^2 = 3$, then

(a) $$ab^2 + bc^2 + ca^2 \leq abc + 2;$$

(b) $$\frac{a}{b+2} + \frac{b}{c+2} + \frac{c}{a+2} \leq 1.$$

1.4. If $a, b, c \geq 1$, then

(a) $$2(ab^2 + bc^2 + ca^2) + 3 \geq 3(ab + bc + ca);$$

(b) $$ab^2 + bc^2 + ca^2 + 6 \geq 3(a + b + c).$$

1.5. If a, b, c are nonnegative real numbers such that

$$a+b+c=3, \qquad a \ge b \ge c,$$

then

(a) $$a^2b+b^2c+c^2a \ge ab+bc+ca;$$

(b) $$8(ab^2+bc^2+ca^2)+3abc \le 27;$$

(c) $$\frac{18}{a^2b+b^2c+c^2a} \le \frac{1}{abc}+5.$$

1.6. If a, b, c are nonnegative real numbers such that

$$a^2+b^2+c^2=3, \qquad a \ge b \ge c,$$

then

$$ab^2+bc^2+ca^2 \le \frac{3}{4}(ab+bc+ca+1).$$

1.7. If a, b, c are nonnegative real numbers such that $a^2+b^2+c^2=3$, then

$$a^2b^3+b^2c^3+c^2a^3 \le 3.$$

1.8. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$a^4b^2+b^4c^2+c^4a^2+4 \ge a^3b^3+b^3c^3+c^3a^3.$$

1.9. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

(a) $$ab^2+bc^2+ca^2+abc \le 4;$$

(b) $$\frac{a}{4-b}+\frac{b}{4-c}+\frac{c}{4-a} \le 1;$$

(c) $$ab^3+bc^3+ca^3+(ab+bc+ca)^2 \le 12;$$

(d) $$\frac{ab^2}{1+a+b}+\frac{bc^2}{1+b+c}+\frac{ca^2}{1+c+a} \le 1.$$

1.10. If a, b, c are positive real numbers, then

$$\frac{1}{a(a+2b)}+\frac{1}{b(b+2c)}+\frac{1}{c(c+2a)} \ge \frac{3}{ab+bc+ca}.$$

1.11. If a, b, c are positive real numbers such that $a + b + c = 3$, then

$$\frac{a}{b^2 + 2c} + \frac{b}{c^2 + 2a} + \frac{c}{a^2 + 2b} \geq 1.$$

1.12. If a, b, c are positive real numbers such that $a + b + c \geq 3$, then

$$\frac{a-1}{b+1} + \frac{b-1}{c+1} + \frac{c-1}{a+1} \geq 0.$$

1.13. If a, b, c are positive real numbers such that $a + b + c = 3$, then

(a)
$$\frac{1}{a^2b + 2} + \frac{1}{b^2c + 2} + \frac{1}{c^2a + 2} \geq 1;$$

(b)
$$\frac{1}{a^3b + 2} + \frac{1}{b^3c + 2} + \frac{1}{c^3a + 2} \geq 1.$$

1.14. If a, b, c are positive real numbers such that $a + b + c = 3$, then

$$\frac{ab}{9 - 4bc} + \frac{bc}{9 - 4ca} + \frac{ca}{9 - 4ab} \leq \frac{3}{5}.$$

1.15. If a, b, c are positive real numbers such that $a + b + c = 3$, then

(a)
$$\frac{a^2}{2a + b^2} + \frac{b^2}{2b + c^2} + \frac{c^2}{2c + a^2} \geq 1;$$

(b)
$$\frac{a^2}{a + 2b^2} + \frac{b^2}{b + 2c^2} + \frac{c^2}{c + 2a^2} \geq 1.$$

1.16. Let a, b, c be positive real numbers such that $a + b + c = 3$. Then,

$$\frac{1}{a + b^2 + c^3} + \frac{1}{b + c^2 + a^3} + \frac{1}{c + a^2 + b^3} \leq 1.$$

1.17. If a, b, c are positive real numbers, then

$$\frac{1 + a^2}{1 + b + c^2} + \frac{1 + b^2}{1 + c + a^2} + \frac{1 + c^2}{1 + a + b^2} \geq 2.$$

1.18. If a, b, c are nonnegative real numbers, then

$$\frac{a}{4a+4b+c} + \frac{b}{4b+4c+a} + \frac{c}{4c+4a+b} \le \frac{1}{3}.$$

1.19. If a, b, c are positive real numbers, then

$$\frac{a+b}{a+7b+c} + \frac{b+c}{b+7c+a} + \frac{c+a}{c+7a+b} \ge \frac{2}{3}.$$

1.20. If a, b, c are positive real numbers, then

$$\frac{a+b}{a+3b+c} + \frac{b+c}{b+3c+a} + \frac{c+a}{c+3a+b} \ge \frac{6}{5}.$$

1.21. If a, b, c are positive real numbers, then

$$\frac{2a+b}{2a+c} + \frac{2b+c}{2b+a} + \frac{2c+a}{2c+b} \ge 3.$$

1.22. If a, b, c are positive real numbers, then

$$\frac{a(a+b)}{a+c} + \frac{b(b+c)}{b+a} + \frac{c(c+a)}{c+b} \le \frac{3(a^2+b^2+c^2)}{a+b+c}.$$

1.23. If a, b, c are real numbers, then

$$\frac{a^2-bc}{4a^2+b^2+4c^2} + \frac{b^2-ca}{4b^2+c^2+4a^2} + \frac{c^2-ab}{4c^2+a^2+4b^2} \ge 0.$$

1.24. If a, b, c are real numbers, then

(a) $$a(a+b)^3 + b(b+c)^3 + c(c+a)^3 \ge 0;$$

(b) $$a(a+b)^5 + b(b+c)^5 + c(c+a)^5 \ge 0.$$

1.25. If a, b, c are real numbers, then

$$3(a^4+b^4+c^4) + 4(a^3b+b^3c+c^3a) \ge 0.$$

1.26. If a, b, c are positive real numbers, then

$$\frac{(a-b)(2a+b)}{(a+b)^2} + \frac{(b-c)(2b+c)}{(b+c)^2} + \frac{(c-a)(2c+a)}{(c+a)^2} \geq 0.$$

1.27. If a, b, c are positive real numbers, then

$$\frac{(a-b)(2a+b)}{a^2+ab+b^2} + \frac{(b-c)(2b+c)}{b^2+bc+c^2} + \frac{(c-a)(2c+a)}{c^2+ca+a^2} \geq 0.$$

1.28. If a, b, c are positive real numbers, then

$$\frac{(a-b)(3a+b)}{a^2+b^2} + \frac{(b-c)(3b+c)}{b^2+c^2} + \frac{(c-a)(3c+a)}{c^2+a^2} \geq 0.$$

1.29. Let a, b, c be positive real numbers such that $abc = 1$. Then,

$$\frac{1}{1+a+b^2} + \frac{1}{1+b+c^2} + \frac{1}{1+c+a^2} \leq 1.$$

1.30. Let a, b, c be positive real numbers such that $abc = 1$. Then,

$$\frac{a}{(a+1)(b+2)} + \frac{b}{(b+1)(c+2)} + \frac{c}{(c+1)(a+2)} \geq \frac{1}{2}.$$

1.31. If a, b, c are positive real numbers such that $ab + bc + ca = 3$, then

$$(a+2b)(b+2c)(c+2a) \geq 27.$$

1.32. If a, b, c are positive real numbers such that $ab + bc + ca = 3$, then

$$\frac{a}{a+a^3+b} + \frac{b}{b+b^3+c} + \frac{c}{c+c^3+a} \leq 1.$$

1.33. If a, b, c are positive real numbers such that $a \geq b \geq c$ and $ab + bc + ca = 3$, then

$$\frac{1}{a+2b} + \frac{1}{b+2c} + \frac{1}{c+2a} \geq 1.$$

1.34. If $a,b,c\in[0,1]$, then

$$\frac{a}{4b^2+5}+\frac{b}{4c^2+5}+\frac{c}{4a^2+5}\geq\frac{1}{3}.$$

1.35. If $a,b,c\in\left[\frac{1}{3},3\right]$, then

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\geq\frac{7}{5}.$$

1.36. If $a,b,c\in\left[\frac{1}{\sqrt{2}},\sqrt{2}\right]$, then

$$\frac{3}{a+2b}+\frac{3}{b+2c}+\frac{3}{c+2a}\geq\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a}.$$

1.37. If a,b,c are nonnegative real numbers, no two of which are zero, then

$$\frac{4abc}{ab^2+bc^2+ca^2+abc}+\frac{a^2+b^2+c^2}{ab+bc+ca}\geq 2.$$

1.38. If a,b,c are nonnegative real numbers such that $a+b+c=3$, then

$$\frac{1}{ab^2+8}+\frac{1}{bc^2+8}+\frac{1}{ca^2+8}\geq\frac{1}{3}.$$

1.39. If a,b,c are nonnegative real numbers such that $a+b+c=3$, then

$$\frac{ab}{bc+3}+\frac{bc}{ca+3}+\frac{ca}{ab+3}\leq\frac{3}{4}.$$

1.40. If a,b,c are nonnegative real numbers such that $a+b+c=3$, then

(a) $$\frac{a}{b^2+3}+\frac{b}{c^2+3}+\frac{c}{a^2+3}\geq\frac{3}{4};$$

(b) $$\frac{a}{b^3+1}+\frac{b}{c^3+1}+\frac{c}{a^3+1}\geq\frac{3}{2}.$$

1.41. Let a, b, c be positive real numbers, and let

$$x = a + \frac{1}{b} - 1, \quad y = b + \frac{1}{c} - 1, \quad z = c + \frac{1}{a} - 1.$$

Prove that

$$xy + yz + zx \geq 3.$$

1.42. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(a - \frac{1}{b} - \sqrt{2}\right)^2 + \left(b - \frac{1}{c} - \sqrt{2}\right)^2 + \left(c - \frac{1}{a} - \sqrt{2}\right)^2 \geq 6.$$

1.43. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left|1 + a - \frac{1}{b}\right| + \left|1 + b - \frac{1}{c}\right| + \left|1 + c - \frac{1}{a}\right| > 2.$$

1.44. If a, b, c are different positive real numbers, then

$$\left|1 + \frac{a}{b-c}\right| + \left|1 + \frac{b}{c-a}\right| + \left|1 + \frac{c}{a-b}\right| > 2.$$

1.45. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(2a - \frac{1}{b} - \frac{1}{2}\right)^2 + \left(2b - \frac{1}{c} - \frac{1}{2}\right)^2 + \left(2c - \frac{1}{a} - \frac{1}{2}\right)^2 \geq \frac{3}{4}.$$

1.46. Let

$$x = a + \frac{1}{b} - \frac{5}{4}, \quad y = b + \frac{1}{c} - \frac{5}{4}, \quad z = c + \frac{1}{a} - \frac{5}{4},$$

where $a \geq b \geq c > 0$. Prove that

$$xy + yz + zx \geq \frac{27}{16}.$$

1.47. Let a, b, c be positive real numbers, and let

$$E = \left(a + \frac{1}{a} - \sqrt{3}\right)\left(b + \frac{1}{b} - \sqrt{3}\right)\left(c + \frac{1}{c} - \sqrt{3}\right);$$

$$F = \left(a + \frac{1}{b} - \sqrt{3}\right)\left(b + \frac{1}{c} - \sqrt{3}\right)\left(c + \frac{1}{a} - \sqrt{3}\right).$$

Prove that $E \geq F$.

1.48. If a, b, c are positive real numbers such that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=5$, then

$$\frac{b}{a}+\frac{c}{b}+\frac{a}{c} \geq \frac{17}{4}.$$

1.49. If a, b, c are positive real numbers, then

(a) $$1+\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 2\sqrt{1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}}\ ;$$

(b) $$1+2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq \sqrt{1+16\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)}\ ;$$

(c) $$3+\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 2\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\ .$$

1.50. If a, b, c are positive real numbers, then

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+15\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right) \geq 16\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right).$$

1.51. If a, b, c are positive real numbers such that $abc=1$, then

(a) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq a+b+c;$$

(b) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{3}{2}(a+b+c-1);$$

(c) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+2 \geq \frac{5}{3}(a+b+c).$$

1.52. If a, b, c are positive real numbers such that $a^2+b^2+c^2=3$, then

(a) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 2+\frac{3}{ab+bc+ca};$$

(b) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \frac{9}{a+b+c}.$$

1.53. If a, b, c are positive real numbers such that $a^2+b^2+c^2=3$, then

$$6\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+5(ab+bc+ca) \geq 33.$$

1.54. If a, b, c are positive real numbers such that $a+b+c=3$, then

(a) $$6\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3 \ge 7(a^2+b^2+c^2);$$

(b) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge a^2+b^2+c^2.$$

1.55. If a, b, c are positive real numbers, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+2 \ge \frac{14(a^2+b^2+c^2)}{(a+b+c)^2}.$$

1.56. Let a, b, c be positive real numbers such that $a+b+c=3$, and let

$$x = 3a+\frac{1}{b}, \quad y = 3b+\frac{1}{c}, \quad z = 3c+\frac{1}{a}.$$

Prove that

$$xy+yz+zx \ge 48.$$

1.57. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\frac{a+1}{b}+\frac{b+1}{c}+\frac{c+1}{a} \ge 2(a^2+b^2+c^2).$$

1.58. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+3 \ge 2(a^2+b^2+c^2).$$

1.59. If a, b, c are positive real numbers, then

$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}+2(ab+bc+ca) \ge 3(a^2+b^2+c^2).$$

1.60. If a, b, c are positive real numbers such that $a^4+b^4+c^4=3$, then

(a) $$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \ge 3;$$

(b) $$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \ge \frac{3}{2}.$$

1.61. If a, b, c are positive real numbers, then

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \ge \frac{3(a^3+b^3+c^3)}{a^2+b^2+c^2}.$$

1.62. If a, b, c are positive real numbers, then

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+a+b+c \ge 2\sqrt{(a^2+b^2+c^2)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}.$$

1.63. If a, b, c are positive real numbers, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+32\left(\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\right) \ge 51.$$

1.64. Find the largest positive real number K such that the inequalities below hold for any positive real numbers a, b, c:

(a) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3 \ge K\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}\right);$$

(b) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3+K\left(\frac{a}{2a+b}+\frac{b}{2b+c}+\frac{c}{2c+a}-1\right) \ge 0.$$

1.65. If $a, b, c \in \left[\frac{1}{2}, 2\right]$, then

(a) $$8\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \ge 5\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)+9;$$

(b) $$20\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \ge 17\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

1.66. If a, b, c are positive real numbers such that $a \le b \le c$, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge \frac{2a}{b+c}+\frac{2b}{c+a}+\frac{2c}{a+b}.$$

1.67. Let a, b, c be positive real numbers such that $abc = 1$.

(a) If $a \le b \le c$, then
$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge a^{3/2} + b^{3/2} + c^{3/2};$$

(b) If $a \le 1 \le b \le c$, then
$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge a^{\sqrt{3}} + b^{\sqrt{3}} + c^{\sqrt{3}}.$$

1.68. If k and a, b, c are positive real numbers, then
$$\frac{1}{(k+1)a+b} + \frac{1}{(k+1)b+c} + \frac{1}{(k+1)c+a} \ge \frac{1}{ka+b+c} + \frac{1}{kb+c+a} + \frac{1}{kc+a+b}.$$

1.69. If a, b, c are positive real numbers, then

(a) $$\frac{a}{\sqrt{2a+b}} + \frac{b}{\sqrt{2b+c}} + \frac{c}{\sqrt{2c+a}} \le \sqrt{a+b+c};$$

(b) $$\frac{a}{\sqrt{a+2b}} + \frac{b}{\sqrt{b+2c}} + \frac{c}{\sqrt{c+2a}} \ge \sqrt{a+b+c}.$$

1.70. Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that
$$a\sqrt{\frac{a+2b}{3}} + b\sqrt{\frac{b+2c}{3}} + c\sqrt{\frac{c+2a}{3}} \le 3.$$

1.71. If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then
$$a\sqrt{1+b^3} + b\sqrt{1+c^3} + c\sqrt{1+a^3} \le 5.$$

1.72. If a, b, c are positive real numbers such that $abc = 1$, then

(a) $$\sqrt{\frac{a}{b+3}} + \sqrt{\frac{b}{c+3}} + \sqrt{\frac{c}{a+3}} \ge \frac{3}{2};$$

(b) $$\sqrt[3]{\frac{a}{b+7}} + \sqrt[3]{\frac{b}{c+7}} + \sqrt[3]{\frac{c}{a+7}} \ge \frac{3}{2}.$$

1.73. If a,b,c are positive real numbers, then

$$\left(1+\frac{4a}{a+b}\right)^2+\left(1+\frac{4b}{b+c}\right)^2+\left(1+\frac{4c}{c+a}\right)^2\geq 27.$$

1.74. If a,b,c are positive real numbers, then

$$\sqrt{\frac{2a}{a+b}}+\sqrt{\frac{2b}{b+c}}+\sqrt{\frac{2c}{c+a}}\leq 3.$$

1.75. If a,b,c are nonnegative real numbers, then

$$\sqrt{\frac{a}{4a+5b}}+\sqrt{\frac{b}{4b+5c}}+\sqrt{\frac{c}{4c+5a}}\leq 1.$$

1.76. If a,b,c are positive real numbers, then

$$\frac{a}{\sqrt{4a^2+ab+4b^2}}+\frac{b}{\sqrt{4b^2+bc+4c^2}}+\frac{c}{\sqrt{4c^2+ca+4a^2}}\leq 1.$$

1.77. If a,b,c are positive real numbers, then

$$\sqrt{\frac{a}{a+b+7c}}+\sqrt{\frac{b}{b+c+7a}}+\sqrt{\frac{c}{c+a+7b}}\geq 1.$$

1.78. If a,b,c are nonnegative real numbers, no two of which are zero, then

(a) $$\sqrt{\frac{a}{3b+c}}+\sqrt{\frac{b}{3c+a}}+\sqrt{\frac{c}{3a+b}}\geq\frac{3}{2};$$

(b) $$\sqrt{\frac{a}{2b+c}}+\sqrt{\frac{b}{2c+a}}+\sqrt{\frac{c}{2a+b}}\geq\sqrt[4]{8}.$$

1.79. If a,b,c are positive real numbers such that $ab+bc+ca=3$, then

(a) $$\frac{1}{(a+b)(3a+b)}+\frac{1}{(b+c)(3b+c)}+\frac{1}{(c+a)(3c+a)}\geq\frac{3}{8};$$

(b) $$\frac{1}{(2a+b)^2}+\frac{1}{(2b+c)^2}+\frac{1}{(2c+a)^2}\geq\frac{1}{3}.$$

1.80. If a, b, c are nonnegative real numbers, then

$$a^4+b^4+c^4+15(a^3b+b^3c+c^3a) \geq \frac{47}{4}(a^2b^2+b^2c^2+c^2a^2).$$

1.81. If a, b, c are nonnegative real numbers such that $a+b+c=4$, then

$$a^3b+b^3c+c^3a \leq 27.$$

1.82. Let a, b, c be nonnegative real numbers such that

$$a^2+b^2+c^2=\frac{10}{3}(ab+bc+ca).$$

Prove that

$$a^4+b^4+c^4 \geq \frac{82}{27}(a^3b+b^3c+c^3a).$$

1.83. If a, b, c are positive real numbers, then

$$\frac{a^3}{2a^2+b^2}+\frac{b^3}{2b^2+c^2}+\frac{c^3}{2c^2+a^2} \geq \frac{a+b+c}{3}.$$

1.84. If a, b, c are positive real numbers, then

$$\frac{a^4}{a^3+b^3}+\frac{b^4}{b^3+c^3}+\frac{c^4}{c^3+a^3} \geq \frac{a+b+c}{2}.$$

1.85. If a, b, c are positive real numbers such that $abc=1$, then

(a) $$3\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)+4\left(\frac{b}{a^2}+\frac{c}{b^2}+\frac{a}{c^2}\right) \geq 7(a^2+b^2+c^2);$$

(b) $$8\left(\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}\right)+5\left(\frac{b}{a^3}+\frac{c}{b^3}+\frac{a}{c^3}\right) \geq 13(a^3+b^3+c^3).$$

1.86. If a, b, c are positive real numbers, then

$$\frac{ab}{b^2+bc+c^2}+\frac{bc}{c^2+ca+a^2}+\frac{ca}{a^2+ab+b^2} \leq \frac{a^2+b^2+c^2}{ab+bc+ca}.$$

1.87. If a, b, c are positive real numbers, then

$$\frac{a-b}{b(2b+c)}+\frac{b-c}{c(2c+a)}+\frac{c-a}{a(2a+b)} \ge 0.$$

1.88. If a, b, c are positive real numbers, then

(a) $$\frac{a^2+6bc}{ab+2bc}+\frac{b^2+6ca}{bc+2ca}+\frac{c^2+6ab}{ca+2ab} \ge 7;$$

(b) $$\frac{a^2+7bc}{ab+bc}+\frac{b^2+7ca}{bc+ca}+\frac{c^2+7ab}{ca+ab} \ge 12.$$

1.89. If a, b, c are positive real numbers, then

(a) $$\frac{ab}{2b+c}+\frac{bc}{2c+a}+\frac{ca}{2a+b} \le \frac{a^2+b^2+c^2}{a+b+c};$$

(b) $$\frac{ab}{b+c}+\frac{bc}{c+a}+\frac{ca}{a+b} \le \frac{3(a^2+b^2+c^2)}{2(a+b+c)};$$

(c) $$\frac{ab}{4b+5c}+\frac{bc}{4c+5a}+\frac{ca}{4a+5b} \le \frac{a^2+b^2+c^2}{3(a+b+c)}.$$

1.90. If a, b, c are positive real numbers, then

(a) $$a\sqrt{b^2+8c^2}+b\sqrt{c^2+8a^2}+c\sqrt{a^2+8b^2} \le (a+b+c)^2;$$

(b) $$a\sqrt{b^2+3c^2}+b\sqrt{c^2+3a^2}+c\sqrt{a^2+3b^2} \le a^2+b^2+c^2+ab+bc+ca.$$

1.91. If a, b, c are positive real numbers, then

(a) $$\frac{1}{a\sqrt{a+2b}}+\frac{1}{b\sqrt{b+2c}}+\frac{1}{c\sqrt{c+2a}} \ge \sqrt{\frac{3}{abc}};$$

(b) $$\frac{1}{a\sqrt{a+8b}}+\frac{1}{b\sqrt{b+8c}}+\frac{1}{c\sqrt{c+8a}} \ge \sqrt{\frac{1}{abc}}.$$

1.92. If a, b, c are positive real numbers, then

$$\frac{a}{\sqrt{5a+4b}}+\frac{b}{\sqrt{5b+4c}}+\frac{c}{\sqrt{5c+4a}} \le \sqrt{\frac{a+b+c}{3}}.$$

1.93. If a, b, c are positive real numbers, then

(a) $$\frac{a}{\sqrt{a+b}}+\frac{b}{\sqrt{b+c}}+\frac{c}{\sqrt{c+a}} \geq \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{2}};$$

(b) $$\frac{a}{\sqrt{a+b}}+\frac{b}{\sqrt{b+c}}+\frac{c}{\sqrt{c+a}} \geq \sqrt[4]{\frac{27(ab+bc+ca)}{4}}.$$

1.94. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$\sqrt{3a+b^2}+\sqrt{3b+c^2}+\sqrt{3c+a^2} \geq 6.$$

1.95. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$\sqrt{a^2+b^2+2bc}+\sqrt{b^2+c^2+2ca}+\sqrt{c^2+a^2+2ab} \geq 2(a+b+c).$$

1.96. If a, b, c are nonnegative real numbers, then

$$\sqrt{a^2+b^2+7bc}+\sqrt{b^2+c^2+7ca}+\sqrt{c^2+a^2+7ab} \geq 3\sqrt{3(ab+bc+ca)}.$$

1.97. If a, b, c are positive real numbers, then

$$\frac{a^2+3ab}{(b+c)^2}+\frac{b^2+3bc}{(c+a)^2}+\frac{c^2+3ca}{(a+b)^2} \geq 3.$$

1.98. If a, b, c are positive real numbers, then

$$\frac{a^2b+1}{a(b+1)}+\frac{b^2c+1}{b(c+1)}+\frac{c^2a+1}{c(a+1)} \geq 3.$$

1.99. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\sqrt{a^3+3b}+\sqrt{b^3+3c}+\sqrt{c^3+3a} \geq 6.$$

1.100. If a, b, c are positive real numbers such that $abc=1$, then

$$\sqrt{\frac{a}{a+6b+2bc}}+\sqrt{\frac{b}{b+6c+2ca}}+\sqrt{\frac{c}{c+6a+2ab}} \geq 1.$$

1.101. If a, b, c are positive real numbers such that $abc = 1$, then

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 \ge 6(a+b+c-1).$$

1.102. If a, b, c are positive real numbers, then

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a} \ge \frac{a+b+c}{a+b+c-\sqrt[3]{abc}}.$$

1.103. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$a\sqrt{b^2+b+1}+b\sqrt{c^2+c+1}+c\sqrt{a^2+a+1} \le 3\sqrt{3}.$$

1.104. If a, b, c are positive real numbers, then

$$\frac{1}{b(a+2b+3c)^2}+\frac{1}{c(b+2c+3a)^2}+\frac{1}{a(c+2a+3b)^2} \le \frac{1}{12abc}.$$

1.105. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

(a) $$\frac{a^2+9b}{b+c}+\frac{b^2+9c}{c+a}+\frac{c^2+9a}{a+b} \ge 15;$$

(b) $$\frac{a^2+3b}{a+b}+\frac{b^2+3c}{b+c}+\frac{c^2+3a}{c+a} \ge 6.$$

1.106. If $a, b, c \in [0, 1]$, then

(a) $$\frac{bc}{2ab+1}+\frac{ca}{2bc+1}+\frac{ab}{2ca+1} \le 1.$$

(b) $$\frac{a}{ab+1}+\frac{b}{bc+1}+\frac{c}{ca+1} \le \frac{3}{2}.$$

1.107. If a, b, c are nonnegative real numbers, then

$$a^4+b^4+c^4+5(a^3b+b^3c+c^3a) \ge 6(a^2b^2+b^2c^2+c^2a^2).$$

1.108. If a, b, c are positive real numbers, then

$$a^5 + b^5 + c^5 - a^4b - b^4c - c^4a \geq 2abc(a^2 + b^2 + c^2 - ab - bc - ca).$$

1.109. If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3$, then

$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+a} \geq \frac{3}{2}.$$

1.110. If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then

$$a\sqrt{a+b} + b\sqrt{b+c} + c\sqrt{c+a} \geq 3\sqrt{2}.$$

1.111. If a, b, c are positive real numbers such that $a + b + c = 3$, then

$$\frac{a}{2b^2+c} + \frac{b}{2c^2+a} + \frac{c}{2a^2+b} \geq 1.$$

1.112. If a, b, c are positive real numbers such that $a + b + c = ab + bc + ca$, then

$$\frac{1}{a^2+b+1} + \frac{1}{b^2+c+1} + \frac{1}{c^2+a+1} \leq 1.$$

1.113. If a, b, c are positive real numbers, then

$$\frac{1}{(a+2b+3c)^2} + \frac{1}{(b+2c+3a)^2} + \frac{1}{(c+2a+3b)^2} \leq \frac{1}{4(ab+bc+ca)}.$$

1.114. If a, b, c are positive real numbers, then

$$\sqrt{\frac{a}{a+b+2c}} + \sqrt{\frac{b}{b+c+2a}} + \sqrt{\frac{c}{c+a+2b}} \leq \frac{3}{2}.$$

1.115. If a, b, c are positive real numbers, then

$$\sqrt{\frac{5a}{a+b+3c}} + \sqrt{\frac{5b}{b+c+3a}} + \sqrt{\frac{5c}{c+a+3b}} \leq 3.$$

1.116. If a, b, $c \in [0,\ 1]$, then

$$ab^2 + bc^2 + ca^2 + \frac{5}{4} \geq a + b + c.$$

1.117. If a, b, c are nonnegative real numbers such that

$$a + b + c = 3, \qquad a \leq b \leq 1 \leq c,$$

then

$$a^2b + b^2c + c^2a \leq 3.$$

1.118. Let a, b, c be nonnegative real numbers such that

$$a + b + c = 3, \qquad a \leq 1 \leq b \leq c.$$

Prove that

(a) $$a^2b + b^2c + c^2a \geq ab + bc + ca;$$

(b) $$a^2b + b^2c + c^2a \geq abc + 2;$$

(c) $$\frac{1}{abc} + 2 \geq \frac{9}{a^2b + b^2c + c^2a};$$

(d) $$ab^2 + bc^2 + ca^2 \geq 3.$$

1.119. If a, b, c are nonnegative real numbers such that

$$a + b + c = 3, \qquad a \leq 1 \leq b \leq c,$$

then

(a) $$\frac{5 - 2a}{1 + b} + \frac{5 - 2b}{1 + c} + \frac{5 - 2c}{1 + a} \geq \frac{9}{2};$$

(b) $$\frac{3 - 2b}{1 + a} + \frac{3 - 2c}{1 + b} + \frac{3 - 2a}{1 + c} \leq \frac{3}{2}.$$

1.120. If a, b, c are nonnegative real numbers such that

$$ab + bc + ca = 3, \qquad a \leq 1 \leq b \leq c,$$

then

(a) $$a^2b + b^2c + c^2a \geq 3;$$

(b) $$ab^2 + bc^2 + ca^2 + 3(\sqrt{3} - 1)abc \geq 3\sqrt{3}.$$

1.121. If a, b, c are nonnegative real numbers such that

$$a^2 + b^2 + c^2 = 3, \qquad a \le 1 \le b \le c,$$

then

(a)
$$a^2b + b^2c + c^2a \ge 2abc + 1;$$

(b)
$$2(ab^2 + bc^2 + ca^2) \ge 3abc + 3.$$

1.122. If a, b, c are nonnegative real numbers such that

$$ab + bc + ca = 3, \qquad a \le b \le 1 \le c,$$

then

$$ab^2 + bc^2 + ca^2 + 3abc \ge 6.$$

1.123. If a, b, c are nonnegative real numbers such that

$$a^2 + b^2 + c^2 = 3, \qquad a \le b \le 1 \le c,$$

then

$$2(a^2b + b^2c + c^2a) \le 3abc + 3.$$

1.124. If a, b, c are nonnegative real numbers such that

$$a^2 + b^2 + c^2 = 3, \qquad a \le b \le 1 \le c,$$

then

$$2(a^3b + b^3c + c^3a) \le abc + 5.$$

1.125. If a, b, c are real numbers, then

$$(a^2 + b^2 + c^2)^2 \ge 3(a^3b + b^3c + c^3a).$$

1.126. If a, b, c are real numbers, then

$$a^4 + b^4 + c^4 + ab^3 + bc^3 + ca^3 \ge 2(a^3b + b^3c + c^3a).$$

1.127. If a, b, c are positive real numbers, then

(a)
$$\frac{a^2}{ab+2c^2}+\frac{b^2}{bc+2a^2}+\frac{c^2}{ca+2b^2}\geq 1;$$

(b)
$$\frac{a^3}{a^2b+2c^3}+\frac{b^3}{b^2c+2a^3}+\frac{c^3}{c^2a+2b^3}\geq 1.$$

1.128. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\frac{a}{ab+1}+\frac{b}{bc+1}+\frac{c}{ca+1}\geq \frac{3}{2}.$$

1.129. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\frac{a}{3a+b^2}+\frac{b}{3b+c^2}+\frac{c}{3c+a^2}\leq \frac{3}{2}.$$

1.130. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\frac{a}{b^2+c}+\frac{b}{c^2+a}+\frac{c}{a^2+b}\geq \frac{3}{2}.$$

1.131. If a, b, c are positive real numbers such that $abc=1$, then

$$\frac{a}{b^3+2}+\frac{b}{c^3+2}+\frac{c}{a^3+2}\geq 1.$$

1.132. Let a, b, c be positive real numbers such that

$$a^m+b^m+c^m=3,$$

where $m>0$. Prove that

$$\frac{a^{m-1}}{b}+\frac{b^{m-1}}{c}+\frac{c^{m-1}}{a}\geq 3.$$

1.133. If a, b, c are positive real numbers, then

(a)
$$\frac{1}{4a}+\frac{1}{4b}+\frac{1}{4c}+\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq 3\left(\frac{1}{3a+b}+\frac{1}{3b+c}+\frac{1}{3c+a}\right);$$

(b)
$$\frac{1}{4a}+\frac{1}{4b}+\frac{1}{4c}+\frac{1}{a+3b}+\frac{1}{b+3c}+\frac{1}{c+3a}\geq 2\left(\frac{1}{3a+b}+\frac{1}{3b+c}+\frac{1}{3c+a}\right).$$

1.134. If a, b, c are positive real numbers such that $a^6 + b^6 + c^6 = 3$, then

$$\frac{a^5}{b} + \frac{b^5}{c} + \frac{c^5}{a} \ge 3.$$

1.135. If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3$, then

$$\frac{a^3}{a + b^5} + \frac{b^3}{b + c^5} + \frac{c^3}{c + a^5} \ge \frac{3}{2}.$$

1.136. If a, b, c are real numbers such that $a^2 + b^2 + c^2 = 3$, then

$$a^2b + b^2c + c^2a + 3 \ge a + b + c + ab + bc + ca.$$

1.137. If a, b, c are positive real numbers such that $a + b + c = 3$, then

$$\frac{12}{a^2b + b^2c + c^2a} \le 3 + \frac{1}{abc}.$$

1.138. If a, b, c are positive real numbers such that $ab + bc + ca = 3$, then

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \ge a^2 + b^2 + c^2.$$

1.139. If a, b, c are positive real numbers such that $a + b + c = 3$, then

$$\frac{24}{a^2b + b^2c + c^2a} + \frac{1}{abc} \ge 9.$$

1.140. Let a, b, c be nonnegative real numbers such that

$$2(a^2 + b^2 + c^2) = 5(ab + bc + ca).$$

Prove that

(a) $$8(a^4 + b^4 + c^4) \ge 17(a^3b + b^3c + c^3a);$$

(b) $$16(a^4 + b^4 + c^4) \ge 34(a^3b + b^3c + c^3a) + 81abc(a + b + c).$$

1.141. Let a, b, c be nonnegative real numbers such that

$$2(a^2+b^2+c^2) = 5(ab+bc+ca).$$

Prove that

(a) $$2(a^3b+b^3c+c^3a) \ge a^2b^2+b^2c^2+c^2a^2+abc(a+b+c);$$

(b) $$11(a^4+b^4+c^4) \ge 17(a^3b+b^3c+c^3a)+129abc(a+b+c);$$

(c) $$a^3b+b^3c+c^3a \le \frac{14+\sqrt{102}}{8}(a^2b^2+b^2c^2+c^2a^2).$$

1.142. If a, b, c are real numbers such that

$$a^3b+b^3c+c^3a \le 0,$$

then

$$a^2+b^2+c^2 \ge k(ab+bc+ca),$$

where

$$k = \frac{1+\sqrt{21+8\sqrt{7}}}{2} \approx 3.7468.$$

1.143. If a, b, c are real numbers such that

$$a^3b+b^3c+c^3a \ge 0,$$

then

$$a^2+b^2+c^2+k(ab+bc+ca) \ge 0,$$

where

$$k = \frac{-1+\sqrt{21+8\sqrt{7}}}{2} \approx 2.7468.$$

1.144. If a, b, c are real numbers such that

$$k(a^2+b^2+c^2) = ab+bc+ca, \qquad k \in \left(\frac{-1}{2}, 1\right),$$

then

$$\alpha_k \le \frac{a^3b+b^3c+c^3}{(a^2+b^2+c^2)^2} \le \beta_k,$$

where

$$27\alpha_k = 1+13k-5k^2-2(1-k)(1+2k)\sqrt{\frac{7(1-k)}{1+2k}},$$

$$27\beta_k = 1+13k-5k^2+2(1-k)(1+2k)\sqrt{\frac{7(1-k)}{1+2k}}.$$

1.145. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\frac{a^2}{4a+b^2}+\frac{b^2}{4b+c^2}+\frac{c^2}{4c+a^2}\ge\frac{3}{5}.$$

1.146. If a, b, c are positive real numbers, then

$$\frac{a^2+bc}{a+b}+\frac{b^2+ca}{b+c}+\frac{c^2+ab}{c+a}\le\frac{(a+b+c)^3}{3(ab+bc+ca)}.$$

1.147. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\sqrt{ab^2+bc^2}+\sqrt{bc^2+ca^2}+\sqrt{ca^2+ab^2}\le 3\sqrt{2}.$$

1.148. If a, b, c are positive real numbers such that $a^5+b^5+c^5=3$, then

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\ge 3.$$

1.149. Let $P(a, b, c)$ be a cyclic homogeneous polynomial of degree three. The inequality

$$P(a, b, c)\ge 0$$

holds for all $a, b, c\ge 0$ if and only if the following two conditions are fulfilled:

(a) $P(1, 1, 1)\ge 0$;

(b) $P(0, b, c)\ge 0$ for all $b, c\ge 0$.

1.150. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$8(a^2b+b^2c+c^2a)+9\ge 11(ab+bc+ca).$$

1.151. If a, b, c are nonnegative real numbers such that $a+b+c=6$, then

$$a^3+b^3+c^3+8(a^2b+b^2c+c^2a)\ge 166.$$

1.152. If a, b, c are positive real numbers such that $abc\ge 1$, then

$$a^{\frac{a}{b}}b^{\frac{b}{c}}c^{\frac{c}{a}}\ge 1.$$

1.153. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+7 \geq \frac{17}{3}\left(\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\right).$$

1.154. Let a, b, c be nonnegative real numbers, no two of which are zero. If $0 \leq k \leq 5$, then

$$\frac{ka+b}{a+c}+\frac{kb+c}{b+a}+\frac{kc+a}{c+b} \geq \frac{3}{2}(k+1).$$

1.155. Let a, b, c be nonnegative real numbers, no two of which are zero. If $k \leq \frac{23}{8}$, then

$$\frac{ka+b}{2a+c}+\frac{kb+c}{2b+a}+\frac{kc+a}{2c+b} \geq k+1.$$

1.156. Let a, b, c be nonnegative real numbers. Prove that

(a) if $k \leq 1-\frac{2}{5\sqrt{5}}$, then

$$\frac{ka+b}{2a+b+c}+\frac{kb+c}{a+2b+c}+\frac{kc+a}{a+b+2c} \geq \frac{3}{4}(k+1).$$

(b) if $k \geq 1+\frac{2}{5\sqrt{5}}$, then

$$\frac{ka+b}{2a+b+c}+\frac{kb+c}{a+2b+c}+\frac{kc+a}{a+b+2c} \leq \frac{3}{4}(k+1).$$

1.157. If a, b, c are positive real numbers such that $a \leq b \leq c$, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3 \geq 2\left(\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\right).$$

1.158. If $a \geq b \geq c \geq 0$, then

$$\frac{3a+b}{2a+c}+\frac{3b+c}{2b+a}+\frac{3c+a}{2c+b} \geq 4.$$

1.159. If $a \geq b \geq c \geq 0$ and $ab+bc+ca=2$, then

$$\sqrt{a+ab}+\sqrt{b+bc}+\sqrt{c+ca} \geq 3.$$

1.160. If $a \geq b \geq c$ are nonnegative numbers such that $ab+bc+ca=3$, then

$$\sqrt{a+2ab}+\sqrt{b+2bc}+\sqrt{c+2ca} \geq 4.$$

1.161. If a, b, c are nonnegative real numbers such that $ab+bc+ca=3$, then

$$\sqrt{a+3b}+\sqrt{b+3c}+\sqrt{c+3a} \geq 6.$$

1.162. If a, b, c are the lengths of the sides of a triangle, then

$$10\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) > 9\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

1.163. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{a}{3a+b-c}+\frac{b}{3b+c-a}+\frac{c}{3c+a-b} \geq 1.$$

1.164. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{a^2-b^2}{a^2+bc}+\frac{b^2-c^2}{b^2+ca}+\frac{c^2-a^2}{c^2+ab} \leq 0.$$

1.165. If a, b, c are the lengths of the sides of a triangle, then

$$a^2(a+b)(b-c)+b^2(b+c)(c-a)+c^2(c+a)(a-b) \geq 0.$$

1.166. If a, b, c are the lengths of the sides of a triangle, then

$$a^2b+b^2c+c^2a \geq \sqrt{abc(a+b+c)(a^2+b^2+c^2)}.$$

1.167. If a, b, c are the lengths of the sides of a triangle, then

$$a^2\left(\frac{b}{c}-1\right)+b^2\left(\frac{c}{a}-1\right)+c^2\left(\frac{a}{b}-1\right)\ge 0.$$

1.168. If a, b, c are the lengths of the sides of a triangle, then

(a) $$a^3b+b^3c+c^3a\ge a^2b^2+b^2c^2+c^2a^2;$$

(b) $$3(a^3b+b^3c+c^3a)\ge (ab+bc+ca)(a^2+b^2+c^2);$$

(c) $$\frac{a^3b+b^3c+c^3}{3}\ge\left(\frac{a+b+c}{3}\right)^4.$$

1.169. If a, b, c are the lengths of the sides of a triangle, then

$$2\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)\ge\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}+3.$$

1.170. If a, b, c are the lengths of the sides of a triangle such that $a<b<c$, then

$$\frac{a^2}{a^2-b^2}+\frac{b^2}{b^2-c^2}+\frac{c^2}{c^2-a^2}\le 0.$$

1.171. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3\ge 2\left(\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\right).$$

1.172. Let a, b, c be the lengths of the sides of a triangle. If $k\ge 2$, then

$$a^kb(a-b)+b^kc(b-c)+c^ka(c-a)\ge 0.$$

1.173. Let a, b, c be the lengths of the sides of a triangle. If $k\ge 1$, then

$$3(a^{k+1}b+b^{k+1}c+c^{k+1}a)\ge (a+b+c)(a^kb+b^kc+c^ka).$$

1.174. Let a, b, c, d be positive real numbers such that $a+b+c+d=4$. Prove that

$$\frac{a}{3+b}+\frac{b}{3+c}+\frac{c}{3+d}+\frac{d}{3+a}\geq 1.$$

1.175. Let a, b, c, d be positive real numbers such that $a+b+c+d=4$. Prove that

$$\frac{a}{1+b^2}+\frac{b}{1+c^2}+\frac{c}{1+d^2}+\frac{d}{1+a^2}\geq 2.$$

1.176. If a, b, c, d are nonnegative real numbers such that $a+b+c+d=4$, then

$$a^2bc+b^2cd+c^2da+d^2ab\leq 4.$$

1.177. If a, b, c, d are nonnegative real numbers such that $a+b+c+d=4$, then

$$a(b+c)^2+b(c+d)^2+c(d+a)^2+d(a+b)^2\leq 16.$$

1.178. If a, b, c, d are positive real numbers, then

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+a}+\frac{d-a}{a+b}\geq 0.$$

1.179. If a, b, c, d are positive real numbers, then

(a) $$\frac{a-b}{a+2b+c}+\frac{b-c}{b+2c+d}+\frac{c-d}{c+2d+a}+\frac{d-a}{d+2a+b}\geq 0;$$

(b) $$\frac{a}{2a+b+c}+\frac{b}{2b+c+d}+\frac{c}{2c+d+a}+\frac{d}{2d+a+b}\leq 1.$$

1.180. If a, b, c, d are positive real numbers such that $abcd=1$, then

$$\frac{1}{a(a+b)}+\frac{1}{b(b+c)}+\frac{1}{c(c+d)}+\frac{1}{d(d+a)}\geq 2.$$

1.181. If a, b, c, d are positive real numbers, then

$$\frac{1}{a(1+b)}+\frac{1}{b(1+c)}+\frac{1}{c(1+d)}+\frac{1}{d(1+a)}\geq \frac{16}{1+8\sqrt{abcd}}.$$

1.182. If a,b,c,d are nonnegative real numbers such that $a^2+b^2+c^2+d^2=4$, then

(a) $$3(a+b+c+d)\ge 2(ab+bc+cd+da)+4;$$

(b) $$a+b+c+d-4\ge(2-\sqrt{2})(ab+bc+cd+da-4).$$

1.183. Let a,b,c,d be positive real numbers.

(a) If $a,b,c,d\ge 1$, then

$$\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{d}\right)\left(d+\frac{1}{a}\right)\ge(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right);$$

(b) If $abcd=1$, then

$$\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{d}\right)\left(d+\frac{1}{a}\right)\ge(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

1.184. If a,b,c,d are positive real numbers, then

$$\left(1+\frac{a}{a+b}\right)^2+\left(1+\frac{b}{b+c}\right)^2+\left(1+\frac{c}{c+d}\right)^2+\left(1+\frac{d}{d+a}\right)^2>7.$$

1.185. If a,b,c,d are positive real numbers, then

$$\frac{a^2-bd}{b+2c+d}+\frac{b^2-ca}{c+2d+a}+\frac{c^2-db}{d+2a+b}+\frac{d^2-ac}{a+2b+c}\ge 0.$$

1.186. If a,b,c,d are positive real numbers such that $a\le b\le c\le d$, then

$$\sqrt{\frac{2a}{a+b}}+\sqrt{\frac{2b}{b+c}}+\sqrt{\frac{2c}{c+d}}+\sqrt{\frac{2d}{d+a}}\le 4.$$

1.187. Let a,b,c,d be nonnegative real numbers, and let

$$x=\frac{a}{b+c},\quad y=\frac{b}{c+d},\quad z=\frac{c}{d+a},\quad t=\frac{d}{a+b}.$$

Prove that

(a) $$\sqrt{xz}+\sqrt{yt}\le 1;$$

(b) $$x+y+z+t+4(xz+yt)\ge 4.$$

1.188. If a, b, c, d are nonnegative real numbers, then

$$\left(1+\frac{2a}{b+c}\right)\left(1+\frac{2b}{c+d}\right)\left(1+\frac{2c}{d+a}\right)\left(1+\frac{2d}{a+b}\right) \ge 9.$$

1.189. Let a, b, c, d be nonnegative real numbers. If $k > 0$, then

$$\left(1+\frac{ka}{b+c}\right)\left(1+\frac{kb}{c+d}\right)\left(1+\frac{kc}{d+a}\right)\left(1+\frac{kd}{a+b}\right) \ge (1+k)^2.$$

1.190. If a, b, c, d are positive real numbers such that $a+b+c+d=4$, then

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{cd}+\frac{1}{da} \ge a^2+b^2+c^2+d^2.$$

1.191. If a, b, c, d are positive real numbers, then

$$\frac{a^2}{(a+b+c)^2}+\frac{b^2}{(b+c+d)^2}+\frac{c^2}{(c+d+a)^2}+\frac{d^2}{(d+a+b)^2} \ge \frac{4}{9}.$$

1.192. If a, b, c, d are positive real numbers such that $a+b+c+d=3$, then

$$ab(b+c)+bc(c+d)+cd(d+a)+da(a+b) \le 4.$$

1.193. If $a \ge b \ge c \ge d \ge 0$ and $a+b+c+d=2$, then

$$ab(b+c)+bc(c+d)+cd(d+a)+da(a+b) \le 1.$$

1.194. Let a, b, c, d be nonnegative real numbers such that $a+b+c+d=4$. If $k \ge \dfrac{37}{27}$, then

$$ab(b+kc)+bc(c+kd)+cd(d+ka)+da(a+kb) \le 4(1+k).$$

1.195. If a, b, c, d are nonnegative real numbers such that $a+b+c+d=4$, then

$$\sqrt{\frac{3a}{b+2}}+\sqrt{\frac{3b}{c+2}}+\sqrt{\frac{3c}{d+2}}+\sqrt{\frac{3d}{a+2}} \le 4.$$

1.196. Let a, b, c, d be positive real numbers such that $a \le b \le c \le d$. Prove that

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right) \ge 4+\frac{a}{c}+\frac{c}{a}+\frac{b}{d}+\frac{d}{b}.$$

1.197. Let a, b, c, d be positive real numbers such that

$$a \le b \le c \le d, \qquad abcd = 1.$$

Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} \ge ab+bc+cd+da.$$

1.198. Let a, b, c, d be positive real numbers such that

$$a \le b \le c \le d, \qquad abcd = 1.$$

Prove that

$$4+\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} \ge 2(a+b+c+d).$$

1.199. Let $A = \{a_1, a_2, a_3, a_4\}$ be a set of real numbers such that

$$a_1+a_2+a_3+a_4 = 0.$$

Prove that there exists a permutation $\{a, b, c, d\}$ of A such that

$$a^2+b^2+c^2+d^2+3(ab+bc+cd+da) \ge 0.$$

1.200. If a, b, c, d, e are positive real numbers, then

$$\frac{a}{a+2b+2c}+\frac{b}{b+2c+2d}+\frac{c}{c+2d+2e}+\frac{d}{d+2e+2a}+\frac{e}{e+2a+2b} \ge 1.$$

1.201. Let a, b, c, d, e be positive real numbers such that $a+b+c+d+e = 5$. Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}+\frac{e}{a} \le 1+\frac{4}{abcde}.$$

1.202. If a, b, c, d, e are real numbers such that $a+b+c+d+e = 0$, then

$$\frac{-\sqrt{5}-1}{4} \le \frac{ab+bc+cd+de+ea}{a^2+b^2+c^2+d^2+e^2} \le \frac{\sqrt{5}-1}{4}.$$

1.203. Let a, b, c, d, e be positive real numbers such that

$$a^2+b^2+c^2+d^2+e^2=5.$$

Prove that

$$\frac{a^2}{b+c+d}+\frac{b^2}{c+d+e}+\frac{c^2}{d+e+a}+\frac{d^2}{e+a+b}+\frac{e^2}{a+b+c}\geq\frac{5}{3}.$$

1.204. Let a,b,c,d,e be nonnegative real numbers such that $a+b+c+d+e=5$. Prove that

$$(a^2+b^2)(b^2+c^2)(c^2+d^2)(d^2+e^2)(e^2+a^2)\leq\frac{729}{2}.$$

1.205. If $a,b,c,d,e\in[1,5]$, then

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+e}+\frac{d-e}{e+a}+\frac{e-a}{a+b}\geq 0.$$

1.206. If $a,b,c,d,e,f\in[1,3]$, then

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+e}+\frac{d-e}{e+f}+\frac{e-f}{f+a}+\frac{f-a}{a+b}\geq 0.$$

1.207. If $a_1,\ a_2,\ \ldots,\ a_n$ $(n\geq 3)$ are positive real numbers, then

$$\sum_{i=1}^{n}\frac{a_i}{a_{i-1}+2a_i+a_{i+1}}\leq\frac{n}{4},$$

where $a_0=a_n$ and $a_{n+1}=a_1$.

1.208. Let $a_1,\ a_2,\ \ldots,\ a_n$ $(n\geq 3)$ be positive real numbers such that $a_1a_2\cdots a_n=1$. Prove that

$$\frac{1}{n-2+a_1+a_2}+\frac{1}{n-2+a_2+a_3}+\cdots+\frac{1}{n-2+a_n+a_1}\leq 1.$$

1.209. If $a_1,a_2,...,a_n\geq 1$, then

$$\prod\left(a_1+\frac{1}{a_2}+n-2\right)\geq n^{n-2}(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right);$$

1.210. If $a_1, a_2, ..., a_n \ge 1$, then

$$\left(a_1 + \frac{1}{a_1}\right)\left(a_2 + \frac{1}{a_2}\right)\cdots\left(a_n + \frac{1}{a_n}\right) + 2^n \ge 2\left(1 + \frac{a_1}{a_2}\right)\left(1 + \frac{a_2}{a_3}\right)\cdots\left(1 + \frac{a_n}{a_1}\right).$$

1.211. Let k and n be positive integers with $k < n$, and let $a_1, a_2, \dots, a_n$ be real numbers such that $a_1 \le a_2 \le \cdots \le a_n$. Then

$$(a_1 + a_2 + \cdots + a_n)^2 \ge n(a_1 a_{k+1} + a_2 a_{k+2} + \cdots + a_n a_{n+k})$$

(where $a_{n+i} = a_i$ for any positive integer i) in the following cases:

(a) $n = 2k$;

(b) $n = 4k$.

1.212. If $a_1,\ a_2,\ \dots,\ a_n \in [1,\ 2]$, then

$$\sum_{i=1}^{n} \frac{3}{a_i + 2a_{i+1}} \ge \sum_{i=1}^{n} \frac{2}{a_i + a_{i+1}},$$

where $a_{n+1} = a_1$.

1.213. If $a_1, a_2, \dots, a_n$ $(n \ge 3)$ are real numbers such that $a_1 \ge a_2 \ge \cdots \ge a_n$ and

$$a_1 a_2 + a_2 a_3 + \cdots + a_n a_1 = n,$$

then

$$(3 - a_1)^2 + (3 - a_2)^2 + \cdots + (3 - a_n)^2 \ge 4n.$$

1.214. Let a, b, c, d be positive real numbers such that $ab + bc + cd + da = 4$.

(a) If $a \ge b \ge 1 \ge c \ge d$, then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + 8 \ge 3(a + b + c + d).$$

(b) If $a \ge b \ge c \ge 1 \ge d$, then the inequality above holds true.

1.215. If a, b, c, d are nonnegative real numbers such that

$$ab + bc + cd + da = 4, \qquad a \ge b \ge c \ge d,$$

then

$$\frac{4}{3} \le \frac{1}{a+2} + \frac{1}{b+2} + \frac{1}{c+2} + \frac{1}{d+2} \le \frac{3}{2}.$$

1.216. If $n \ge 6$ and $a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n$ such that $a_1a_2 + a_2a_3 + \cdots + a_na_1 = n$, then

$$\frac{1}{a_1+3} + \frac{1}{a_2+3} + \cdots + \frac{1}{a_n+3} \ge \frac{n}{4}.$$

1.217. If x_1, x_2, x_3, x_4, x_5 are nonnegative real numbers such that

$$x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 = 5,$$

then

$$\frac{1}{5x_1+4} + \frac{1}{5x_2+4} + \frac{1}{5x_3+4} + \frac{1}{5x_4+4} + \frac{1}{5x_5+4} \ge \frac{5}{9}.$$

1.218. If a, b, c, d, e are nonnegative real numbers such that

$$ab + bc + cd + de + ea = 5,$$

then

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} + \frac{1}{e+1} \ge \frac{5}{2}.$$

1.219. If $a_1, a_2, \ldots, a_8$ are nonnegative real numbers such that $a_1a_2 + a_2a_3 + \cdots + a_8a_1 = 8$, then

$$\frac{1}{5a_1+3} + \frac{1}{5a_2+3} + \cdots + \frac{1}{5a_8+3} \ge 1.$$

1.220. If a, b, c, d, e are nonnegative real numbers such that

$$ab + bc + cd + de + ea = 5, \qquad a \ge b \ge c \ge 1 \ge d \ge e,$$

then

$$\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3} + \frac{1}{d+3} + \frac{1}{e+3} \ge \frac{5}{4}.$$

1.221. Prove that 3 is the largest positive value of the constant k such that

$$\frac{1}{a+k} + \frac{1}{b+k} + \frac{1}{c+k} + \frac{1}{d+k} + \frac{1}{e+k} \ge \frac{5}{1+k}$$

for any $a \ge b \ge c \ge d \ge 1 \ge e \ge 0$ satisfying $ab + bc + cd + de + ea = 5$.

1.222. If a, b, c, d are nonnegative real numbers such that

$$ab + ac + ad + bc + bd + cd = 6,$$

then

$$\frac{1}{ab+3} + \frac{1}{bc+3} + \frac{1}{cd+3} + \frac{1}{da+3} \geq 1.$$

1.223. If a, b, c, d are nonnegative real numbers such that

$$ab + ac + ad + bc + bd + cd = 6, \qquad a \geq b \geq c \geq d,$$

then

$$\frac{1}{ab+5} + \frac{1}{bc+5} + \frac{1}{cd+5} + \frac{1}{da+5} \geq \frac{2}{3}.$$

1.224. If a, b, c, d are nonnegative real numbers such that

$$ab + bc + cd + da = 4, \qquad a \geq b \geq c \geq d,$$

then

$$\frac{1}{ab+4} + \frac{1}{ac+4} + \frac{1}{ad+4} + \frac{1}{bc+4} + \frac{1}{bd+4} + \frac{1}{cd+4} \geq \frac{6}{5}.$$

1.225. If a, b, c, d are nonnegative real numbers such that

$$ab + bc + cd + da = 4, \qquad a \geq b \geq c \geq d,$$

then

$$\frac{1}{ab+7} + \frac{1}{ac+7} + \frac{1}{ad+7} + \frac{1}{bc+7} + \frac{1}{bd+7} + \frac{1}{cd+7} \geq \frac{3}{4}.$$

1.226. If a, b, c, d are nonnegative real numbers such that

$$ab + bc + cd + da = 4,$$

then

$$\frac{4\sqrt{2}}{3} \leq \frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} + \frac{1}{d^2+1} < 3.$$

1.227. If a, b, c, d are nonnegative real numbers such that

$$ab + bc + cd + da = 4, \qquad a \geq b \geq 1 \geq c \geq d,$$

then

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} + \frac{1}{d^2+1} \geq 2.$$

1.228. If a,b,c,d are nonnegative real numbers such that

$$ab+bc+cd+da=4,$$

then

$$2 \le \frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}+\frac{1}{d+1} \le 3.$$

1.229. If a,b,c,d,e are nonnegative real numbers such that $ab+bc+cd+de+ea=1$, then

$$3 < \frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}+\frac{1}{d+1}+\frac{1}{e+1} \le 4.$$

1.230. If a,b,c,d,e,f are nonnegative real numbers such that

$$ab+bc+cd+de+ef+fa=6,$$

then

$$(2a+1)^2+(2b+1)^2+(2c+1)^2+(2d+1)^2+(2e+1)^2+(2f+1)^2 \ge 54.$$

1.231. Prove that 4 is the largest positive value of the constant k such that

$$a_1^2+a_2^2+\cdots+a_n^2-n \ge k(a_1+a_2+\cdots+a_n-n)$$

for all odd integers $n \ge 3$ and nonnegative real numbers a_i which satisfy $a_1a_2+a_2a_3+\cdots+a_na_1=n$.

1.232. If a,b,c,d,e are positive real numbers such that

$$ab+bc+cd+de+ea=5,$$

then

$$5\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right) \ge 4(a+b+c+d+e)+5.$$

1.233. If a,b,c,d,e are positive real numbers such that

$$ab+bc+cd+de+ea=5, \qquad a \ge b \ge c \ge 1 \ge d \ge e,$$

then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+10 \ge 3(a+b+c+d+e).$$

1.234. For given $n \ge 3$, prove that 3 is the largest positive value of the constant k such that

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-n \ge k(a_1+a_2+\cdots+a_n-n)$$

for any $a_1 \ge a_2 \ge \cdots \ge a_{n-1} \ge 1 \ge a_n > 0$ with $a_1a_2+a_2a_3+\cdots+a_{n-1}a_n+a_na_1=n$.

1.235. If a,b,c,d,e,f are nonnegative real numbers such that

$$ab+bc+cd+de+ef+fa=6, \qquad a \ge b \ge c \ge d \ge e \ge f,$$

then

$$\frac{1}{a+3}+\frac{1}{b+3}+\frac{1}{c+3}+\frac{1}{d+3}+\frac{1}{e+3}+\frac{1}{f+3} \ge \frac{3}{2}.$$

1.236. Let $a_1,a_2,\ldots,a_n$ be positive real numbers such that

$$a_1a_2+a_2a_3+\cdots+a_na_1=n, \qquad a_1 \ge a_2 \ge \cdots \ge a_n.$$

Prove that:

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n} \ge a_1+a_2+\cdots+a_n.$$

1.237. If $n \ge 3$ and $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$, then

$$\sqrt{\frac{1}{n}\sum_{cyclic} a_1a_2} \ge \sqrt[n-1]{\frac{1}{n}\sum_{cyclic} a_1a_2\cdots a_{n-1}}.$$

1.238. Let a,b,c,d,e be nonnegative real numbers satisfying $ab+bc+cd+de+ea=5$. Prove that:

(a) $$(a+2)^2+(b+2)^2+(c+2)^2+(d+2)^2+(e+2)^2 \ge 45.$$

(b) $$a^{3/2}+b^{3/2}+c^{3/2}+d^{3/2}+e^{3/2} \ge 5.$$

1.239. If a,b,c,d are nonnegative real numbers such that $ab+bc+cd+da \ge 4$, then

$$(a^2+1)(b^2+1)(c^2+1)(d^2+1) \ge (a+b+c+d)^2.$$

1.240. Let a, b, c, d, e be real numbers such that $a \geq b \geq c \geq d \geq e \geq 0$ and $ab + bc + cd + de + ea = 5$. Prove that

$$a^{5/4} + b^{5/4} + c^{5/4} + d^{5/4} + e^{5/4} \geq 5.$$

1.241. If $a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n \geq 0$ such that $a_1 + a_2 + \cdots + a_n = n$, then

$$a_1a_2 + a_2a_3 + \cdots + a_na_1 \leq n.$$

1.242. If $0 \leq a_1 \leq 1 \leq a_2 \leq \cdots \leq a_n$ such that $a_1 + a_2 + \cdots + a_n = n$, then

$$a_1a_2 + a_2a_3 + \cdots + a_na_1 \leq n.$$

1.243. Suppose $n \geq 4$ and $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. If $a_1 = a_2$ and $a_{n-1} = a_n$, then

$$n(a_1a_2 + a_2a_3 + \cdots + a_na_1) \geq (a_1 + a_2 + \cdots + a_n)^2.$$

1.244. If a, b, c, d, e are positive real numbers such that $a \geq b \geq c \geq d \geq e$ and $ab + bc + cd + de + ea = 5$, then

$$a^2 + b^2 + c^2 + d^2 + e^2 + 5(a + b + c + d + e) \geq 30.$$

1.245. If $a \geq b \geq 1 \geq c \geq d \geq e \geq f \geq 0$ such that $ab + bc + cd + de + ef + fa = 6$, then

$$(2a + 3)^2 + (2b + 3)^2 + (2c + 3)^2 + (2d + 3)^2 + (2e + 3)^2 + (2f + 3)^2 \geq 150.$$

1.246. If $a \geq b \geq c \geq d \geq e \geq 0$, then

$$\sqrt{\frac{ab + bc + cd + de + ea}{5}} \geq \sqrt[3]{\frac{abc + bcd + cde + dea + eab}{5}}.$$

1.247. Let a, b, c, d be nonnegative real numbers such that

$$\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3} + \frac{1}{d+3} = 1.$$

Prove that there is a permutation (x_1, x_2, x_3, x_4) of the sequence (a, b, c, d) such that

$$x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 \geq 4.$$

1.248. Let $a_1 \geq a_2 \geq \cdots \geq a_9 \geq 0$ such that $a_1 + a_2 + \cdots + a_9 = 2$. Prove that

$$a_1a_2 + a_2a_3 + \cdots + a_9a_1 \leq 1.$$

1.249. Let n be a natural number, $n \geq 3$. Prove that there is a real number $q_n > 1$ such that

$$\frac{a_1}{a_2+a_3} + \frac{a_2}{a_3+a_4} + \cdots + \frac{a_n}{a_1+a_2} \geq \frac{n}{2}$$

for any real numbers $a_1, a_2, \ldots, a_n \in [1/q_n, q_n]$.

1.250. If a, b, c, d are positive real numbers and $0 \leq x \leq 1$, then

$$\sum_{cyclic} \frac{a}{a+(3-x)b+xc} \geq 1.$$

1.251. Prove that 18 is the largest positive value of the constant k such that

$$\frac{1}{ab^2+k} + \frac{1}{bc^2+k} + \frac{1}{ca^2+k} \geq \frac{3}{1+k}$$

for all $a \geq b \geq c \geq 0$ such that $a+b+c=3$.

1.252. Let $a = b \geq c \geq d \geq 0$ such that $ab+bc+cd+da=4$. Prove that

$$a^2+b^2+c^2+d^2+28 \geq 8(a+b+c+d).$$

1.253. If x_1, x_2, x_3, x_4, x_5 are positive real numbers such that

$$x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 = 5,$$

then

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} + \frac{25}{x_1+x_2+x_3+x_4+x_5} \geq 10.$$

1.254. Prove that $\frac{7}{6}$ is the least positive value of the power exponent k such that

$$x_1^k + x_2^k + x_3^k + x_4^k + x_5^k \geq 5$$

for any nonnegative real numbers x_i with at most one $x_i < 1$ and $x_1x_2+x_2x_3+x_3x_4+x_4x_5+x_5x_1 = 5$.

1.255. Let a, b, c, d be nonnegative real numbers such that at most one of them is larger than 1 and $ab + bc + cd + da \leq 4$. Prove that

$$a^2 + b^2 + c^2 + d^2 + 16 \geq 5(a + b + c + d).$$

1.256. Prove that $[-32, 17]$ is the range of values of the real constant k such that

$$(a + b + c + d)^4 + 4k(a + b + c + d) \geq (16 + k)(a + b)^2(c + d)^2$$

for all nonnegative real numbers a, b, c, d with $a \geq b \geq c \geq d$ and $abc + bcd + cda + dab = 4$.

1.2 Solutions

P 1.1. *If a, b, c are nonnegative real numbers such that $a+b+c=3$, then*

$$ab^2+bc^2+ca^2 \le 4.$$

(Canada, 1999)

First Solution. Assume that $a=\max\{a,b,c\}$. Since

$$ab^2+bc^2+ca^2 \le ab\cdot\frac{a+b}{2}+abc+ca^2=\frac{a(a+b)(b+2c)}{2},$$

it suffices to show that

$$a(a+b)(b+2c)\le 8.$$

By the AM-GM inequality, we have

$$a(a+b)(b+2c)\le\left[\frac{a+(a+b)+(b+2c)}{3}\right]^3=8\left(\frac{a+b+c}{3}\right)^3=8.$$

The equality holds for $a=2$, $b=0$, $c=1$ (and any cyclic permutation).

Second Solution. Let (x,y,z) be a permutation of (a,b,c) such that

$$x\ge y\ge z.$$

Since

$$xy\ge zx\ge yz,$$

by the rearrangement inequality, we have

$$\begin{aligned} ab^2+bc^2+ca^2 &= b\cdot ab+c\cdot bc+a\cdot ca \\ &\le x\cdot xy+y\cdot zx+z\cdot yz \\ &= y(x^2+xz+z^2). \end{aligned}$$

Using this result and the AM-GM inequality, we get

$$\begin{aligned} ab^2+bc^2+ca^2 \le y(x+z)^2 &= 4y\cdot\frac{x+z}{2}\cdot\frac{x+z}{2} \\ &\le 4\left(\frac{y+\frac{x+z}{2}+\frac{x+z}{2}}{3}\right)^3 \\ &= 4\left(\frac{x+y+z}{3}\right)^3=4. \end{aligned}$$

Third Solution. Without loss of generality, assume that b is between a and c; that is,

$$(b-a)(b-c)\le 0,\quad b^2+ac\le b(a+c).$$

Since

$$ab^2 + bc^2 + ca^2 = a(b^2 + ac) + bc^2 \le ab(a + c) + bc^2 = b(a^2 + ac + c^2)$$
$$\le b(a + c)^2 = b(3 - b)^2,$$

it suffices to show that

$$b(3 - b)^2 \le 4.$$

Indeed,

$$b(3 - b)^2 - 4 = (b - 1)^2(b - 4) \le (b - 1)^2(b - 3) = -(b - 1)^2(a + c) \le 0.$$

Fourth Solution. Write the inequality in the homogeneous form

$$4(a + b + c)^3 \ge 27(ab^2 + bc^2 + ca^2),$$

which is equivalent to

$$4(a^3 + b^3 + c^3) + 12(a + b)(b + c)(c + a) \ge 27(ab^2 + bc^2 + ca^2),$$

$$4\sum a^3 + 12\left(\sum a^2b + \sum ab^2 + 2abc\right) \ge 27\sum ab^2,$$

$$4\sum a^3 + 12\sum a^2b + 24abc \ge 15\sum ab^2.$$

On the other hand, the obvious inequality

$$\sum a(2a - pb - qc)^2 \ge 0$$

is equivalent to

$$4\sum a^3 + (q^2 - 4p)\sum a^2b + 6pqabc \ge (4q - p^2)\sum ab^2.$$

Setting $p = 1$ and $q = 4$ leads to the desired inequality; in addition,

$$4(a + b + c)^3 - 27(ab^2 + bc^2 + ca^2) = \sum a(2a - b - 4c)^2 \ge 0.$$

□

P 1.2. *If a, b, c are positive real numbers such that $a + b + c = 3$, then*

$$(ab + bc + ca)(ab^2 + bc^2 + ca^2) \le 9.$$

Solution. Let (x, y, z) be a permutation of (a, b, c) such that $x \ge y \ge z$. As shown in the second solution of P 1.1,

$$ab^2 + bc^2 + ca^2 \le y(x^2 + xz + z^2).$$

Consequently, it suffices to show that

$$y(xy + yz + zx)(x^2 + xz + z^2) \le 9.$$

By the AM-GM inequality, we get

$$4(xy + yz + zx)(x^2 + xz + z^2) \le (xy + yz + zx + x^2 + xz + z^2)^2$$

$$= (x + z)^2(x + y + z)^2 = 9(x + z)^2.$$

Thus, we still have to show that

$$y(x + z)^2 \le 4.$$

This follows from the AM-GM inequality, as follows:

$$2y(x + z)^2 \le \left[\frac{2y + (x + z) + (x + z)}{3}\right]^3 = 8.$$

The equality holds for $a = b = c = 1$.

□

P 1.3. *If a, b, c are nonnegative real numbers such that $a^2 + b^2 + c^2 = 3$, then*

(a) $$ab^2 + bc^2 + ca^2 \le abc + 2;$$

(b) $$\frac{a}{b + 2} + \frac{b}{c + 2} + \frac{c}{a + 2} \le 1.$$

(Vasile Cîrtoaje, 2005)

Solution. (a) ***First Solution.*** Without loss of generality, assume that b is between a and c; that is,

$$(b - a)(b - c) \le 0, \quad b^2 + ac \le b(a + c).$$

Since

$$ab^2 + bc^2 + ca^2 = a(b^2 + ac) + bc^2 \le ab(a + c) + bc^2 = b(a^2 + c^2) + abc,$$

it suffices to show that

$$b(a^2 + c^2) \le 2.$$

We have

$$2 - b(a^2 + c^2) = 2 - b(3 - b^2) = (b - 1)^2(b + 2) \ge 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 0$, $b = 1$, $c = \sqrt{2}$ (or any cyclic permutation).

Second Solution. Let (x, y, z) be a permutation of (a, b, c) such that $x \ge y \ge z$. As shown in the second solution of P 1.1,

$$ab^2 + bc^2 + ca^2 \le y(x^2 + xz + z^2).$$

Therefore, it suffices to show that

$$y(x^2 + xz + z^2) \le xyz + 2,$$

which can be written as

$$y(x^2 + z^2) \le 2.$$

Indeed,

$$2 - y(x^2 + z^2) = 2 - y(3 - y^2) = (y - 1)^2(y + 2) \ge 0.$$

(b) Write the inequality as follows:

$$\sum a(a + 2)(c + 2) \le (a + 2)(b + 2)(c + 2),$$

$$ab^2 + bc^2 + ca^2 + 2(a^2 + b^2 + c^2) \le abc + 8,$$

$$ab^2 + bc^2 + ca^2 \le abc + 2.$$

The last inequality is just the inequality in (a).

□

P 1.4. *If $a, b, c \ge 1$, then*

(a)
$$2(ab^2 + bc^2 + ca^2) + 3 \ge 3(ab + bc + ca);$$

(b)
$$ab^2 + bc^2 + ca^2 + 6 \ge 3(a + b + c).$$

Solution. (a) ***First Solution.*** From

$$a(b - 1)^2 + b(c - 1)^2 + c(a - 1)^2 \ge 0,$$

we get

$$ab^2 + bc^2 + ca^2 \ge 2(ab + bc + ca) - (a + b + c).$$

Using this inequality gives

$$2(ab^2 + bc^2 + ca^2) + 3 - 3(ab + bc + ca) \ge (ab + bc + ca) - 2(a + b + c) + 3$$

$$= (a - 1)(b - 1) + (b - 1)(c - 1) + (c - 1)(a - 1) \ge 0.$$

The equality holds for $a = b = c = 1$.

Second Solution. From

$$\sum b(a - 1)(b - 1) \ge 0,$$

we get

$$ab^2 + bc^2 + ca^2 \geq a^2 + b^2 + c^2 + ab + bc + ca - (a + b + c).$$

Thus, it suffices to show that

$$2(a^2 + b^2 + c^2) + 2(ab + bc + ca) - 2(a + b + c) + 3 \geq 3(ab + bc + ca),$$

which is equivalent to

$$2(a^2 + b^2 + c^2) - 2(a + b + c) + 3 \geq ab + bc + ca,$$

$$(a - 1)^2 + (b - 1)^2 + (c - 1)^2 + (a^2 + b^2 + c^2 - ab - bc - ca) \geq 0,$$

$$2(a - 1)^2 + 2(b - 1)^2 + 2(c - 1)^2 + (a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0.$$

(b) The inequality in (b) follows by summing the inequality in (a) and the obvious inequality

$$3(a - 1)(b - 1) + 3(b - 1)(c - 1) + 3(c - 1)(a - 1) \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 1.5. *If a, b, c are nonnegative real numbers such that*

$$a + b + c = 3, \qquad a \geq b \geq c,$$

then

(a)
$$a^2b + b^2c + c^2a \geq ab + bc + ca;$$

(b)
$$8(ab^2 + bc^2 + ca^2) + 3abc \leq 27;$$

(c)
$$\frac{18}{a^2b + b^2c + c^2a} \leq \frac{1}{abc} + 5.$$

Solution. (a) Write the inequality in the homogeneous form

$$3(a^2b + b^2c + c^2a) \geq (a + b + c)(ab + bc + ca),$$

which is equivalent to

$$a^2b + b^2c + c^2a - 3abc \geq ab^2 + bc^2 + ca^2 - a^2b - b^2c - c^2a.$$

This inequality is true because

$$a^2b + b^2c + c^2a - 3abc \geq 0$$

(by the AM-GM inequality) and

$$ab^2 + bc^2 + ca^2 - a^2b - b^2c - c^2a = (a - b)(b - c)(c - a) \leq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 3$ and $b = c = 0$.

(b) Write the inequality in the homogeneous form

$$(a+b+c)^3 \geq 8(ab^2+bc^2+ca^2)+3abc,$$

$$\sum a^3 + 3abc + 3\sum a^2b \geq 5\sum ab^2,$$

$$\sum a^3 + 3abc - \left(\sum ab^2 + \sum a^2b\right) \geq 4\left(\sum ab^2 - \sum a^2b\right),$$

$$\sum a^3 + 3abc - \sum ab(a+b) \geq 4(a-b)(b-c)(c-a).$$

The inequality is true since

$$(a-b)(b-c)(c-a) \leq 0$$

and, by Schur's inequality of degree three,

$$\sum a^3 + 3abc - \sum ab(a+b) \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = b = 3/2$ and $c = 0$.

(c) Since

$$ab^2+bc^2+ca^2-a^2b-b^2c-c^2a = (a-b)(b-c)(c-a) \leq 0,$$

it suffices to prove the symmetric inequality

$$\frac{36}{(a^2b+b^2c+c^2a)+(ab^2+bc^2+ca^2)} \leq \frac{1}{abc}+5,$$

which is equivalent to

$$\frac{36}{(a+b+c)(ab+bc+ca)-3abc} \leq \frac{1}{abc}+5,$$

$$\frac{12}{ab+bc+ca-abc} \leq \frac{1}{abc}+5,$$

$$\frac{12}{a(b+c)-(a-1)bc} \leq \frac{1}{a \cdot bc}+5,$$

$$\frac{12}{a(3-a)-(a-1)bc} \leq \frac{1}{a \cdot bc}+5.$$

Since $a-1 \geq 0$ and

$$4bc \leq (b+c)^2 = (3-a)^2,$$

it suffices to show that

$$\frac{48}{4a(3-a)-(a-1)(3-a)^2} \leq \frac{4}{a(3-a)^2}+5,$$

which is equivalent to

$$\frac{48}{(3-a)(3+a^2)} \le \frac{4}{a(3-a)^2} + 5,$$

$$5a^5 - 30a^4 + 60a^3 - 38a^2 - 9a + 12 \ge 0,$$

$$(a-1)^2(5a^3 - 20a^2 + 15a + 12) \ge 9.$$

We need to show that $1 \le a \le 3$ involves

$$5a^3 - 20a^2 + 15a + 12 \ge 0.$$

If $1 \le a \le 2$, then

$$5a^3 - 20a^2 + 15a + 12 = 5a(a-2)^2 + (12 - 5a) > 0.$$

If $2 \le a \le 3$, then

$$5a^3 - 20a^2 + 15a + 12 = 5(a-2)^3 + 10a^2 - 45a + 52 \ge 10a^2 - 45a + 52 > 0$$
$$= 10\left(a - \frac{9}{4}\right)^2 + \frac{11}{8} > 0.$$

The equality holds for $a = b = c = 1$.

□

P 1.6. *If a, b, c are nonnegative real numbers such that*

$$a^2 + b^2 + c^2 = 3, \qquad a \ge b \ge c,$$

then

$$ab^2 + bc^2 + ca^2 \le \frac{3}{4}(ab + bc + ca + 1).$$

Solution. Let us denote

$$p = a + b + c, \qquad q = ab + bc + ca.$$

From $a^2 + b^2 + c^2 = 3$, it follows that

$$2q = p^2 - 3.$$

In addition, from the known inequalities

$$(a + b + c)^2 \ge a^2 + b^2 + c^2$$

and

$$3(a^2 + b^2 + c^2) \ge (a + b + c)^2,$$

we get

$$\sqrt{3} \le p \le 3.$$

Since

$$ab^2 + bc^2 + ca^2 - a^2b - b^2c - c^2a = (a-b)(b-c)(c-a) \le 0,$$

it suffices to show that

$$ab^2 + bc^2 + ca^2 + (a^2b + b^2c + c^2a) \le \frac{3}{2}(ab + bc + ca + 1).$$

which is equivalent to

$$pq \le 3abc + \frac{3}{2}(q+1),$$

$$6abc + 3(q+1) \ge 2pq.$$

Consider two cases: $\sqrt{3} \le p \le \frac{12}{5}$ and $\frac{12}{5} \le p \le 3$.

Case 1: $\sqrt{3} \le p \le \frac{12}{5}$. Since

$$6abc + 3(q+1) - 2pq \ge 3(q+1) - 2pq = 3 - (2p-3)q = \frac{1}{2}[6 - (2p-3)(p^2-3)],$$

it suffices to show that

$$(2p-3)(p^2-3) \le 6.$$

Indeed, we have

$$(2p-3)(p^2-3) \le \left(\frac{24}{5} - 3\right)\left(\frac{144}{25} - 3\right) = \frac{621}{125} < 6.$$

Case 2: $\frac{12}{5} \le p \le 3$. According to Schur's inequality of degree three, we have

$$p^3 + 9abc \ge 4pq.$$

Thus, it suffices to prove that

$$2(4pq - p^3) + 9(q+1) \ge 6pq,$$

which is equivalent to

$$(2p+9)q - 2p^3 + 9 \ge 0,$$

$$(2p+9)(p^2-3) - 4p^3 + 18 \ge 0,$$

$$-2p^3 + 9p^2 - 6p - 9 \ge 0,$$

$$(3-p)(2p^2 - 3p - 3) \ge 0.$$

This inequality is true since $3 - p \ge 0$ and

$$2p^2 - 3p - 3 \ge \frac{24}{5}p - 3p - 3 = \frac{9}{5}p - 3 \ge \frac{9}{5} \cdot \frac{12}{5} - 3 > 0.$$

The equality holds for $a = b = c = 1$.

□

P 1.7. *If a, b, c are nonnegative real numbers such that $a^2+b^2+c^2=3$, then*

$$a^2b^3+b^2c^3+c^2a^3 \le 3.$$

(Vasile Cîrtoaje, 2005)

Solution. Let (x, y, z) be a permutation of (a, b, c) such that

$$x \ge y \ge z.$$

Since

$$x^2y^2 \ge z^2x^2 \ge y^2z^2,$$

the rearrangement inequality yields

$$a^2b^3+b^2c^3+c^2a^3 = b\cdot a^2b^2 + c\cdot b^2c^2 + a\cdot c^2a^2 \le x\cdot x^2y^2 + y\cdot z^2x^2 + z\cdot y^2z^2$$

$$= y(x^3y+z^2x^2+yz^3) \le y\left(x^2\cdot\frac{x^2+y^2}{2} + z^2x^2 + z^2\cdot\frac{y^2+z^2}{2}\right)$$

$$= \frac{y(x^2+z^2)(x^2+y^2+z^2)}{2} = \frac{3y(x^2+z^2)}{2}.$$

Thus, it suffices to show that

$$y(x^2+z^2) \le 2$$

for $x^2+y^2+z^2=3$. By the AM-GM inequality, we get

$$6 = 2y^2+(x^2+z^2)+(x^2+z^2) \ge 3\sqrt[3]{2y^2(x^2+z^2)^2}.$$

The equality holds for $a=b=c=1$.

□

P 1.8. *If a, b, c are nonnegative real numbers such that $a+b+c=3$, then*

$$a^4b^2+b^4c^2+c^4a^2+4 \ge a^3b^3+b^3c^3+c^3a^3.$$

Solution. Write the inequality as

$$a^2(a^2b^2+c^4-ab^3-ac^3)+4 \ge b^2c^2(bc-b^2).$$

Since

$$2\sum(a^2b^2+c^4-ab^3-ac^3) = \sum[a^4+b^4+2a^2b^2-2ab(a^2+b^2)]$$
$$= \sum(a^2+b^2)(a-b)^2 \ge 0,$$

we may assume (without loss of generality) that

$$a^2b^2 + c^4 - ab^3 - ac^3 \geq 0.$$

Thus, it suffices to show that

$$4 \geq b^2c^2(bc - b^2).$$

Since

$$bc - b^2 \leq \frac{c^2}{4},$$

it is enough to prove that

$$16 \geq b^2c^4.$$

From

$$3 = a + b + c \geq b + \frac{c}{2} + \frac{c}{2} \geq 3\sqrt[3]{b\left(\frac{c}{2}\right)^2},$$

the conclusion follows. The equality holds for $a = 0$, $b = 1$, $c = 2$ (or any cyclic permutation). □

P 1.9. *If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then*

(a) $$ab^2 + bc^2 + ca^2 + abc \leq 4;$$

(b) $$\frac{a}{4-b} + \frac{b}{4-c} + \frac{c}{4-a} \leq 1;$$

(c) $$ab^3 + bc^3 + ca^3 + (ab + bc + ca)^2 \leq 12;$$

(d) $$\frac{ab^2}{1+a+b} + \frac{bc^2}{1+b+c} + \frac{ca^2}{1+c+a} \leq 1.$$

Solution. (a) ***First Solution.*** Without loss of generality, assume that b is between a and c; that is,

$$(b - a)(b - c) \leq 0, \qquad b^2 + ca \leq b(c + a).$$

Using this result and the AM-GM inequality, we have

$$ab^2 + bc^2 + ca^2 + abc = a(b^2 + ca) + bc^2 + abc \leq ab(c + a) + bc^2 + abc$$

$$= b(a + c)^2 = \frac{1}{2} \cdot 2b \cdot (a + c) \cdot (a + c) \leq \frac{1}{2}\left[\frac{2b + (a + c) + (a + c)}{3}\right]^3 = 4.$$

The equality holds for $a = b = c = 1$, and also for $a = 0$, $b = 1$, $c = 2$ (or any cyclic permutation).

Second Solution. Let (x, y, z) be a permutation of (a, b, c) such that

$$x \geq y \geq z.$$

As shown in the second solution of P 1.1,

$$ab^2 + bc^2 + ca^2 \le y(x^2 + xz + z^2);$$

hence

$$ab^2 + bc^2 + ca^2 + abc \le y(x + z)^2.$$

Thus, it suffices to show that $x + y + z = 3$ involves

$$y(x + z)^2 \le 4.$$

According to the AM-GM inequality, we have

$$\frac{1}{4}y(x+z)^2 = y \cdot \frac{x+z}{2} \cdot \frac{x+z}{2} \le \left(\frac{y + \frac{x+z}{2} + \frac{x+z}{2}}{3} \right)^3 = 1.$$

Third Solution. Write the inequality in the homogeneous form

$$4(a + b + c)^3 \ge 27(ab^2 + bc^2 + ca^2 + abc).$$

Without loss of generality, suppose that $a = min\{a, b, c\}$. Putting $b = a + x$ and $c = a + y$, where $x, y \ge 0$, the inequality can be restated as

$$9(x^2 - xy + y^2)a + (2x - y)^2(x + 4y) \ge 0,$$

which is obviously true.

(b) ***First Solution.*** Write the inequality in the homogeneous form

$$\sum \frac{a}{4a + b + 4c} \le \frac{1}{3}.$$

Multiplying by $a + b + c$, the inequality becomes as follows:

$$\sum \frac{a^2 + ab + ac}{4a + b + 4c} \le \frac{a + b + c}{3},$$

$$\sum \left(\frac{a^2 + ab + ac}{4a + b + 4c} - \frac{a}{4} \right) \le \frac{a + b + c}{12},$$

$$\sum \frac{9ab}{4a + b + 4c} \le a + b + c.$$

Since

$$\frac{9}{4a + b + 4c} = \frac{9}{(2a + c) + (2a + c) + (2c + b)} \le \frac{1}{2a + c} + \frac{1}{2a + c} + \frac{1}{2c + b}$$
$$= \frac{2}{2a + c} + \frac{1}{2c + b},$$

we have

$$\sum \frac{9ab}{4a+b+4c} \le \sum \frac{2ab}{2a+c} + \sum \frac{ab}{2c+b} = \sum \frac{2ab}{2a+c} + \sum \frac{bc}{2a+c}$$
$$= \sum \frac{2ab+bc}{2a+c} = \sum b = a+b+c.$$

The equality holds for $a=b=c=1$, and also for $a=0$, $b=1$, $c=2$ (or any cyclic permutation).

Second Solution. Write the inequality as follows:

$$\sum a(4-a)(4-c) \le (4-a)(4-b)(4-c),$$

$$32 + \sum ab^2 + abc \le 4\left(\sum a^2 + 2\sum ab\right),$$

$$32 + \sum ab^2 + abc \le 4\left(\sum a\right)^2,$$

$$ab^2 + bc^2 + ca^2 + abc \le 4.$$

The last inequality is just the inequality in (a).

(c) Using the inequality in (a), we get

$$(a+b+c)(ab^2+bc^2+ca^2+abc) \le 12,$$

which is equivalent to the desired inequality

$$ab^3 + bc^3 + ca^3 + (ab+bc+ca)^2 \le 12.$$

(d) Let $q = ab+bc+ca$. Since

$$\sum ab^2(1+b+c)(1+c+a) = \sum ab^2(4+q+c+c^2) = (4+q)\sum ab^2 + (3+q)abc$$

and

$$\prod(1+a+b) = 1 + \sum(a+b) + \sum(b+c)(c+a) + \prod(a+b)$$
$$= 7 + 3q + \sum c^2 + (3q - abc) = 16 + 4q - abc,$$

the inequality is equivalent to

$$(4+q)\sum ab^2 + (3+q)abc \le 16 + 4q - abc,$$

$$(4+q)\left(\sum ab^2 + abc - 4\right) \le 0.$$

According to (a), the desired inequality is clearly true.

Remark. The following statement is also valid:

- *If a, b, c are nonnegative real numbers such that $a+b+c=3$, then*

$$ab^2+bc^2+ca^2+abc+(a-1)^2(b-1)^2(c-1)^2 \le 4,$$

with equality for $a=b=c=1$, and also for $a=0$, $b=1$, $c=2$ (or any cyclic permutation).

Having in view the second solution of (a), it is enough to show that

$$(a-1)^2(b-1)^2(c-1)^2 \le (4-b)(1-b)^2,$$

where b is between a and c. This is true if

$$|(a-1)(c-1)| \le \sqrt{4-b}.$$

Assuming that $a \le c$ (hence $a \le b \le c$, $a \le 1$, $c \ge 1$), the inequality can be written as follows:

$$(1-a)(c-1) \le \sqrt{4-b},$$

$$a+c-1 \le ac+\sqrt{4-b},$$

$$2-b \le ac+\sqrt{4-b}.$$

This is true if

$$2-b \le \sqrt{4-b}.$$

Indeed,

$$\sqrt{4-b}-(2-b) = \frac{4-b-(2-b)^2}{\sqrt{4-b}+2-b} = \frac{b(3-b)}{\sqrt{4-b}+2-b}$$

$$= \frac{b(a+c)}{\sqrt{4-b}+2-b}) \ge 0.$$

□

P 1.10. *If a, b, c are positive real numbers, then*

$$\frac{1}{a(a+2b)}+\frac{1}{b(b+2c)}+\frac{1}{c(c+2a)} \ge \frac{3}{ab+bc+ca}.$$

First Solution. Write the inequality as

$$\sum \frac{a(b+c)+bc}{a(a+2b)} \ge 3,$$

$$\sum \frac{b+c}{a+2b}+\sum \frac{bc}{a(a+2b)} \ge 3.$$

It suffices to show that

$$\sum \frac{b+c}{a+2b} \ge 2$$

and

$$\sum \frac{bc}{a(a+2b)} \geq 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{b+c}{a+2b} \geq \frac{\left[\sum(b+c)\right]^2}{\sum(b+c)(a+2b)} = \frac{4\left(\sum a\right)^2}{2\sum a^2 + 4\sum ab} = 2$$

and

$$\sum \frac{bc}{a(a+2b)} \geq \frac{\left(\sum bc\right)^2}{abc\sum(a+2b)} = \frac{\left(\sum bc\right)^2}{3abc\sum a} = 1 + \frac{\sum a^2(b-c)^2}{6abc\sum a} \geq 1.$$

The equality holds for $a = b = c$.

Second Solution. We apply the Cauchy-Schwarz inequality in the following way

$$\sum \frac{1}{a(a+2b)} \geq \frac{\left(\sum c\right)^2}{\sum ac^2(a+2b)} = \frac{\left(\sum a\right)^2}{\sum a^2b^2 + 2abc\sum a}.$$

Thus, it suffices to show that

$$\frac{\left(\sum a\right)^2}{\sum a^2b^2 + 2abc\sum a} \geq \frac{3}{\sum ab},$$

which is equivalent to

$$\left(\sum ab\right)\left(\sum a^2 + 2\sum ab\right) \geq 3\sum a^2b^2 + 6abc\sum a,$$

$$\sum ab(a^2+b^2) \geq \sum a^2b^2 + abc\sum a.$$

The latter inequality follows by summing the obvious inequalities

$$\sum ab(a^2+b^2) \geq 2\sum a^2b^2$$

and

$$\sum a^2b^2 \geq abc\sum a.$$

□

P 1.11. *If a, b, c are positive real numbers such that $a+b+c=3$, then*

$$\frac{a}{b^2+2c} + \frac{b}{c^2+2a} + \frac{c}{a^2+2b} \geq 1.$$

Solution. Using the Cauchy-Schwarz inequality, we get

$$\sum \frac{a}{b^2+2c} \ge \frac{\left(\sum a\right)^2}{\sum a(b^2+2c)} = 1 + \frac{\sum a^2 - \sum ab^2}{\sum ab^2 + 2\sum ab}.$$

Thus, it suffices to show that

$$\sum a^2 - \sum ab^2 \ge 0.$$

Write this inequality in the homogeneous form

$$(a+b+c)(a^2+b^2+c^2) \ge 3(ab^2+bc^2+ca^2),$$

which is equivalent to the obvious inequality

$$a(a-c)^2 + b(b-a)^2 + c(c-b)^2 \ge 0.$$

The equality holds for $a=b=c=1$.

□

P 1.12. *If a,b,c are positive real numbers such that $a+b+c \ge 3$, then*

$$\frac{a-1}{b+1} + \frac{b-1}{c+1} + \frac{c-1}{a+1} \ge 0.$$

Solution. Write the inequality as

$$(a^2-1)(c+1) + (b^2-1)(a+1) + (c^2-1)(b+1) \ge 0,$$

$$ab^2+bc^2+ca^2+a^2+b^2+c^2 \ge a+b+c+3.$$

From

$$a(b-1)^2 + b(c-1)^2 + c(a-1)^2 \ge 0,$$

we get

$$ab^2+bc^2+ca^2 \ge 2(ab+bc+ca) - (a+b+c).$$

Using this inequality yields

$$ab^2+bc^2+ca^2+a^2+b^2+c^2-a-b-c-3 \ge (a+b+c)^2 - 2(a+b+c) - 3$$

$$= (a+b+c-3)(a+b+c+1) \ge 0.$$

The equality holds for $a=b=c=1$.

□

P 1.13. *If a, b, c are positive real numbers such that $a+b+c=3$, then*

(a)
$$\frac{1}{a^2b+2}+\frac{1}{b^2c+2}+\frac{1}{c^2a+2}\ge 1;$$

(b)
$$\frac{1}{a^3b+2}+\frac{1}{b^3c+2}+\frac{1}{c^3a+2}\ge 1.$$

Solution. By the AM-GM inequality, we have

$$1=\left(\frac{a+b+c}{3}\right)^3\ge abc.$$

On the other hand, according to the inequality in P 1.9-(a):

$$ab^2+bc^2+ca^2\le 4-abc.$$

(a) By expanding, the inequality can be restated as

$$a^3b^3c^3+abc(ab^2+bc^2+ca^2)\le 4.$$

It is true if

$$a^3b^3c^3+abc(4-abc)\le 4,$$

which is equivalent to

$$(abc-1)(a^2b^2c^2+4)\ge 0.$$

The equality occurs for $a=b=c=1$.

(b) By expanding, the inequality becomes

$$a^4b^4c^4+abc(a^2b^3+b^2c^3+c^2a^3)\le 4.$$

Let $p=a+b+c$, $q=ab+bc+ca$ and $r=abc$. Using the inequality from P 1.9-(a), we have

$$a^2b^3+b^2c^3+c^2a^3=(ab^2+bc^2+ca^2)(ab+bc+ca)-abc(a^2+b^2+c^2+ab+bc+ca)$$

$$\le (4-r)q-r(9-q)=4q-9r.$$

Thus, it suffices to show that

$$r^4+r(4q-9r)\le 4.$$

By Schur's inequality, we have

$$4q\le \frac{p^3+9r}{p}=9+3r.$$

Therefore,

$$r^4+r(4q-9r))-4\le r^4+r(9-6r)-4=r^4-6r^2+9r-4=(r-1)(r^3+r^2-5r+4)$$

$$= (r-1)[r^3 + (1-r)(4-r)] \leq 0.$$

The equality occurs for $a = b = c = 1$.

Remark 1. We can generalize the inequality in (a) as follows (see P 1.38):

- *If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then the inequality*

$$\frac{1}{a^2b+k} + \frac{1}{b^2c+k} + \frac{1}{c^2a+k} \geq \frac{3}{1+k}$$

holds for $0 \leq k \leq 8$, with equality for $a = b = c = 1$. If $k = 8$, then the equality occurs again for $a = 0$, $b = 1$, $c = 2$ (or any cyclic permutation).

Remark 2. We claim that the following *open generalization* of the inequality in (b) is true:

- *If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then the inequality*

$$\frac{1}{a^3b+k} + \frac{1}{b^3c+k} + \frac{1}{c^3a+k} \geq \frac{3}{1+k}$$

holds for $0 \leq k \leq k_0$, where $k_0 = \dfrac{1458}{473} \approx 3.08245$. For $k = k_0$, the equality occurs when $a = b = c = 1$, and also when $a = \dfrac{9}{4}$, $b = \dfrac{3}{4}$ and $c = 0$ (or any cyclic permutation).

□

P 1.14. *If a, b, c are positive real numbers such that $a + b + c = 3$, then*

$$\frac{ab}{9-4bc} + \frac{bc}{9-4ca} + \frac{ca}{9-4ab} \leq \frac{3}{5}.$$

Solution. We have

$$\sum \frac{ab}{9-4bc} \leq \sum \frac{ab}{9-(b+c)^2} = \sum \frac{b}{3+b+c} = \sum \frac{b}{a+2b+2c}$$

$$= \frac{1}{2}\sum \left[1 - \frac{a+2c}{a+2b+2c}\right] = \frac{3}{2} - \frac{1}{2}\sum \frac{a+2c}{a+2b+2c}.$$

Thus, it suffices to show that

$$\sum \frac{a+2c}{a+2b+2c} \geq \frac{9}{5}.$$

Using the Cauchy-Schwarz inequality, we get

$$\sum \frac{a+2c}{a+2b+2c} \geq \frac{[\sum(a+2c)]^2}{\sum(a+2c)(a+2b+2c)} = \frac{9(a+b+c)^2}{5(a+b+c)^2} = \frac{9}{5}.$$

The equality holds for $a = b = c = 1$.

□

P 1.15. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

(a)
$$\frac{a^2}{2a+b^2}+\frac{b^2}{2b+c^2}+\frac{c^2}{2c+a^2}\ge 1;$$

(b)
$$\frac{a^2}{a+2b^2}+\frac{b^2}{b+2c^2}+\frac{c^2}{c+2a^2}\ge 1.$$

Solution. (a) By the Cauchy-Schwarz inequality, we have

$$\sum\frac{a^2}{2a+b^2}\ge\frac{\left(\sum a^2\right)^2}{\sum a^2(2a+b^2)}=\frac{\sum a^4+2\sum a^2b^2}{2\sum a^3+\sum a^2b^2}.$$

Thus, it suffices to prove that

$$\sum a^4+\sum a^2b^2\ge 2\sum a^3,$$

which is equivalent to the homogeneous inequalities

$$3\sum a^4+3\sum a^2b^2\ge 2\left(\sum a\right)\left(\sum a^3\right),$$

$$\sum a^4+3\sum a^2b^2-2\sum ab(a^2+b^2)\ge 0,$$

$$\sum(a-b)^4\ge 0.$$

The equality holds for $a=b=c=1$.

(b) By the Cauchy-Schwarz inequality, we get

$$\sum\frac{a^2}{a+2b^2}\ge\frac{\left(\sum a^2\right)^2}{\sum a^2(a+2b^2)}=\frac{\sum a^4+2\sum a^2b^2}{\sum a^3+2\sum a^2b^2}.$$

Thus, it suffices to prove that

$$\sum a^4\ge\sum a^3.$$

We have

$$\sum a^4-\sum a^3=\sum(a^4-a^3-a+1)=\sum(a-1)(a^3-1)\ge 0.$$

The equality holds for $a=b=c=1$.

□

P 1.16. *Let a,b,c be positive real numbers such that $a+b+c=3$. Then,*

$$\frac{1}{a+b^2+c^3}+\frac{1}{b+c^2+a^3}+\frac{1}{c+a^2+b^3}\le 1.$$

(Vasile Cîrtoaje, 2009)

Solution *(by Vo Quoc Ba Can).* By the Cauchy-Schwarz inequality, we have

$$\sum \frac{1}{a+b^2+c^3} \le \sum \frac{a^3+b^2+c}{(a^2+b^2+c^2)^2} = \frac{\sum a^3 + \sum a^2 + 3}{(a^2+b^2+c^2)^2}.$$

Therefore, it suffices to show that

$$(a^2+b^2+c^2)^2 \ge a^3+b^3+c^3+(a^2+b^2+c^2)+3,$$

or, equivalently,

$$(a^2+b^2+c^2)^2 + \sum a^2(3-a) \ge 4(a^2+b^2+c^2)+3.$$

Let us denote $t = a^2+b^2+c^2$. Applying again the Cauchy-Schwarz inequality, we get

$$\sum a^2(3-a) \ge \frac{[\sum a(3-a)]^2}{\sum(3-a)} = \frac{(9-a^2-b^2-c^2)^2}{6}.$$

Thus, it is enough to show that

$$t^2 + \frac{(9-t)^2}{6} \ge 4t+3.$$

This inequality reduces to $(t-3)^2 \ge 0$. The equality occurs for $a=b=c=1$.

□

P 1.17. *If a, b, c are positive real numbers, then*

$$\frac{1+a^2}{1+b+c^2} + \frac{1+b^2}{1+c+a^2} + \frac{1+c^2}{1+a+b^2} \ge 2.$$

Solution. From

$$1+b+c^2 \le 1 + \frac{1+b^2}{2} + c^2,$$

we have

$$\frac{1+a^2}{1+b+c^2} \ge \frac{2(1+a^2)}{1+b^2+2(1+c^2)}.$$

Thus, it suffices to show that

$$\frac{x}{y+2z} + \frac{y}{z+2x} + \frac{z}{x+2y} \ge 1,$$

where

$$x = 1+a^2, \quad y = 1+b^2, \quad z = 1+c^2.$$

Using the Cauchy-Schwarz inequality gives

$$\frac{x}{y+2z}+\frac{y}{z+2x}+\frac{z}{x+2y} \geq \frac{(x+y+z)^2}{x(y+2z)+y(z+2x)+z(x+2y)}$$
$$= \frac{(x+y+z)^2}{3(xy+yz+zx)} \geq 1.$$

The equality occurs for $a=b=c=1$.

□

P 1.18. *If a,b,c are nonnegative real numbers, then*

$$\frac{a}{4a+4b+c}+\frac{b}{4b+4c+a}+\frac{c}{4c+4a+b} \leq \frac{1}{3}.$$

(Pham Kim Hung, 2007)

Solution. If two of a,b,c are zero, then the inequality is trivial. Otherwise, multiplying by $4(a+b+c)$, the inequality becomes as follows:

$$\sum \frac{4a(a+b+c)}{4a+4b+c} \leq \frac{4}{3}(a+b+c),$$

$$\sum \left[\frac{4a(a+b+c)}{4a+4b+c} - a\right] \leq \frac{1}{3}(a+b+c),$$

$$\sum \frac{ca}{4a+4b+c} \leq \frac{1}{9}(a+b+c).$$

By the AM-HM inequality, we get

$$\frac{9}{4a+4b+c} = \frac{9}{(2b+c)+2(2a+b)} \leq \frac{1}{2b+c}+\frac{2}{2a+b}.$$

Therefore,

$$\sum \frac{ca}{4a+4b+c} \leq \frac{1}{9}\sum ca\left(\frac{1}{2b+c}+\frac{2}{2a+b}\right)$$
$$= \frac{1}{9}\left(\sum \frac{ca}{2b+c}+\sum \frac{2ab}{2b+c}\right) = \frac{1}{9}\sum a,$$

as desired. The equality occurs for $a=b=c$, and also for $a=2b$ and $c=0$ (or any cyclic permutation).

□

P 1.19. *If a,b,c are positive real numbers, then*

$$\frac{a+b}{a+7b+c}+\frac{b+c}{b+7c+a}+\frac{c+a}{c+7a+b} \geq \frac{2}{3}.$$

Solution. Write the inequality as

$$\sum\left(\frac{a+b}{a+7b+c}-\frac{1}{k}\right)\ge\frac{2}{3}-\frac{3}{k},\quad k>0,$$

$$\sum\frac{(k-1)a+(k-7)b-c}{a+7b+c}\ge\frac{2k-9}{3}.$$

Consider that all fractions in the left hand side are nonnegative and apply the Cauchy-Schwarz inequality, as follows:

$$\sum\frac{(k-1)a+(k-7)b-c}{a+7b+c}\ge\frac{[(k-1)\sum a+(k-7)\sum b-\sum c]^2}{\sum(a+7b+c)[(k-1)a+(k-7)b-c]}$$
$$=\frac{(2k-9)^2\left(\sum a\right)^2}{(8k-51)\sum a^2+2(5k-15)\sum ab}.$$

We choose $k=12$ to have $8k-51=5k-15$, hence

$$(8k-51)\sum a^2+2(5k-15)\sum ab=45\left(\sum a\right)^2.$$

For this value of k, the desired inequality

$$\sum\frac{(k-1)a+(k-7)b-c}{a+7b+c}\ge\frac{2k-9}{3}$$

can be restated as

$$\sum\frac{11a+5b-c}{a+7b+c}\ge 5.$$

Without loss of generality, assume that $a=\max\{a,b,c\}$. Consider further two cases.

Case 1: $11b+5c-a\ge 0$. By the Cauchy-Schwarz inequality, we have

$$\sum\frac{11a+5b-c}{a+7b+c}\ge\frac{[\sum(11a+5b-c)]^2}{\sum(a+7b+c)(11a+5b-c)}=\frac{225\left(\sum a\right)^2}{45\left(\sum a\right)^2}=5.$$

Case 2: $11b+5c-a<0$. We have

$$\sum\frac{a+b}{a+7b+c}>\frac{a+b}{a+7b+c}=\frac{2}{3}+\frac{a-11b-2c}{3(a+7b+c)}>\frac{2}{3}.$$

Thus, the proof is completed. The equality holds for $a=b=c$.

□

P 1.20. *If a,b,c are positive real numbers, then*

$$\frac{a+b}{a+3b+c}+\frac{b+c}{b+3c+a}+\frac{c+a}{c+3a+b}\ge\frac{6}{5}.$$

(Vasile Cîrtoaje, 2007)

Solution. Due to homogeneity, we may assume that

$$a+b+c=1,$$

when the inequality becomes

$$\sum \frac{1-c}{1+2b} \ge \frac{6}{5},$$

$$5\sum(1-c)(1+2c)(1+2a) \ge 6(2a+1)(2b+1)(2c+1),$$

$$5\left(4+6\sum ab-4\sum a^2b\right) = 6\left(3+4\sum ab+8abc\right),$$

$$1+3\sum ab \ge 10\sum a^2b+24abc,$$

$$(a+b+c)^3+3(a+b+c)(ab+bc+ca) \ge 10(a^2b+b^2c+c^a)+24abc,$$

$$\sum a^3+6\sum ab^2 \ge 4\sum a^2b+9abc,$$

$$\left[2\sum a^3-\sum ab(a+b)\right]+3\left[\sum ab(a+b)-6abc\right]+10\left(\sum ab^2-\sum a^2b\right) \ge 0,$$

$$\sum(a+b)(a-b)^2+3\sum c(a-b)^2+10\left(\sum ab^2-\sum a^2b\right) \ge 0,$$

$$\sum(a+b+3c)(a-b)^2+10(a-b)(b-c)(c-a) \ge 0.$$

Assume that

$$a=\min\{a,b,c\},$$

and use the substitution

$$b=a+x, \qquad c=a+y, \qquad x,y \ge 0.$$

The inequality becomes

$$(5a+x+3y)x^2+(5a+x+y)(x-y)^2+(5a+3x+y)y^2-10xy(x-y) \ge 0.$$

Clearly, it suffices to consider the case $a=0$, when the inequality becomes

$$x^3-4x^2y+6xy^2+y^3 \ge 0.$$

Indeed, we have

$$x^3-4x^2y+6xy^2+y^3 = x(x-2y)^2+2xy^2+y^3 \ge 0.$$

The equality holds for $a=b=c$.

□

P 1.21. *If a, b, c are positive real numbers, then*

$$\frac{2a+b}{2a+c}+\frac{2b+c}{2b+a}+\frac{2c+a}{2c+b}\ge 3.$$

(Pham Kim Hung, 2007)

Solution. Without loss of generality, assume that $a=\max\{a,b,c\}$. There are two cases to consider.

Case 1: $a\le 2b+2c$. Write the inequality as

$$\sum\left(\frac{2a+b}{2a+c}-\frac{1}{2}\right)\ge\frac{3}{2},$$

$$\sum\frac{2a+2b-c}{2a+c}\ge 3.$$

Since

$$2a+2b-c>0,\quad 2b+2c-a\ge 0,\quad 2c+2a-b>0,$$

we may apply the Cauchy-Schwarz inequality to get

$$\sum\frac{2a+2b-c}{2a+c}\ge\frac{\left[\sum(2a+2b-c)\right]^2}{\sum(2a+2b-c)(2a+c)}=\frac{9\left(\sum a\right)^2}{3\left(\sum a\right)^2}=3.$$

Case 2: $a>2b+2c$. Since

$$2a+c-(2b+a)=(a-2b-2c)+3c>0,$$

we have

$$\frac{2a+b}{2a+c}+\frac{2b+c}{2b+a}>\frac{2a+b}{2a+c}+\frac{2b+c}{2a+c}=1+\frac{3b}{2a+c}>1.$$

Therefore, it suffices to show that

$$\frac{2c+a}{2c+b}\ge 2.$$

Indeed,

$$\frac{2c+a}{2c+b}>\frac{2c+2b+2c}{2c+b}=2.$$

Thus, the proof is completed. The equality holds for $a=b=c$.

□

P 1.22. *If a, b, c are positive real numbers, then*

$$\frac{a(a+b)}{a+c}+\frac{b(b+c)}{b+a}+\frac{c(c+a)}{c+b}\le\frac{3(a^2+b^2+c^2)}{a+b+c}.$$

(Pham Huu Duc, 2007)

Solution. Write the inequality as

$$\sum \frac{a(a+b)(a+b+c)}{a+c} \le 3(a^2+b^2+c^2),$$

$$\sum \frac{ab(a+b)+a(a+b)(a+c)}{a+c} \le 3(a^2+b^2+c^2),$$

$$\sum \frac{ab(a+b)}{a+c} \le 2(a^2+b^2+c^2)-(ab+bc+ca).$$

Let (x,y,z) be a permutation of (a,b,c) such that $x \ge y \ge z$. Since

$$x+y \ge z+x \ge y+z$$

and

$$xy(x+y) \ge zx(z+x) \ge yz(y+z),$$

by the rearrangement inequality, we have

$$\sum \frac{ab(a+b)}{a+c} \le \frac{xy(x+y)}{y+z} + \frac{zx(z+x)}{z+x} + \frac{yz(y+z)}{x+y}.$$

Consequently, it suffices to show that

$$\frac{xy(x+y)}{y+z} + \frac{yz(y+z)}{x+y} \le 2(x^2+y^2+z^2)-xy-yz-2zx.$$

Write this inequality as follows:

$$xy\left(\frac{x+y}{y+z}-1\right) + yz\left(\frac{y+z}{x+y}-1\right) \le 2(x^2+y^2+z^2-xy-yz-zx),$$

$$\frac{xy(x-z)}{y+z} + \frac{yz(z-x)}{x+y} \le (x-y)^2+(y-z)^2+(z-x)^2,$$

$$\frac{y(x+y+z)(z-x)^2}{(x+y)(y+z)} \le (x-y)^2+(y-z)^2+(z-x)^2.$$

Since

$$y(x+y+z) < (x+y)(y+z),$$

the last inequality is clearly true. The equality holds for $a=b=c$.

□

P 1.23. *If a,b,c are real numbers, then*

$$\frac{a^2-bc}{4a^2+b^2+4c^2} + \frac{b^2-ca}{4b^2+c^2+4a^2} + \frac{c^2-ab}{4c^2+a^2+4b^2} \ge 0.$$

(Vasile Cîrtoaje, 2006)

Solution. Since

$$\frac{4(a^2-bc)}{4a^2+b^2+4c^2} = 1 - \frac{(b+2c)^2}{4a^2+b^2+4c^2},$$

we may rewrite the inequality as

$$\frac{(b+2c)^2}{4a^2+b^2+4c^2} + \frac{(c+2a)^2}{4b^2+c^2+4a^2} + \frac{(a+2b)^2}{4c^2+a^2+4b^2} \le 3.$$

Using the Cauchy-Schwarz inequality gives

$$\frac{(b+2c)^2}{4a^2+b^2+4c^2} = \frac{(b+2c)^2}{(2a^2+b^2)+2(2c^2+a^2)} \le \frac{b^2}{2a^2+b^2} + \frac{2c^2}{2c^2+a^2}.$$

Therefore,

$$\sum \frac{(b+2c)^2}{4a^2+b^2+4c^2} \le \sum \frac{b^2}{2a^2+b^2} + \sum \frac{2c^2}{2c^2+a^2} = \sum \frac{b^2}{2a^2+b^2} + \sum \frac{2a^2}{2a^2+b^2} = 3.$$

The equality occurs when

$$a(2b^2+c^2) = b(2c^2+a^2) = c(2a^2+b^2);$$

that is, when $a=b=c$, and also when $a=2b=4c$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- *Let a,b,c be real numbers. If $k>0$, then*

$$\frac{a^2-bc}{2ka^2+b^2+k^2c^2} + \frac{b^2-ca}{2kb^2+c^2+k^2a^2} + \frac{c^2-ab}{2kc^2+a^2+k^2b^2} \ge 0,$$

with equality for $a=b=c$, and also for $a=kb=k^2c$ (or any cyclic permutation).

□

P 1.24. *If a,b,c are real numbers, then*

(a) $$a(a+b)^3 + b(b+c)^3 + c(c+a)^3 \ge 0;$$

(b) $$a(a+b)^5 + b(b+c)^5 + c(c+a)^5 \ge 0.$$

(Vasile Cîrtoaje, 1989)

Solution. (a) Using the substitution

$$b+c=2x, \quad c+a=2y, \quad a+b=2z,$$

which are equivalent to

$$a=y+z-x, \quad b=z+x-y, \quad c=x+y-z,$$

the inequality becomes in succession

$$x^4+y^4+z^4+xy^3+yz^3+zx^3 \ge x^3y+y^3z+z^3x,$$

$$\sum(x^4+2xy^3-2x^3y+y^4) \ge 0,$$

$$\sum(x^2-xy-y^2)^2+\sum x^2y^2 \ge 0,$$

the last being clearly true. The equality occurs for $a=b=c=0$.

(b) Using the same substitution, the inequality turns into

$$x^6+y^6+z^6+xy^5+yz^5+zx^5 \ge x^5y+y^5z+z^5x,$$

which is equivalent to

$$\sum[x^6+y^6-2xy(x^4-y^4)] \ge 0,$$

$$\sum[(x^2+y^2)(x^4-x^2y^2+y^4)-2xy(x^2+y^2)(x^2-y^2)] \ge 0,$$

$$\sum(x^2+y^2)(x^2-xy-y^2)^2 \ge 0.$$

The equality occurs for $a=b=c=0$.

□

P 1.25. *If a,b,c are real numbers, then*

$$3(a^4+b^4+c^4)+4(a^3b+b^3c+c^3a) \ge 0.$$

(Vasile Cîrtoaje, 2005)

Solution. If a,b,c are nonnegative, then the inequality is trivial. Since the inequality remains unchanged by replacing a,b,c with $-a,-b,-c$, respectively, it suffices to consider the case when only one of a,b,c is negative; let $c<0$. Replacing now c with $-c$, the inequality can be restated as

$$3(a^4+b^4+c^4)+4a^3b \ge 4(b^3c+c^3a),$$

where $a,b,c \ge 0$. It is enough to prove that

$$3(a^4+b^4+c^4+a^3b) \ge 4(b^3c+c^3a).$$

Case 1: $a \le b$. Since $a^3b \ge a^4$, it suffices to show that

$$6a^4+3b^4+3c^4 \ge 4(b^3c+ac^3).$$

Using the AM-GM inequality yields

$$3b^4+c^4 \ge 4\sqrt[4]{b^{12}c^4}=4b^3c.$$

Therefore, it suffices to show that

$$6a^4 + 2c^4 \geq 4ac^3.$$

Indeed, we have

$$3a^4 + c^4 = 3a^4 + \frac{1}{3}c^4 + \frac{1}{3}c^4 + \frac{1}{3}c^4 \geq 4\sqrt[4]{\frac{a^4c^{12}}{9}} = \frac{4}{\sqrt{3}}ac^3 \geq 2ac^3.$$

Case 2: $a \geq b$. Since $3a^3b \geq 3b^4$, it suffices to show that

$$3a^4 + 6b^4 + 3c^4 \geq 4(b^3c + ac^3).$$

By the AM-GM inequality, we get

$$6b^4 + \frac{c^4}{8} = 2b^4 + 2b^4 + 2b^4 + \frac{c^4}{8} \geq 4\sqrt[4]{b^{12}c^4} = 4b^3c.$$

Thus, we still have to show that

$$3a^4 + \frac{23}{8}c^4 \geq 4ac^3.$$

We will prove the sharper inequality

$$3a^4 + \frac{5}{2}c^4 \geq 4ac^3.$$

Indeed, we have

$$3a^4 + \frac{5}{2}c^4 = 3a^4 + \frac{5}{6}c^4 + \frac{5}{6}c^4 + \frac{5}{6}c^4 \geq 4\sqrt[4]{\frac{125a^4c^{12}}{72}} \geq 4ac^3.$$

The equality occurs for $a = b = c = 0$.

□

P 1.26. *If a, b, c are positive real numbers, then*

$$\frac{(a-b)(2a+b)}{(a+b)^2} + \frac{(b-c)(2b+c)}{(b+c)^2} + \frac{(c-a)(2c+a)}{(c+a)^2} \geq 0.$$

(Vasile Cîrtoaje, 2006)

Solution. Since

$$\frac{(a-b)(2a+b)}{(a+b)^2} = \frac{2a^2 - b(a+b)}{(a+b)^2} = \frac{2a^2}{(a+b)^2} - \frac{b}{a+b},$$

we can write the inequality as

$$2\sum\left(\frac{a}{a+b}\right)^2-\sum\frac{b}{a+b}\ge 0.$$

According to P 1.1 in Volume 2, we have

$$\begin{aligned}2\sum\left(\frac{a}{a+b}\right)^2&=\sum\left(\frac{a}{a+b}\right)^2+\sum\left(\frac{b}{b+c}\right)^2\\&=\sum\left[\frac{1}{(1+b/a)^2}+\frac{1}{(1+c/b)^2}\right]\\&\ge\sum\frac{1}{1+c/a}=\sum\frac{a}{a+c}=\sum\frac{b}{b+a}.\end{aligned}$$

Therefore,

$$2\sum\left(\frac{a}{a+b}\right)^2-\sum\frac{b}{a+b}\ge\sum\frac{b}{b+a}-\sum\frac{b}{a+b}=0.$$

The equality holds for $a=b=c$.

□

P 1.27. *If a,b,c are positive real numbers, then*

$$\frac{(a-b)(2a+b)}{a^2+ab+b^2}+\frac{(b-c)(2b+c)}{b^2+bc+c^2}+\frac{(c-a)(2c+a)}{c^2+ca+a^2}\ge 0.$$

(Vasile Cîrtoaje, 2006)

Solution. Since

$$\frac{(a-b)(2a+b)}{a^2+ab+b^2}=\frac{3a^2-(a^2+ab+b^2)}{a^2+ab+b^2}=\frac{3a^2}{a^2+ab+b^2}-1,$$

we can write the inequality as

$$\sum\frac{a^2}{a^2+ab+b^2}\ge 1,$$

$$\sum\frac{1}{1+b/a+(b/a)^2}\ge 1.$$

Clearly, this inequality follows immediately from P 1.45 in Volume 2. The equality holds for $a=b=c$.

□

P 1.28. *If a, b, c are positive real numbers, then*

$$\frac{(a-b)(3a+b)}{a^2+b^2} + \frac{(b-c)(3b+c)}{b^2+c^2} + \frac{(c-a)(3c+a)}{c^2+a^2} \ge 0.$$

(Vasile Cîrtoaje, 2006)

Solution. Since

$$(a-b)(3a+b) = (a-b)^2 + 2(a^2-b^2),$$

we can write the inequality as

$$\sum \frac{(a-b)^2}{a^2+b^2} + 2\sum \frac{a^2-b^2}{a^2+b^2} \ge 0.$$

Using the identity

$$\sum \frac{a^2-b^2}{a^2+b^2} + \prod \frac{a^2-b^2}{a^2+b^2} = 0,$$

the inequality becomes

$$\sum \frac{(a-b)^2}{a^2+b^2} \ge 2\prod \frac{a^2-b^2}{a^2+b^2}.$$

By the AM-GM inequality, we have

$$\sum \frac{(a-b)^2}{a^2+b^2} \ge 3\sqrt[3]{\prod \frac{(a-b)^2}{a^2+b^2}}.$$

Thus, it suffices to show that

$$3\sqrt[3]{\prod \frac{(a-b)^2}{a^2+b^2}} \ge 2\prod \frac{a^2-b^2}{a^2+b^2},$$

which is equivalent to

$$27\prod \frac{(a-b)^2}{a^2+b^2} \ge 8\prod \frac{(a^2-b^2)^3}{(a^2+b^2)^3}.$$

This inequality is true if

$$27\prod (a^2+b^2)^2 \ge \prod (a-b)(a+b)^3.$$

Assume that $a = \max\{a, b, c\}$. For the nontrivial case $a > c > b$, we can get this inequality by multiplying the inequalities

$$3(a^2+b^2)^2 \ge 2(a-b)(a+b)^3,$$

$$3(c^2+b^2)^2 \ge 2(c-b)(c+b)^3,$$

$$3(a^2+c^2)^2 \ge 2(a-c)(a+c)^3.$$

These inequalities are true because

$$3(a^2+b^2)^2 - 2(a-b)(a+b)^3 = a^2(a-2b)^2 + b^2(2a^2+4ab+5b^2) > 0.$$

The equality holds for $a = b = c$.

□

P 1.29. *Let a, b, c be positive real numbers such that $abc = 1$. Then,*

$$\frac{1}{1+a+b^2} + \frac{1}{1+b+c^2} + \frac{1}{1+c+a^2} \le 1.$$

(Vasile Cîrtoaje, 2005)

Solution. Using the substitution

$$a = x^3, \quad b = y^3, \quad c = z^3,$$

we have to show that $xyz = 1$ involves

$$\frac{1}{1+x^3+y^6} + \frac{1}{1+y^3+z^6} + \frac{1}{1+z^3+x^6} \le 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{1}{1+x^3+y^6} \le \sum \frac{z^4+x+y^{-2}}{(z^2+x^2+y^2)^2} = \frac{\sum(z^4+x^2yz+x^2z^2)}{(x^2+y^2+z^2)^2}.$$

So, it remains to show that

$$(x^2+y^2+z^2)^2 \ge \sum x^4 + xyz\sum x + \sum x^2y^2,$$

which is equivalent to the known inequality

$$\sum x^2y^2 \ge xyz \sum x.$$

The equality occurs for $a = b = c = 1$.

Remark. Actually, the following generalization holds:

- *Let a, b, c be positive real numbers such that $abc = 1$. If $k \ge 0$, then*

$$\frac{1}{1+a+b^k} + \frac{1}{1+b+c^k} + \frac{1}{1+c+a^k} \le 1.$$

□

P 1.30. *Let a, b, c be positive real numbers such that $abc = 1$. Then,*

$$\frac{a}{(a+1)(b+2)} + \frac{b}{(b+1)(c+2)} + \frac{c}{(c+1)(a+2)} \ge \frac{1}{2}.$$

Solution. Using the substitution

$$a = \frac{x}{y}, \quad b = \frac{y}{z}, \quad c = \frac{z}{x},$$

where x, y, z are positive real numbers, the inequality can be restated as

$$\frac{zx}{(x+y)(y+2z)} + \frac{xy}{(y+z)(z+2x)} + \frac{yz}{(z+x)(x+2y)} \ge \frac{1}{2}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{zx}{(x+y)(y+2z)} \ge \frac{\left(\sum zx\right)^2}{\sum zx(x+y)(y+2z)} = \frac{1}{2}.$$

The equality occurs for $a = b = c = 1$.

□

P 1.31. *If a, b, c are positive real numbers such that $ab + bc + ca = 3$, then*

$$(a+2b)(b+2c)(c+2a) \ge 27.$$

(Michael Rozenberg, 2007)

Solution. Write the inequality in the homogeneous form

$$A + B \ge 0,$$

where

$$\begin{aligned} A &= (a+2b)(b+2c)(c+2a) - 3(a+b+c)(ab+bc+ca) \\ &= (a-b)(b-c)(c-a) \end{aligned}$$

and

$$B = 3(ab+bc+ca)[a+b+c-\sqrt{3(ab+bc+ca)}].$$

Since

$$\begin{aligned} B &= \frac{3(ab+bc+ca)[(a-b)^2+(b-c)^2+(c-a)^2]}{2(a+b+c+\sqrt{3(ab+bc+ca)}]} \\ &\ge \frac{3(ab+bc+ca)[(a-b)^2+(b-c)^2+(c-a)^2]}{4(a+b+c)}, \end{aligned}$$

it suffices to show that

$$4(a+b+c)(a-b)(b-c)(c-a) + 3(ab+bc+ca)[(a-b)^2+(b-c)^2+(c-a)^2] \ge 0.$$

Consider $c = \min\{a, b, c\}$, and use the substitution

$$a = c + x, \quad b = c + y, \quad x, y \geq 0.$$

The inequality becomes

$$-4xy(x - y)(3c + x + y) + 6(x^2 - xy + y^2)[3c^2 + 2(x + y)c + xy] \geq 0,$$

which is equivalent to

$$9(x^2 - xy + y^2)c^2 + 6Cc + D \geq 0,$$

where

$$C = x^3 - x^2y + xy^2 + y^3 \geq x(x^2 - xy + y^2),$$

$$D = xy(x^2 + 5y^2 - 3xy) \geq (2\sqrt{5} - 3)x^2y^2.$$

Since $C \geq 0$ and $D \geq 0$, the inequality is obvious. The equality holds for $a = b = c = 1$.

□

P 1.32. *If a, b, c are positive real numbers such that $ab + bc + ca = 3$, then*

$$\frac{a}{a + a^3 + b} + \frac{b}{b + b^3 + c} + \frac{c}{c + c^3 + a} \leq 1.$$

(Andrei Ciupan, 2005)

Solution. Write the inequality as

$$\frac{1}{1 + a^2 + b/a} + \frac{1}{1 + b^2 + c/b} + \frac{1}{1 + c^2 + a/c} \leq 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{1}{1 + a^2 + b/a} \leq \sum \frac{c^2 + 1 + ab}{(c + a + b)^2} = 1.$$

The equality holds for $a = b = c = 1$.

□

P 1.33. *If a, b, c are positive real numbers such that $a \geq b \geq c$ and $ab + bc + ca = 3$, then*

$$\frac{1}{a + 2b} + \frac{1}{b + 2c} + \frac{1}{c + 2a} \geq 1.$$

Solution. According to the well known inequality

$$x+y+z \geq \sqrt{3(xy+yz+zx)},$$

where x, y, z are positive real numbers, it suffices to prove that

$$\frac{1}{(a+2b)(b+2c)}+\frac{1}{(b+2c)(c+2a)}+\frac{1}{(c+2a)(a+2b)} \geq \frac{1}{3}.$$

This is equivalent to the following inequalities

$$9(a+b+c) \geq (a+2b)(b+2c)(c+2a),$$

$$3(a+b+c)(ab+bc+ca) \geq (a+2b)(b+2c)(c+2a),$$

$$a^2b+b^2c+c^2a \geq ab^2+bc^2+ca^2,$$

$$(a-b)(b-c)(a-c) \geq 0.$$

The last inequality is clearly true for $a \geq b \geq c$. The equality occurs for $a=b=c=1$.

□

P 1.34. *If* $a, b, c \in [0,1]$, *then*

$$\frac{a}{4b^2+5}+\frac{b}{4c^2+5}+\frac{c}{4a^2+5} \leq \frac{1}{3}.$$

Solution. Let

$$E(a,b,c)=\frac{a}{4b^2+5}+\frac{b}{4c^2+5}+\frac{c}{4a^2+5}.$$

We have

$$\begin{aligned} E(a,b,c)-E(1,b,c) &= \frac{a-1}{4b^2+5}+c\left(\frac{1}{4a^2+5}-\frac{1}{9}\right) \\ &= (1-a)\left[\frac{4c(1+a)}{9(4a^2+5)}-\frac{1}{4b^2+5}\right] \\ &\leq (1-a)\left[\frac{4(1+a)}{9(4a^2+5)}-\frac{1}{9}\right] \\ &= \frac{-(1-a)(1-2a)^2}{9(4a^2+5)} \leq 0, \end{aligned}$$

and, similarly,

$$E(a,b,c)-E(a,1,c) \leq 0, \qquad E(a,b,c)-E(a,b,1) \leq 0.$$

Therefore,

$$E(a,b,c) \leq E(1,b,c) \leq E(1,1,c) \leq E(1,1,1)=\frac{1}{3}.$$

The equality occurs for $a=b=c=1$, and also for $a=\frac{1}{2}$ and $b=c=1$ (or any cyclic permutation).

□

P 1.35. *If* $a,b,c \in \left[\frac{1}{3}, 3\right]$, *then*

$$\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \ge \frac{7}{5}.$$

Solution. Assume that $a = \max\{a,b,c\}$ and show that

$$E(a,b,c) \ge E(a,b,\sqrt{ab}\,) \ge \frac{7}{5},$$

where

$$E(a,b,c) = \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a}.$$

We have

$$E(a,b,c) - E(a,b,\sqrt{ab}\,) = \frac{b}{b+c} + \frac{c}{c+a} - \frac{2\sqrt{b}}{\sqrt{a}+\sqrt{b}}$$
$$= \frac{\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{ab}-c\right)^2}{(b+c)(c+a)\left(\sqrt{a}+\sqrt{b}\right)} \ge 0.$$

Substituting $x = \sqrt{\frac{a}{b}}$, the hypothesis $a,b,c \in \left[\frac{1}{3}, 3\right]$ involves $x \in \left[\frac{1}{3}, 3\right]$. Then,

$$E(a,b,\sqrt{ab}) - \frac{7}{5} = \frac{a}{a+b} + \frac{2\sqrt{b}}{\sqrt{a}+\sqrt{b}} - \frac{7}{5}$$
$$= \frac{x^2}{x^2+1} + \frac{2}{x+1} - \frac{7}{5}$$
$$= \frac{3-7x+8x^2-2x^3}{5(x+1)(x^2+1)}$$
$$= \frac{(3-x)[x^2+(1-x)^2]}{5(x+1)(x^2+1)} \ge 0.$$

The equality holds for $a = 3$, $b = \frac{1}{3}$ and $c = 1$ (or any cyclic permutation).

□

P 1.36. *If* $a,b,c \in \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right]$, *then*

$$\frac{3}{a+2b} + \frac{3}{b+2c} + \frac{3}{c+2a} \ge \frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a}.$$

Solution. Write the inequality as

$$\sum\left(\frac{3}{a+2b}-\frac{2}{a+b}+\frac{1}{ka}-\frac{1}{kb}\right)\ge 0, \qquad k>0,$$

$$\sum\frac{-(a-b)[a^2-(k-3)ab+2b^2]}{kab(a+2b)(a+b)}\ge 0.$$

Choosing $k=6$, the inequality becomes

$$\sum\frac{(a-b)^2(2b-a)}{6ab(a+2b)(a+b)}\ge 0.$$

Since

$$2b-a\ge\frac{2}{\sqrt{2}}-\sqrt{2}=0,$$

the conclusion follows. The equality holds for $a=b=c$.

□

P 1.37. *If a,b,c are nonnegative real numbers, no two of which are zero, then*

$$\frac{4abc}{ab^2+bc^2+ca^2+abc}+\frac{a^2+b^2+c^2}{ab+bc+ca}\ge 2.$$

(Vo Quoc Ba Can, 2009)

First Solution. Without loss of generality, assume that b is between a and c; that is,

$$b^2+ca\le b(c+a).$$

Then,

$$\begin{aligned}ab^2+bc^2+ca^2+abc &= a(b^2+ca)+bc^2+abc\\ &\le ab(c+a)+bc^2+abc\\ &= b(a+c)^2,\end{aligned}$$

and it suffices to prove that

$$\frac{4ac}{(a+c)^2}+\frac{a^2+b^2+c^2}{ab+bc+ca}\ge 2.$$

This inequality is equivalent to

$$[a^2+c^2-b(a+c)]^2\ge 0.$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. Let (x, y, z) be a permutation of (a, b, c) such that $x \ge y \ge z$. As we have shown in the second solution of P 1.1,

$$ab^2 + bc^2 + ca^2 \le y(x^2 + xz + z^2);$$

hence

$$ab^2 + bc^2 + ca^2 + abc \le y(x + z)^2.$$

Thus, it suffices to prove that

$$\frac{4xyz}{y(x+z)^2} + \frac{x^2+y^2+z^2}{xy+yz+zx} \ge 2,$$

which is equivalent to

$$\frac{x^2+y^2+z^2}{xy+yz+zx} \ge \frac{2(x^2+z^2)}{x+z)^2},$$

$$(x^2 + z^2)^2 - 2y(x + z)(x^2 + z^2) + y^2(x + z)^2 \ge 0,$$

$$(x^2 + z^2 - xy - yz)^2 \ge 0.$$

□

P 1.38. *If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then*

$$\frac{1}{ab^2+8} + \frac{1}{bc^2+8} + \frac{1}{ca^2+8} \ge \frac{1}{3}.$$

(Vasile Cîrtoaje, 2007)

Solution. By expanding, we can write the inequality as

$$64 \ge r^3 + 16A + 5rB,$$

$$64 \ge r^3 + (16 - 5r)A + 5r(A + B),$$

where

$$r = abc, \quad A = ab^2 + bc^2 + ca^2, \quad B = a^2b + b^2c + c^2a.$$

By the AM-GM inequality, we have

$$r \le \left(\frac{a+b+c}{3}\right)^3 = 1.$$

On the other hand, by the inequality (a) in P 1.9, we get

$$A \le 4 - r,$$

and by Schur's inequality, we have

$$(a + b + c)^3 + 9abc \ge 4(a + b + c)(ab + bc + ca),$$

which is equivalent to

$$A+B \le \frac{27-3r}{4}.$$

Therefore, it suffices to prove that

$$64 \ge r^3 + (16-5r)(4-r) + \frac{5r(27-3r)}{4}.$$

We can write this inequality in the obvious form

$$r(1-r)(9+4r) \ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=0$, $b=1$, $c=2$ (or any cyclic permutation).

□

P 1.39. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$\frac{ab}{bc+3} + \frac{bc}{ca+3} + \frac{ca}{ab+3} \le \frac{3}{4}.$$

(Vasile Cîrtoaje, 2008)

Solution. Using the inequality (a) in P 1.9, namely

$$a^2b + b^2c + c^2a \le 4 - abc,$$

we have

$$\begin{aligned}\sum ab(ca+3)(ab+3) &= abc\sum a^2b + 9abc + 3\sum a^2b^2 + 9\sum ab \\ &\le 13abc - a^2b^2c^2 + 3\sum a^2b^2 + 9\sum ab.\end{aligned}$$

On the other hand,

$$(ab+3)(bc+3)(ca+3) = a^2b^2c^2 + 9abc + 9\sum ab + 27.$$

Therefore, it suffices to prove that

$$4\left(13abc - a^2b^2c^2 + 3\sum a^2b^2 + 9\sum ab\right) \le 3\left(a^2b^2c^2 + 9abc + 9\sum ab + 27\right),$$

which is equivalent to

$$7a^2b^2c^2 + 81 \ge 25abc + 12\sum a^2b^2 + 9\sum ab,$$

$$7r^2 + 47r \ge 3(q+3)(4q-9),$$

where

$$q = ab + bc + ca, \quad r = abc, \quad q \le 3, \quad r \le 1.$$

Since

$$7r^2 + 47r \ge 9r^2 + 45r,$$

it suffices to show that

$$3r^2 + 15r \ge (q+3)(4q-9).$$

Consider the non-trivial case

$$\frac{9}{4} < q \le 3,$$

and apply the fourth degree Schur's inequality

$$r \ge \frac{(p^2-q)(4q-p^2)}{6p} = \frac{(9-q)(4q-9)}{18}.$$

It remains to show that

$$\frac{(9-q)^2(4q-9)^2}{108} + \frac{5(9-q)(4q-9)}{6} \ge (q+3)(4q-9),$$

which is equivalent to

$$(4q-9)(3-q)(69q-4q^2-81) \ge 0.$$

This is true because

$$69q - 4q^2 - 81 = (3-q)(4q-9) + 6(8q-9) > 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 0$ and $b = c = \frac{3}{2}$ (or any cyclic permutation).

□

P 1.40. *If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then*

(a) $$\frac{a}{b^2+3} + \frac{b}{c^2+3} + \frac{c}{a^2+3} \ge \frac{3}{4};$$

(b) $$\frac{a}{b^3+1} + \frac{b}{c^3+1} + \frac{c}{a^3+1} \ge \frac{3}{2}.$$

(Vasile Cîrtoaje and Bin Zhao, 2005)

Solution. (a) By the AM-GM inequality, we have

$$b^2 + 3 = b^2 + 1 + 1 + 1 \ge 4\sqrt[4]{b^2 \cdot 1^3} = 4\sqrt{b}.$$

Therefore,

$$\frac{3a}{b^2+3} = a - \frac{ab^2}{b^2+3} \ge a - \frac{ab^2}{4\sqrt{b}} = a - \frac{1}{4}ab\sqrt{b}.$$

Taking account of this inequality and the similar ones, it suffices to prove that

$$ab\sqrt{b}+bc\sqrt{c}+ca\sqrt{a}\le 3.$$

This inequality follows immediately by replacing a,b,c with $\sqrt{a},\sqrt{b},\sqrt{c}$ in the inequality in P 1.7. The equality holds for $a=b=c=1$.

(b) Using the AM-GM Inequality gives

$$\frac{a}{b^3+1}=a-\frac{ab^3}{b^3+1}\ge a-\frac{ab^3}{2b\sqrt{b}}=a-\frac{1}{2}ab\sqrt{b},$$

and, similarly,

$$\frac{b}{c^3+1}\ge b-\frac{1}{2}bc\sqrt{c},\qquad \frac{c}{a^3+1}\ge c-\frac{1}{2}ca\sqrt{a}.$$

Thus, it suffices to show that

$$ab\sqrt{b}+bc\sqrt{c}+ca\sqrt{a}\le 3,$$

which follows from the inequality in P 1.7. The equality holds for $a=b=c=1$.

Open problem. *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. If*

$$0<k\le 3+2\sqrt{3},$$

then

$$\frac{a}{b^2+k}+\frac{b}{c^2+k}+\frac{c}{a^2+k}\ge\frac{3}{1+k}.$$

For $k=3+2\sqrt{3}$, the equality occurs when $a=b=c=1$, and again when $a=0$, $b=3-\sqrt{3}$ and $c=\sqrt{3}$ (or any cyclic permutation thereof).

□

P 1.41. *Let a,b,c be positive real numbers, and let*

$$x=a+\frac{1}{b}-1,\qquad y=b+\frac{1}{c}-1,\qquad z=c+\frac{1}{a}-1.$$

Prove that

$$xy+yz+zx\ge 3.$$

(Vasile Cîrtoaje, 1991)

First Solution. Among x,y,z, there are two numbers either less than or equal to 1, or larger than or equal to 1. Let y and z be these numbers; that is,

$$(y-1)(z-1)\ge 0.$$

Since

$$xy+yz+zx-3=(y-1)(z-1)+(x+1)(y+z)-4,$$

it suffices to show that

$$(x+1)(y+z) \geq 4.$$

Since

$$y+z = b+\frac{1}{a}+c+\frac{1}{c}-2 \geq b+\frac{1}{a},$$

we have

$$(x+1)(y+z)-4 \geq (x+1)\left(b+\frac{1}{a}\right)-4 = ab+\frac{1}{ab}-2 \geq 0.$$

The equality holds for $a=b=c=1$.

Second Solution. Without loss of generality, assume that $x = \max\{x,y,z\}$. Then,

$$\begin{aligned} x &\geq \frac{1}{3}(x+y+z) = \frac{1}{3}\left[\left(a+\frac{1}{a}\right)+\left(b+\frac{1}{b}\right)+\left(c+\frac{1}{c}\right)-3\right] \\ &\geq \frac{1}{3}(2+2+2-3) = 1. \end{aligned}$$

On the other hand, from

$$\begin{aligned} (x+1)(y+1)(z+1) &= abc+\frac{1}{abc}+a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ &\geq 2+a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ &= 5+x+y+z, \end{aligned}$$

we get

$$xyz+xy+yz+zx \geq 4.$$

Since

$$y+z = \frac{1}{a}+b+\frac{(c-1)^2}{c} > 0,$$

two cases are possible: $yz \leq 0$ and $y,z > 0$.

Case 1: $yz \leq 0$. Since $xyz \leq 0$, it follows that

$$xy+yz+zx \geq 4-xyz \geq 4 > 3.$$

Case 2: $y,z > 0$. We need to show that $d \geq 1$, where

$$d = \sqrt{\frac{xy+yz+zx}{3}}.$$

By the AM-GM inequality, we have $d^3 \geq xyz$. Thus, from $xyz+xy+yz+zx \geq 4$, we get

$$d^3+3d^2 \geq 4,$$

$$(d-1)(d+2)^2 \geq 0,$$

hence $d \geq 1$.

□

P 1.42. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\left(a - \frac{1}{b} - \sqrt{2}\right)^2 + \left(b - \frac{1}{c} - \sqrt{2}\right)^2 + \left(c - \frac{1}{a} - \sqrt{2}\right)^2 \geq 6.$$

Solution *(by Nguyen Van Quy).* Using the substitution

$$a = \frac{y}{x}, \quad b = \frac{x}{z}, \quad c = \frac{z}{y}, \qquad x, y, z > 0,$$

the inequality becomes as follows:

$$\left(\frac{y-z}{x} - \sqrt{2}\right)^2 + \left(\frac{z-x}{y} - \sqrt{2}\right)^2 + \left(\frac{x-y}{z} - \sqrt{2}\right)^2 \geq 6,$$

$$\left(\frac{y-z}{x}\right)^2 + \left(\frac{z-x}{y}\right)^2 + \left(\frac{x-y}{z}\right)^2 - 2\sqrt{2}\left(\frac{y-z}{x} + \frac{z-x}{y} + \frac{x-y}{z}\right) \geq 0,$$

$$\left(\frac{y-z}{x}\right)^2 + \left(\frac{z-x}{y}\right)^2 + \left(\frac{x-y}{z}\right)^2 + \frac{2\sqrt{2}(y-z)(z-x)(x-y)}{xyz} \geq 0.$$

Assume that $x = \max\{x, y, z\}$. For $x \geq z \geq y$, the inequality is clearly true. Consider further that $x \geq y \geq z$ and write the desired inequality as

$$u^2 + v^2 + w^2 \geq 2\sqrt{2}\, uvw,$$

where

$$u = \frac{y-z}{x} \geq 0, \quad v = \frac{x-z}{y} \geq 0, \quad w = \frac{x-y}{z} \geq 0.$$

In addition, we have

$$uv = \left(1 - \frac{z}{y}\right)\left(1 - \frac{z}{x}\right) < 1 \cdot 1 = 1.$$

According to the AM-GM inequality, we get

$$u^2 + v^2 + w^2 \geq 2uv + w^2 \geq 2u^2v^2 + w^2 \geq 2\sqrt{2}\, uvw.$$

This completes the proof. The equality holds for $a = b = c$.

□

P 1.43. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\left|1 + a - \frac{1}{b}\right| + \left|1 + b - \frac{1}{c}\right| + \left|1 + c - \frac{1}{a}\right| > 2.$$

Solution. Using the substitution

$$a = \frac{y}{x}, \quad b = \frac{x}{z}, \quad c = \frac{z}{y}, \qquad x, y, z > 0,$$

the inequality can be restated as

$$\left|1 + \frac{y-z}{x}\right| + \left|1 + \frac{x-y}{z}\right| + \left|1 + \frac{z-x}{y}\right| > 2.$$

Without loss of generality, assume that $x = \max\{x, y, z\}$. We have

$$\left|1 + \frac{y-z}{x}\right| + \left|1 + \frac{x-y}{z}\right| + \left|1 + \frac{z-x}{y}\right| - 2 \ge \left|1 + \frac{y-z}{x}\right| + \left|1 + \frac{x-y}{z}\right| - 2$$

$$= \frac{x+y-z}{x} + \frac{z+x-y}{z} - 2 = \frac{y-z}{x} + \frac{x-y}{z} \ge \frac{y-z}{x} + \frac{x-y}{x} = \frac{x-z}{x} \ge 0.$$

□

P 1.44. *If a, b, c are different positive real numbers, then*

$$\left|1 + \frac{a}{b-c}\right| + \left|1 + \frac{b}{c-a}\right| + \left|1 + \frac{c}{a-b}\right| > 2.$$

(Vasile Cîrtoaje, 2012)

Solution. Without loss of generality, assume that $a = \max\{a, b, c\}$. It suffices to show that

$$\left|1 + \frac{a}{b-c}\right| + \left|1 + \frac{c}{a-b}\right| > 2,$$

which is equivalent to

$$\frac{a+b-c}{|b-c|} + \frac{a-b+c}{a-b} > 2.$$

For $b > c$, this inequality is true since

$$\frac{a+b-c}{|b-c|} + \frac{a-b+c}{a-b} > \frac{a+b-c}{|b-c|} = \frac{a}{b-c} + 1 > 1 + 1 = 2.$$

Also, for $b < c$, we have

$$\frac{a+b-c}{|b-c|} + \frac{a-b+c}{a-b} = \frac{a+b-c}{c-b} + \frac{a-b+c}{a-b}$$

$$= \frac{a}{c-b} + \frac{c}{a-b} > \frac{a}{c-b} + \frac{c-b}{a-b} \ge 2\sqrt{\frac{a}{a-b}} > 2.$$

□

P 1.45. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\left(2a - \frac{1}{b} - \frac{1}{2}\right)^2 + \left(2b - \frac{1}{c} - \frac{1}{2}\right)^2 + \left(2c - \frac{1}{a} - \frac{1}{2}\right)^2 \geq \frac{3}{4}.$$

(Vasile Cîrtoaje, 2012)

Solution. Using the substitution

$$x = 2a - \frac{1}{b}, \quad y = 2b - \frac{1}{c}, \quad z = 2c - \frac{1}{a},$$

we can write the inequality as

$$x^2 + y^2 + z^2 \geq x + y + z.$$

From

$$x + y + z = 2\sum a - \sum \frac{1}{a}$$

and

$$xyz = 7 - 4\sum a + 2\sum \frac{1}{a},$$

it follows that

$$2(x + y + z) + xyz = 7.$$

In addition, from

$$2(|x| + |y| + |z|) + \left(\frac{|x| + |y| + |z|}{3}\right)^3 \geq 2(|x| + |y| + |z|) + |xyz|$$

$$\geq 2(x + y + z) + xyz = 7,$$

we get

$$|x| + |y| + |z| \geq 3.$$

Therefore, we have

$$x^2 + y^2 + z^2 \geq \frac{1}{3}(|x| + |y| + |z|)^2 \geq |x| + |y| + |z| \geq x + y + z.$$

The equality holds for $a = b = c = 1$.

□

P 1.46. *Let*

$$x = a + \frac{1}{b} - \frac{5}{4}, \quad y = b + \frac{1}{c} - \frac{5}{4}, \quad z = c + \frac{1}{a} - \frac{5}{4},$$

where $a \geq b \geq c > 0$. Prove that

$$xy + yz + zx \geq \frac{27}{16}.$$

(Vasile Cîrtoaje, 2011)

Solution. Write the inequality as

$$\sum\left(ab+\frac{1}{ab}\right)+\sum\frac{b}{a}-\frac{5}{2}\sum\left(a+\frac{1}{a}\right)+6\geq 0.$$

Since

$$\sum\frac{b}{a}-\sum\frac{a}{b}=\frac{(a-b)(b-c)(a-c)}{abc}\geq 0,$$

we have

$$2\sum\frac{b}{a}\geq\sum\frac{b}{a}+\sum\frac{a}{b}=\left(\sum a\right)\left(\sum\frac{1}{a}\right)-3.$$

Thus, it suffices to prove the symmetric inequality

$$2\sum\left(ab+\frac{1}{ab}\right)+\left(\sum a\right)\left(\sum\frac{1}{a}\right)-5\sum\left(a+\frac{1}{a}\right)+9\geq 0.$$

Setting

$$p=a+b+c,\quad q=ab+bc+ca,\quad r=abc,$$

we need to show that

$$(2q-5p+9)r+pq-5q+2p\geq 0$$

for all $a,b,c>0$. For fixed p and q, the linear function

$$f(r)=(2q-5p+9)r+pq-5q+2p$$

is minimum when r is either minimum or maximum. Thus, according to P 3.57 in Volume 1, it suffices to prove that $f(r)\geq 0$ for $a=0$ and for $b=c$.

For $a=0$, we need to show that

$$(b+c)bc-5bc+2(b+c)\geq 0.$$

Indeed, putting $x=\sqrt{bc}$, we have

$$(b+c)bc-5bc+2(b+c)\geq 2x^3-5x^2+4x>0.$$

For $b=c$, since

$$p=a+2b,\quad q=2ab+b^2,\quad r=ab^2,$$

the inequality $f(r)\geq 0$ becomes

$$(4ab+2b^2-5a-10b+9)ab^2+(a+2b)(2ab+b^2)-10ab-5b^2+2a+4b\geq 0;$$

that is,

$$Aa^2+2Ba+C\geq 0,$$

where

$$A=b(4b^2-5b+2)>0,\quad B=b^4-5b^3+7b^2-5b+1,\quad C=b(2b^2-5b+4)>0.$$

Let

$$x = b + \frac{1}{b}, \quad x \geq 2.$$

The inequality $B \geq 0$ is equivalent to

$$b^2 + \frac{1}{b^2} - 5\left(b + \frac{1}{b}\right) + 7 \geq 0,$$

$$x^2 - 5x + 5 \geq 0,$$

$$x \geq \frac{5 + \sqrt{5}}{2}.$$

Consider two cases.

Case 1: $x \geq \dfrac{5+\sqrt{5}}{2}$. Since $A > 0$, $B \geq 0$, $C > 0$, we have $Aa^2 + 2Ba + C > 0$.

Case 2: $2 \leq x < \dfrac{5+\sqrt{5}}{2}$. Since $A > 0$, $B < 0$, $C > 0$ and

$$Aa^2 + 2Ba + C = (Aa^2 + C) + 2Ba \geq 2a(\sqrt{AC} + B),$$

we need to show that $AC \geq B^2$, which is equivalent to

$$8\left(b^2 + \frac{1}{b^2}\right) - 30\left(b + \frac{1}{b}\right) + 45 \geq \left[b^2 + \frac{1}{b^2} - 5\left(b + \frac{1}{b}\right) + 7\right]^2,$$

$$8x^2 - 30x + 29 \geq (x^2 - 5x + 5)^2,$$

$$(x-2)^2(x^2 - 6x - 1) \leq 0.$$

This inequality is true for $x \leq 3 + \sqrt{10}$, therefore for $x < (5+\sqrt{5})/2$. Thus, the proof is completed. The equality holds for $a = b = c = 1$.

□

P 1.47. *Let a, b, c be positive real numbers, and let*

$$E = \left(a + \frac{1}{a} - \sqrt{3}\right)\left(b + \frac{1}{b} - \sqrt{3}\right)\left(c + \frac{1}{c} - \sqrt{3}\right);$$

$$F = \left(a + \frac{1}{b} - \sqrt{3}\right)\left(b + \frac{1}{c} - \sqrt{3}\right)\left(c + \frac{1}{a} - \sqrt{3}\right).$$

Prove that $E \geq F$.

(Vasile Cîrtoaje, 2011)

Solution. By expanding, the inequality becomes

$$\sum(a^2-bc)+\sum bc(bc-a^2)\ge\sqrt{3}\sum ab(b-c).$$

Since

$$\sum(a^2-bc)=\sum a^2-\sum ab\ge 0$$

and

$$\sum bc(bc-a^2)=\sum a^2b^2-abc\sum a\ge 0,$$

by the AM-GM inequality, we have

$$\sum(a^2-bc)+\sum bc(bc-a^2)\ge 2\sqrt{\left[\sum(a^2-bc)\right]\left[\sum bc(bc-a^2)\right]}.$$

Thus, it suffices to show that

$$2\sqrt{\left[\sum(a^2-bc)\right]\left[\sum bc(bc-a^2)\right]}\ge\sqrt{3}\sum ab(b-c),$$

which is equivalent to

$$2\sqrt{\left[\sum(a^2-bc)\right]\left[\sum\left(\frac{1}{a^2}-\frac{1}{bc}\right)\right]}\ge\sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right),$$

$$\sqrt{[(a+c-2b)^2+3(c-a)^2]\left[3\left(\frac{1}{b}-\frac{1}{c}\right)^2+\left(\frac{2}{a}-\frac{1}{b}-\frac{1}{c}\right)^2\right]}\ge$$

$$\ge 2\sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right).$$

Applying the Cauchy-Schwarz inequality, it suffices to show that

$$(a+c-2b)\left(\frac{1}{b}-\frac{1}{c}\right)+(c-a)\left(\frac{2}{a}-\frac{1}{b}-\frac{1}{c}\right)\ge 2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right),$$

which is an identity. Thus, the proof is completed. The equality holds when the following two equations are satisfied:

$$a^2+b^2+c^2-ab-bc-ca=a^2b^2+b^2c^2+c^2a^2-abc(a+b+c)$$

and

$$3+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

□

P 1.48. *If a, b, c are positive real numbers such that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=5$, then*

$$\frac{b}{a}+\frac{c}{b}+\frac{a}{c} \geq \frac{17}{4}.$$

(Vasile Cîrtoaje, 2007)

Solution. Making the substitution

$$x=\frac{a}{b}, \quad y=\frac{b}{c}, \quad z=\frac{c}{a},$$

we need to show that if x, y, z are positive real numbers satisfying

$$xyz=1, \qquad x+y+z=5,$$

then

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq \frac{17}{4}.$$

From $(y+z)^2 \geq 4yz$, we get

$$(5-x)^2 \geq \frac{4}{x};$$

therefore,

$$(5-x)+(5-x)+\frac{x}{4} \geq 3\sqrt[3]{(5-x)^2\frac{x}{4}} \geq 3,$$

which involves $x \leq 4$. We have

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{17}{4}=\frac{1}{x}+\frac{y+z}{yz}-\frac{17}{4}=\frac{1}{x}+x(5-x)-\frac{17}{4}$$
$$=\frac{4-17x+20x^2-4x^3}{4x}=\frac{(4-x)(1-2x)^2}{4x} \geq 4.$$

The equality holds when one of x, y, z is 4 and the others are $\frac{1}{2}$; that is, when

$$a=4b=2c$$

(or any cyclic permutation).

□

P 1.49. *If a, b, c are positive real numbers, then*

(a) $$1+\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 2\sqrt{1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}}\ ;$$

(b) $$1+2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq \sqrt{1+16\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)}\ ;$$

(c) $$3+\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 2\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\ .$$

(Vasile Cîrtoaje, 2007)

Solution. Let
$$x=\frac{a}{b}, \quad y=\frac{b}{c}, \quad z=\frac{c}{a}$$
and
$$p=x+y+z, \quad q=xy+yz+zx.$$
By the AM-GM inequality, we have
$$p \ge 3\sqrt[3]{xyz}=3.$$

(a) We need to show that $xyz=1$ involves
$$1+x+y+z \ge 2\sqrt{1+xy+yz+zx},$$
which is equivalent to
$$(1+p)^2 \ge 4+4q$$
or
$$p+3 \ge 2\sqrt{p+q+3}.$$

First Solution. By Schur's inequality of degree three, we have
$$p^3+9 \ge 4pq.$$

Thus,
$$(1+p)^2-4-4q \ge 1+p)^2-4-\left(p^2+\frac{9}{p}\right)=\frac{(p-3)(2p+3)}{p} \ge 0.$$
The equality holds for $a=b=c$.

Second Solution. Without loss of generality, assume that b is between a and c. By the AM-GM inequality, we have
$$2\sqrt{p+q+3}=2\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \le \frac{a+b+c}{b}+b\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

Therefore,
$$p+3-2\sqrt{p+q+3} \ge \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3-\frac{a+b+c}{b}-b\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$
$$=\frac{(a-b)(b-c)}{ab} \ge 0.$$

(b) We have to show that $xyz = 1$ involves

$$1 + 2(x + y + z) \geq \sqrt{1 + 16(xy + yz + zx)},$$

which is equivalent to

$$p^2 + p \geq 4q.$$

By Schur's inequality of degree three, we have

$$p^3 + 9 \geq 4pq.$$

Thus,

$$p^2 + p - 4q \geq p^2 + p - \left(p^2 + \frac{9}{p}\right) = \frac{(p-3)(p+3)}{9} \geq 0.$$

The equality holds for $a = b = c$.

(c) Write the inequality as follows:

$$(3 + x + y + z)^2 \geq 4(3 + x + y + z + xy + yz + zx),$$

$$(x + y + z)^2 + 2(x + y + z) \geq 3 + 4(xy + yz + zx),$$

$$(1 + x + y + z)^2 \geq 4(1 + xy + yz + zx),$$

$$1 + x + y + z \geq 2\sqrt{1 + xy + yz + zx},$$

$$1 + \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 2\sqrt{1 + \frac{b}{a} + \frac{c}{b} + \frac{a}{c}}.$$

Thus, the inequality is equivalent to the inequality in (a).

□

P 1.50. *If a, b, c are positive real numbers, then*

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} + 15\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) \geq 16\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right).$$

Solution. Making the substitution

$$x = \frac{a}{b}, \quad y = \frac{b}{c}, \quad z = \frac{c}{a},$$

we have to show that $xyz = 1$ involves

$$x^2 + y^2 + z^2 + 15(xy + yz + zx) \geq 16(x + y + z),$$

which is equivalent to

$$(x + y + z)^2 - 16(x + y + z) + 13(xy + yz + zx) \geq 0.$$

According to P 3.58 in Volume 1, for fixed $x+y+z$ and $xyz=1$, the expression

$$xy+yz+zx$$

is minimum when two of x,y,z are equal. Therefore, due to symmetry, it suffices to consider that $x=y$. We need to show that

$$(2x+z)^2-16(2x+z)+13(x^2+2xz)\geq 0$$

for $x^2z=1$. Write this inequality as

$$17x^6-32x^5+30x^3-16x^2+1\geq 0,$$

or

$$(x-1)^2g(x)\geq 0,\quad g(x)=17x^4+2x^3-13x^2+2x+1.$$

Since

$$g(x)=(2x-1)^4+x(x^3+34x^2-37x+10),$$

it suffices to show that

$$x^3+34x^2-37x+10\geq 0.$$

There are two cases to consider.

Case 1: $x\in\left(0,\dfrac{1}{2}\right]\cup\left[\dfrac{10}{17},\infty\right)$. We have

$$x^3+34x^2-37x+10>34x^2-37x+10=(2x-1)(17x-10)\geq 0.$$

Case 2: $x\in\left(\dfrac{1}{2},\dfrac{10}{17}\right)$. We have

$$2(x^3+34x^2-37x+10)>2\left(\frac{1}{2}x^2+34x^2-37x+10\right)=69x^2-74x+20.$$

Since $69x^2-74x+20>0$ for all real x, the proof is completed. The equality holds for $a=b=c$.

□

P 1.51. *If a,b,c are positive real numbers such that $abc=1$, then*

(a) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq a+b+c;$$

(b) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq \frac{3}{2}(a+b+c-1);$$

(c) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+2\geq \frac{5}{3}(a+b+c).$$

Solution. (a) We write the inequality as

$$\left(2\frac{a}{b}+\frac{b}{c}\right)+\left(2\frac{b}{c}+\frac{c}{a}\right)+\left(2\frac{c}{a}+\frac{a}{b}\right)\geq 3(a+b+c).$$

In virtue of the AM-GM inequality, we get

$$\left(2\frac{a}{b}+\frac{b}{c}\right)+\left(2\frac{b}{c}+\frac{c}{a}\right)+\left(2\frac{c}{a}+\frac{a}{b}\right)\geq 3\sqrt[3]{\frac{a^2}{bc}}+3\sqrt[3]{\frac{b^2}{ca}}+3\sqrt[3]{\frac{c^2}{ab}}=3(a+b+c).$$

The equality holds for $a=b=c=1$.

(b) Using the substitution

$$a=\frac{y}{x},\quad b=\frac{z}{y},\quad c=\frac{x}{z},$$

where $x,y,z>0$, the inequality can be restated as

$$2(x^3+y^3+z^3)+3xyz\geq 3(x^2y+y^2z+z^2x).$$

First Solution. We get the desired inequality by summing Schur's inequality of degree three

$$x^3+y^3+z^3+3xyz\geq (x^2y+y^2z+z^2x)+(xy^2+yz^2+zx^2)$$

and

$$x^3+y^3+z^3+xy^2+yz^2+zx^2\geq 2(x^2y+y^2z+z^2x).$$

The latter inequality is equivalent to

$$x(x-y)^2+y(y-z)^2+z(z-x)^2\geq 0.$$

The equality holds for $a=b=c=1$.

Second Solution. Multiplying by $x+y+z$, the desired inequality in x,y,z turns into

$$2\sum x^4-\sum x^3y-3\sum x^2y^2+2\sum xy^3\geq 0.$$

Write this inequality as

$$\sum[(1+k)x^4-x^3y-3x^2y^2+2xy^3+(1-k)y^4]\geq 0,$$

$$\sum(x-y)[x^3-3xy^2-y^3+k(x^3+x^2y+xy^2+y^3)]\geq 0.$$

Choosing $k=\frac{3}{4}$, we get the obvious inequality

$$\sum(x-y)^2(7x^2+10xy+y^2)\geq 0.$$

(c) Making the substitution

$$a = \frac{y}{x}, \quad b = \frac{z}{y}, \quad c = \frac{x}{z}, \qquad x, y, z > 0,$$

we need to show that

$$3(x^3 + y^3 + z^3) + 6xyz \ge 5(x^2y + y^2z + z^2x).$$

Assuming that $x = \min\{x, y, z\}$ and substituting

$$y = x + p, \quad z = x + q, \quad p, q \ge 0,$$

the inequality turns into

$$(p^2 - pq + q^2)x + 3p^3 + 3q^3 - 5p^2q \ge 0.$$

This is true since, by the AM-GM inequality, we get

$$6p^3 + 6q^3 = 3p^3 + 3p^3 + 6q^3 \ge 3\sqrt[3]{3p^3 \cdot 3p^3 \cdot 6q^3} = 9\sqrt[3]{2}\ p^2q \ge 10p^2q.$$

The equality holds for $a = b = c = 1$.

Remark. The following stronger inequality holds for $abc = 1$:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge \sqrt{3(a^2 + b^2 + c^2)}.$$

By squaring, the inequality becomes

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} + 2\left(\frac{a}{c} + \frac{b}{a} + \frac{c}{b}\right) \ge 3(a^2 + b^2 + c^2).$$

By the AM-GM inequality, we have

$$\frac{a^2}{b^2} + 2\frac{a}{c} \ge 3\sqrt[3]{\frac{a^4}{b^2c^2}} = 3a^2,$$

$$\frac{b^2}{c^2} + 2\frac{b}{a} \ge 3b^2,$$

$$\frac{c^2}{a^2} + 2\frac{c}{b} \ge 3c^2.$$

Summing this inequalities, the conclusion follows.

□

P 1.52. *If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3$, then*

(a)
$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 2 + \frac{3}{ab + bc + ca};$$

(b)
$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge \frac{9}{a + b + c}.$$

Solution. (a) By the Cauchy-Schwarz inequality, we have

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{(a+b+c)^2}{ab+bc+ca}=2+\frac{3}{ab+bc+ca}.$$

The equality holds for $a=b=c=1$.

(b) Using the inequality in (a), it suffices to show that

$$2+\frac{3}{ab+bc+ca}\geq\frac{9}{a+b+c}.$$

Let

$$t=\frac{a+b+c}{3},\qquad t\leq 1.$$

Since

$$2(ab+bc+ca)=(a+b+c)^2-(a^2+b^2+c^2)=9t^2-3,$$

the inequality becomes

$$2+\frac{2}{3t^2-1}\geq\frac{3}{t},$$

$$(t-1)^2(2t+1)\geq 0.$$

The equality holds for $a=b=c=1$.

□

P 1.53. *If a,b,c are positive real numbers such that $a^2+b^2+c^2=3$, then*

$$6\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+5(ab+bc+ca)\geq 33.$$

Solution. Write the inequality in the homogeneous form

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\geq\frac{5}{2}\left(1-\frac{ab+bc+ca}{a^2+b^2+c^2}\right).$$

We will prove the sharper inequality

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\geq m\left(1-\frac{ab+bc+ca}{a^2+b^2+c^2}\right),$$

where

$$m=4\sqrt{2}-3>\frac{5}{2}.$$

Write this inequality as follows:

$$\left(\sum a^2\right)\left(\sum ab^2\right)+mabc\sum ab-(m+3)abc\sum a^2\geq 0,$$

$$\sum ab^4 + \sum a^3b^2 + (m+1)abc\sum ab - (m+3)abc\sum a^2 \geq 0,$$

$$\sum ab^4 + \sum a^3b^2 + 2(2\sqrt{2}-1)abc\sum ab - 4\sqrt{2}\ abc\sum a^2 \geq 0,$$

On the other hand, from

$$\sum a(a-b)^2(b-kc)^2 \geq 0,$$

we get

$$\sum ab^4 + \sum a^3b^2 + (k^2-2)\sum a^2b^3 + k(4-k)abc\sum ab - 4kabc\sum a^2 \geq 0.$$

Choosing $k = \sqrt{2}$, we get the desired inequality. The equality holds for $a = b = c = 1$.

□

P 1.54. *If a, b, c are positive real numbers such that $a + b + c = 3$, then*

(a) $$6\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) + 3 \geq 7(a^2 + b^2 + c^2);$$

(b) $$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq a^2 + b^2 + c^2.$$

Solution. (a) Write the inequality in the homogeneous form

$$2\left(\sum a\right)^2\left(\sum ab^2\right) + abc\left(\sum a\right)^2 \geq 21abc\sum a^2,$$

which is equivalent to

$$\sum ab^4 + \sum a^3b^2 + 2\sum a^2b^3 + 4abc\sum ab - 8abc\sum a^2 \geq 0.$$

On the other hand, from

$$\sum a(a-b)^2(b-kc)^2 \geq 0,$$

we get

$$\sum ab^4 + \sum a^3b^2 + (k^2-2)\sum a^2b^3 + k(4-k)abc\sum ab - 4kabc\sum a^2 \geq 0.$$

Choosing $k = 2$, we get the desired inequality. The equality holds for $a = b = c = 1$.

(b) We get the desired inequality by adding the inequality in (a) and the obvious inequality

$$a^2 + b^2 + c^2 \geq 3.$$

The equality holds for $a = b = c = 1$.

□

P 1.55. *If a, b, c are positive real numbers, then*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+2 \ge \frac{14(a^2+b^2+c^2)}{(a+b+c)^2}.$$

(Vo Quoc Ba Can, 2010)

Solution. By expanding, the inequality becomes as follows:

$$\left(\sum \frac{a}{b}\right)\left(\sum a^2+2\sum ab\right)+4\sum ab \ge 12\sum a^2,$$

$$\sum \frac{a^3}{b}+\sum \frac{a^2b}{c}+2\sum \frac{ab^2}{c}+7\sum ab \ge 10\sum a^2,$$

$$A+B \ge 10\sum a^2-10\sum ab,$$

where

$$A=\sum \frac{a^3}{b}+\sum \frac{a^2b}{c}-2\sum \frac{ab^2}{c}, \qquad B=4\sum \frac{ab^2}{c}-3\sum ab.$$

Since

$$A=\sum\left(\frac{b^3}{c}+\frac{a^2b}{c}-\frac{2ab^2}{c}\right)=\sum \frac{b(a-b)^2}{c}$$

and

$$B=\sum\left(\frac{4ca^2}{b}-12ca+9bc\right)=\sum \frac{c(2a-3b)^2}{b},$$

we get

$$A+B=\sum\left[\frac{b(a-b)^2}{c}+\frac{c(2a-3b)^2}{b}\right]$$

$$\ge 2\sum(a-b)(2a-3b)=10\sum a^2-10\sum ab.$$

Thus, the proof is completed. For $a \ge b \ge c$, the equality holds for

$$b(a-b)=c(2a-3b), \qquad c(b-c)=a(2b-3c), \qquad a(c-a)=b(2c-3a),$$

which are equivalent to

$$\frac{a}{\sqrt{7}-\tan\frac{\pi}{7}}=\frac{b}{\sqrt{7}-\tan\frac{2\pi}{7}}=\frac{c}{\sqrt{7}-\tan\frac{4\pi}{7}}.$$

Notice that the equality conditions involve

$$a^2+b^2+c^2=2ab+2bc+2ca,$$

hence

$$\sqrt{a}=\sqrt{b}+\sqrt{c}.$$

Remark. Using the inequality in P 1.55, we can prove the weaker inequality

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{7(ab+bc+ca)}{a^2+b^2+c^2}\ge\frac{17}{2},$$

with equality for the same conditions. It suffices to show that

$$\frac{14(a^2+b^2+c^2)}{(a+b+c)^2}-2\ge\frac{17}{2}-\frac{7(ab+bc+ca)}{a^2+b^2+c^2}$$

which is equivalent to

$$(a^2+b^2+c^2-2ab-2bc-2ca)^2\ge 0.$$

Actually, the following statement is valid.

If a,b,c are positive real numbers, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{19(a^2+b^2+c^2)+2(ab+bc+ca)}{a^2+b^2+c^2+6(ab+bc+ca)},$$

with equality for $a=b=c$, and also for

$$\frac{a}{\sqrt{7}-\tan\frac{\pi}{7}}=\frac{b}{\sqrt{7}-\tan\frac{2\pi}{7}}=\frac{c}{\sqrt{7}-\tan\frac{4\pi}{7}}$$

(or any cyclic permutation).

This inequality is stronger than the inequality in P 1.55.

□

P 1.56. *Let a,b,c be positive real numbers such that $a+b+c=3$, and let*

$$x=3a+\frac{1}{b},\quad y=3b+\frac{1}{c},\quad z=3c+\frac{1}{a}.$$

Prove that

$$xy+yz+zx\ge 48.$$

(Vasile Cîrtoaje, 2007)

Solution. Write the inequality as follows:

$$3(ab+bc+ca)+\frac{1}{abc}+\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\ge 13.$$

We get this inequality by adding the inequality P 1.54-(a), namely

$$6\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)+3\ge 7(a^2+b^2+c^2),$$

and the inequality

$$18(ab+bc+ca)+\frac{6}{abc}+7(a^2+b^2+c^2)\ge 81.$$

Since

$$a^2+b^2+c^2=9-2(ab+bc+ca),$$

the latter inequality is equivalent to

$$2(ab+bc+ca)+\frac{3}{abc}\ge 9.$$

By the known inequality

$$(ab+bc+ca)^2\ge 3abc(a+b+c),$$

we get

$$\frac{1}{abc}\ge\frac{9}{(ab+bc+ca)^2}.$$

Thus, it suffices to show that

$$2q+\frac{27}{q^2}\ge 9,$$

where $q=ab+bc+ca$. Indeed, by the AM-GM inequality, we have

$$2q+\frac{27}{q^2}=q+q+\frac{27}{q^2}\ge 3\sqrt[3]{q\cdot q\cdot\frac{27}{q^2}}=9.$$

The equality holds for $a=b=c=1$.

□

P 1.57. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$\frac{a+1}{b}+\frac{b+1}{c}+\frac{c+1}{a}\ge 2(a^2+b^2+c^2).$$

Solution. We get the desired inequality by summing the inequality in P 1.54-(a), namely

$$6\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3\ge 7(a^2+b^2+c^2),$$

and the inequality

$$6\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge 5(a^2+b^2+c^2)+3.$$

Write the latter inequality as $F(a,b,c)\ge 0$, where

$$F(a,b,c)=6\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-5(a^2+b^2+c^2)-3,$$

then assume that
$$a = \max\{a,b,c\}, \qquad b+c \le 2.$$
and show that
$$F(a,b,c) \ge F\left(a, \frac{b+c}{2}, \frac{b+c}{2}\right) \ge 0.$$
Indeed, we have
$$F(a,b,c) - F\left(a, \frac{b+c}{2}, \frac{b+c}{2}\right) = 6\left(\frac{b+c}{bc} - \frac{4}{b+c}\right) - 5\left[b^2+c^2-\frac{1}{2}(b+c)^2\right]$$
$$= (b-c)^2\left[\frac{6}{bc(b+c)} - \frac{5}{2}\right] \ge (b-c)^2\left[\frac{24}{(b+c)^3} - \frac{5}{2}\right] \ge 0.$$
Also,
$$F\left(a, \frac{b+c}{2}, \frac{b+c}{2}\right) = F\left(a, \frac{3-a}{2}, \frac{3-a}{2}\right) = \frac{3(a-1)^2(12-15a+5a^2)}{2a(3-a)} \ge 0.$$
The equality holds for $a = b = c = 1$.

□

P 1.58. *If a, b, c are positive real numbers such that $a+b+c = 3$, then*
$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + 3 \ge 2(a^2+b^2+c^2).$$

(Pham Huu Duc, 2007)

First Solution. Assume that
$$a = \max\{a,b,c\},$$
then homogenize the inequality and write it as follows:
$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + a+b+c \ge \frac{6(a^2+b^2+c^2)}{a+b+c},$$
$$\sum\left(\frac{b^2}{c} - 2b + c\right) \ge 6\left(\frac{a^2+b^2+c^2}{a+b+c} - \frac{a+b+c}{3}\right),$$
$$\sum \frac{(b-c)^2}{c} \ge \frac{2}{a+b+c}\sum(b-c)^2,$$
$$(b-c)^2A + (c-a)^2B + (a-b)^2C \ge 0,$$
where
$$A = \frac{a+b}{c} - 1 > 0, \quad B = \frac{b+c}{a} - 1, \quad C = \frac{c+a}{b} - 1 > 0.$$

By the Cauchy-Schwarz inequality, we have

$$(b-c)^2A+(a-b)^2C \ge \frac{[(b-c)+(a-b)]^2}{\dfrac{1}{A}+\dfrac{1}{C}} = \frac{AC}{A+C}(a-c)^2.$$

Therefore, it suffices to show that

$$\frac{AC}{A+C}+B \ge 0.$$

Indeed, by the third degree Schur's inequality, we get

$$AB+BC+CA = 3+\frac{a^3+b^3+c^3+3abc-ab(a+b)-bc(b+c)-ca(c+a)}{abc} \ge 3.$$

The equality holds for $a=b=c=1$.

Second Solution (by *Michael Rozenberg*). Write the inequality in the homogeneous form

$$\left(\sum a\right)\left(\sum ab^3\right)+abc\left(\sum a\right)^2 \ge 6abc\sum a^2.$$

By expanding, we get

$$\sum(ab^4+a^2b^3+2ab^2c^2-4a^3bc) \ge 0,$$

which is equivalent to

$$\sum a(b^2-2bc+ac)^2 \ge 0.$$

□

P 1.59. *If a,b,c are positive real numbers, then*

$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}+2(ab+bc+ca) \ge 3(a^2+b^2+c^2).$$

(Michael Rozenberg, 2010)

Solution. Write the inequality as

$$\sum\left(\frac{a^3}{b}+ab-2a^2\right) \ge a^2+b^2+c^2-ab-bc-ca,$$

$$\frac{a(a-b)^2}{b}+\frac{b(b-c)^2}{c}+\frac{c(c-a)^2}{a} \ge a^2+b^2+c^2-ab-bc-ca.$$

Assume that $a=\max\{a,b,c\}$.

Case 1: $a \ge b \ge c$. By the Cauchy-Schwarz inequality, we have

$$\frac{a(a-b)^2}{b}+\frac{b(b-c)^2}{c} \ge \frac{[(a-b)+(b-c)]^2}{\frac{b}{a}+\frac{c}{b}} = \frac{ab(a-c)^2}{b^2+ac}.$$

On the other hand,

$$a^2+b^2+c^2-ab-bc-ca=(a-c)^2+(b-a)(b-c)\le(a-c)^2.$$

Therefore, it suffice to show that

$$\frac{ab(a-c)^2}{b^2+ac}+\frac{c(a-c)^2}{a}\ge(a-c)^2,$$

which is true if

$$\frac{ab}{b^2+ac}+\frac{c}{a}\ge 1.$$

This inequality is equivalent to

$$a^2b+b^2c+c^2a-ab^2-ca^2\ge 0,$$

$$bc^2-(a-b)(b-c)(c-a)\ge 0.$$

Case 2: $a\ge c\ge b$. By the Cauchy-Schwarz inequality, we have

$$\frac{b(b-c)^2}{c}+\frac{c(c-a)^2}{a}\ge\frac{[(b-c)+(c-a)]^2}{\frac{c}{b}+\frac{a}{c}}=\frac{bc(a-b)^2}{c^2+ab}.$$

On the other hand,

$$a^2+b^2+c^2-ab-bc-ca=(a-b)^2+(c-a)(c-b)\le(a-b)^2.$$

Therefore, it suffice to show that

$$\frac{a(a-b)^2}{b}+\frac{bc(a-b)^2}{c^2+ab}\ge(a-b)^2,$$

which is equivalent to

$$(a-b)^2(a^2b+b^2c+c^2a-ab^2-bc^2)\ge 0,$$

$$(a-b)^2[ab(a-b)+b^2c+c^2(a-b)]\ge 0.$$

The equality holds for $a=b=c$.

□

P 1.60. *If a,b,c are positive real numbers such that $a^4+b^4+c^4=3$, then*

(a) $$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\ge 3;$$

(b) $$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\ge\frac{3}{2}.$$

(Alexey Gladkich, 2005)

Solution. (a) By Hölder's inequality, we have

$$\left(\sum \frac{a^2}{b}\right)\left(\sum \frac{a^2}{b}\right)\left(\sum a^2b^2\right) \ge \left(\sum a^2\right)^3.$$

Therefore, it suffices to show that

$$\left(\sum a^2\right)^3 \ge 9\sum a^2b^2,$$

which has the homogeneous form

$$\left(\sum a^2\right)^3 \ge 3\left(\sum a^2b^2\right)\sqrt{3\sum a^4}.$$

Using the notation

$$x = \sum a^2, \quad y = \sum a^2b^2,$$

the inequality can be restated as

$$x^3 \ge 3y\sqrt{3(x^2-2y)}.$$

By squaring, the inequality becomes

$$x^6 - 27x^2y^2 + 54y^3 \ge 0,$$

which is true because

$$x^6 - 27x^2y^2 + 54y^3 = (x^2-3y)^2(x^2+6y) \ge 0.$$

The equality holds for $a = b = c = 1$.

(b) By Hölder's inequality, we have

$$\left(\sum \frac{a^2}{b+c}\right)\left(\sum \frac{a^2}{b+c}\right)\left[\sum a^2(b+c)^2\right] \ge \left(\sum a^2\right)^3.$$

Thus, it suffices to prove that

$$\left(\sum a^2\right)^3 \ge \frac{9}{4}\sum a^2(b+c)^2.$$

Using the inequality from the proof of (a), namely

$$\left(\sum a^2\right)^3 \ge 9\sum a^2b^2,$$

we still have to show that

$$\sum a^2b^2 \ge \frac{1}{4}\sum a^2(b+c)^2.$$

This inequality is equivalent to

$$\sum a^2(b-c)^2 \ge 0.$$

The equality holds for $a = b = c = 1$.

□

P 1.61. *If a, b, c are positive real numbers, then*

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \ge \frac{3(a^3+b^3+c^3)}{a^2+b^2+c^2}.$$

(Vo Quoc Ba Can, 2010)

Solution *(by Ta Minh Hoang).* Assume that

$$a = \max\{a, b, c\},$$

and write the inequality as follows:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}-a-b-c \ge \frac{3(a^3+b^3+c^3)}{a^2+b^2+c^2}-a-b-c,$$

$$\sum \frac{(a-b)^2}{b} \ge \frac{1}{a^2+b^2+c^2}\sum (a+b)(a-b)^2,$$

$$(b-c)^2A+(c-a)^2B+(a-b)^2C \ge 0,$$

where

$$A=\frac{a^2+b^2-bc}{c}>0, \quad B=\frac{b^2+c^2-ca}{a}, \quad C=\frac{c^2+a^2-ab}{b}>0.$$

Consider the nontrivial case $B<0$; that is,

$$ac-b^2-c^2>0.$$

From

$$ac-b^2-c^2=c(a-2b)-(b-c)^2,$$

it follows that

$$c(a-2b)>(b-c)^2 \ge 0,$$

hence

$$a>2b.$$

By the Cauchy-Schwarz inequality, we have

$$(b-c)^2A+(a-b)^2C \ge \frac{[(b-c)+(a-b)]^2}{\dfrac{1}{A}+\dfrac{1}{C}} = \frac{AC}{A+C}\,(a-c)^2.$$

Therefore, it suffices to show that $\dfrac{AC}{A+C}+B \ge 0$; that is, $\dfrac{1}{A}+\dfrac{1}{B}+\dfrac{1}{C} \le 0$, or

$$\frac{c}{a^2+b^2-bc}+\frac{b}{c^2+a^2-ab} \le \frac{a}{ca-b^2-c^2}.$$

Case 1: $a \geq b \geq c$. Since

$$\begin{aligned} a^2+b^2-bc-(ca-b^2-c^2) &> a^2+b^2-bc-ca \\ &= a(a-c)+b(b-c) \geq 0, \end{aligned}$$

and

$$\begin{aligned} c^2+a^2-ab-(ca-b^2-c^2) &> a^2+b^2-a(b+c) \\ &\geq a^2+bc-a(b+c) \\ &= (a-b)(a-c) \geq 0, \end{aligned}$$

it suffices to show that $c+b \leq a$. Indeed, we have $a > 2b \geq b+c$.

Case 2: $a \geq c \geq b$. Replacing b and c by c and b, respectively, we need to show that $a \geq b \geq c$ involves

$$\frac{a^2}{c}+\frac{c^2}{b}+\frac{b^2}{a} \geq \frac{3(a^3+b^3+c^3)}{a^2+b^2+c^2}.$$

According to the previous case, we have

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \geq \frac{3(a^3+b^3+c^3)}{a^2+b^2+c^2}.$$

Therefore, it suffices to show that

$$\frac{a^2}{c}+\frac{c^2}{b}+\frac{b^2}{a} \geq \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}.$$

This inequality is equivalent to

$$(a+b+c)(a-b)(b-c)(a-c) \geq 0,$$

which is clearly true for $a \geq b \geq c$.
The proof is completed. The equality holds for $a=b=c=1$.

Remark. A similar inequality is the following:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \geq \frac{(a+b+c)(a^2+b^2+c^2)}{ab+bc+ca}.$$

By expending, the inequality becomes

$$\frac{ab^3}{c}+\frac{bc^3}{a}+\frac{ca^3}{b} \geq a^2b+b^2c+c^2a.$$

$$a^2(b^2-ca)^2+b^2(c^2-ab)^2+c^2(a^2-bc)^2 \geq 0.$$

□

P 1.62. *If a,b,c are positive real numbers, then*

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+a+b+c\ge 2\sqrt{(a^2+b^2+c^2)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}.$$

(Pham Huu Duc, 2006)

Solution. Without loss of generality, we may assume that b is between a and c; that is,

$$(b-a)(b-c)\le 0.$$

Since

$$\begin{aligned}2\sqrt{(a^2+b^2+c^2)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}&=2\sqrt{\frac{a^2+b^2+c^2}{b}\left(a+\frac{b^2}{c}+\frac{bc}{a}\right)}\\&\le\frac{a^2+b^2+c^2}{b}+a+\frac{b^2}{c}+\frac{bc}{a}\\&=\frac{a^2}{b}+\frac{b^2}{c}+a+b+\frac{bc}{a}+\frac{c^2}{b},\end{aligned}$$

it suffices to prove that

$$\frac{c^2}{a}+c\ge\frac{bc}{a}+\frac{c^2}{b}.$$

This is true because

$$\frac{c^2}{a}+c-\frac{bc}{a}-\frac{c^2}{b}=\frac{c(a-b)(b-c)}{ab}\ge 0.$$

The proof is completed. The equality holds for $a=b=c$.

□

P 1.63. *If a,b,c are positive real numbers, then*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+32\left(\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\right)\ge 51.$$

(Vasile Cîrtoaje, 2009)

Solution. Write the inequality as

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+45\ge 32\left(\frac{b}{a+b}+\frac{c}{b+c}+\frac{a}{c+a}\right).$$

Using the substitution

$$x=\frac{a}{b},\quad y=\frac{b}{c},\quad z=\frac{c}{a},$$

which involves $xyz = 1$, the inequality becomes

$$x + y + z + 45 - 32\left(\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1}\right) \ge 0.$$

We get this inequality by summing the inequalities

$$x - \frac{32}{x+1} + 15 \ge 9\ln x,$$

$$y - \frac{32}{y+1} + 15 \ge 9\ln y,$$

$$z - \frac{32}{z+1} + 15 \ge 9\ln z.$$

Let

$$f(x) = x - \frac{32}{x+1} + 15 - 9\ln x, \quad x > 0.$$

From the derivative

$$f'(x) = 1 + \frac{32}{(x+1)^2} - \frac{9}{x} = \frac{(x-1)(x-3)^2}{x(x+1)^2},$$

it follows that $f(x)$ is decreasing for $0 < x \le 1$ and increasing for $x \ge 1$. Therefore, we have $f(x) \ge f(1) = 0$. The equality holds for $a = b = c$.

□

P 1.64. *Find the largest positive real number K such that the inequalities below hold for any positive real numbers a, b, c:*

(a) $$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} - 3 \ge K\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2}\right);$$

(b) $$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} - 3 + K\left(\frac{a}{2a+b} + \frac{b}{2b+c} + \frac{c}{2c+a} - 1\right) \ge 0.$$

(Vasile Cîrtoaje, 2008)

Solution. (a) For

$$a = x^3, \quad b = x, \quad c = 1,$$

the inequality becomes

$$x^2 + x + \frac{1}{x^3} - 3 \ge K\left(\frac{x^3}{x+1} + \frac{x}{1+x^3} + \frac{1}{x^3+x} - \frac{3}{2}\right),$$

$$\frac{(1-K)x^3}{x+1} + \frac{x^2}{x+1} + x + \frac{1}{x^3} - 3 - K\left(\frac{x}{1+x^3} + \frac{1}{x^3+x} - \frac{3}{2}\right) \ge 0.$$

For $x \to \infty$, we get the necessary condition $1 - K \geq 0$. We will show that the original inequality is true for $K = 1$; that is,

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{3}{2} + \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}.$$

Write the inequality as

$$\left(\frac{c}{a} - \frac{c}{a+b}\right) + \left(\frac{a}{b} - \frac{a}{b+c}\right) + \left(\frac{b}{c} - \frac{b}{c+a}\right) \geq \frac{3}{2},$$

$$\frac{bc}{a(a+b)} + \frac{ca}{b(b+c)} + \frac{ab}{c(c+a)} \geq \frac{3}{2}.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{bc}{a(a+b)} + \frac{ca}{b(b+c)} + \frac{ab}{c(c+a)} \geq \frac{(bc+ca+ab)^2}{abc(a+b)+abc(b+c)+abc(c+a)}$$
$$= \frac{(bc+ca+ab)^2}{2abc(a+b+c)} \geq \frac{3}{2}.$$

The equality holds for $a = b = c$.

(b) For $b = 1$ and $c = a^2$, the inequality becomes

$$2a + \frac{1}{a^2} - 3 + K\left(\frac{2a}{2a+1} + \frac{1}{a^2+2} - 1\right) \geq 0,$$

$$\frac{(a-1)^2(2a+1)}{a^2} - \frac{K(a-1)^2}{(2a+1)(a^2+2)} \geq 0.$$

This inequality holds for any positive a if and only if

$$\frac{2a+1}{a^2} - \frac{K}{(2a+1)(a^2+2)} \geq 0.$$

For $a = 1$, this inequality involves $K \leq 27$. We will show that the original inequality is true for $K = 27$. Using the substitution

$$x = \frac{a}{b}, \quad y = \frac{b}{c}, \quad z = \frac{c}{a},$$

which involves $xyz = 1$, the inequality can be restated as

$$x + y + z - 3 - \frac{27}{2}\left(\frac{1}{2x+1} + \frac{1}{2y+1} + \frac{1}{2z+1} - 1\right) \geq 0.$$

First Solution. We get the desired inequality by summing the inequalities

$$x - \frac{27}{2(2x+1)} + \frac{7}{2} \geq 4\ln x,$$

$$y - \frac{27}{2(2y+1)} + \frac{7}{2} \ge 4\ln y,$$

$$z - \frac{27}{2(2z+1)} + \frac{7}{2} \ge 4\ln z.$$

Let

$$f(x) = x - \frac{27}{2(2x+1)} + \frac{7}{2} - 4\ln x, \quad x > 0.$$

From the derivative

$$f'(x) = 1 + \frac{27}{(2x+1)^2} - \frac{4}{x} = \frac{4(x-1)^3}{x(2x+1)^2},$$

it follows that $f(x)$ is decreasing for $0 < x \le 1$ and increasing for $x \ge 1$. Therefore, we have $f(x) \ge f(1) = 0$. The equality holds for $a = b = c$.

Second Solution. Replacing x, y, z by e^x, e^y, e^z, respectively, we need to show that

$$x + y + z = 0$$

involves

$$f(x) + f(y) + f(z) \ge 3f\left(\frac{x+y+z}{3}\right),$$

where

$$f(u) = e^u - \frac{27}{2(2e^u+1)}.$$

If f is convex on $\mathbb{R}$, then this inequality is just Jensen's inequality. Indeed, f is convex because

$$e^{-u}f''(u) = 1 + \frac{27(1-2e^u)}{(2e^u+1)^3} = \frac{4(e^u-1)^2(2e^u+7)}{(2e^u+1)^3} \ge 0.$$

□

P 1.65. *If* $a, b, c \in \left[\frac{1}{2}, 2\right]$, *then*

(a)
$$8\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \ge 5\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) + 9;$$

(b)
$$20\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \ge 17\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right).$$

(Vasile Cîrtoaje, 2008)

Solution. Without loss of generality, assume that

$$a = \max\{a, b, c\}.$$

Let

$$t = \sqrt{\frac{a}{c}}, \quad 1 \le t \le 2.$$

(a) Let

$$E(a,b,c) = 8\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) - 5\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right) - 9.$$

We will show that

$$E(a,b,c) \ge E(a,\sqrt{ac},c) \ge 0.$$

We have

$$E(a,b,c) - E(a,\sqrt{ac},c) = 8\left(\frac{a}{b}+\frac{b}{c}-2\sqrt{\frac{a}{c}}\right) - 5\left(\frac{b}{a}+\frac{c}{b}-2\sqrt{\frac{c}{a}}\right)$$

$$= \frac{(b-\sqrt{ac})^2(8a-5c)}{abc} \ge 0.$$

Also,

$$\begin{aligned} E(a,\sqrt{ac},c) &= 8\left(2\sqrt{\frac{a}{c}}+\frac{c}{a}-3\right) - 5\left(2\sqrt{\frac{c}{a}}+\frac{a}{c}-3\right) \\ &= 8\left(2t+\frac{1}{t^2}-3\right) - 5\left(\frac{2}{t}+t^2-3\right) \\ &= \frac{8}{t^2}(t-1^2(2t+1) - \frac{5}{t}(t-1)^2(t+2) \\ &= \frac{(t-1)^2(4+5t)(2-t)}{t^2} \ge 0. \end{aligned}$$

The equality holds for $a=b=c$, and also for $a=2$, $b=1$ and $c=\frac{1}{2}$ (or any cyclic permutation).

(b) Let

$$E(a,b,c) = 20\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) - 17\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

We will show that

$$E(a,b,c) \ge E(a,\sqrt{ac},c) \ge 0.$$

We have

$$E(a,b,c) - E(a,\sqrt{ac},c) = 20\left(\frac{a}{b}+\frac{b}{c}-2\sqrt{\frac{a}{c}}\right) - 17\left(\frac{b}{a}+\frac{c}{b}-2\sqrt{\frac{c}{a}}\right)$$

$$= \frac{(b-\sqrt{ac})^2(20a-17c)}{abc} \ge 0.$$

Also, we have

$$\begin{aligned}E(a,\sqrt{ac},c) &= 20\left(2\sqrt{\frac{a}{c}}+\frac{c}{a}\right)-17\left(2\sqrt{\frac{c}{a}}+\frac{a}{c}\right)\\ &= 20\left(2t+\frac{1}{t^2}\right)-17\left(\frac{2}{t}+t^2\right)\\ &= \frac{20-34t+40t^3-17t^4}{t^2}\\ &= \frac{(2-t)(17t^3-6t^2-12t+10)}{t^2}.\end{aligned}$$

We need to show that $17t^3-6t^2-12t+10\ge 0$ for $1\le t\le 2$. Indeed, we have

$$17t^3-6t^2-12t+10\ge 11t^2-12t+10>4t^2-12t+9=(2t-3)^2\ge 0.$$

The equality holds for $a=2$, $b=1$ and $c=\frac{1}{2}$ (or any cyclic permutation).

□

P 1.66. *If a,b,c are positive real numbers such that $a\le b\le c$, then*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{2a}{b+c}+\frac{2b}{c+a}+\frac{2c}{a+b}.$$

First Solution. Since

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)=\left(\frac{a}{b}-1\right)\left(\frac{b}{c}-1\right)\left(\frac{c}{a}-1\right)\ge 0,$$

it suffices to show that

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\ge\frac{4a}{b+c}+\frac{4b}{c+a}+\frac{4c}{a+b}.$$

This inequality is equivalent to

$$a\left(\frac{1}{b}+\frac{1}{c}-\frac{4}{b+c}\right)+b\left(\frac{1}{c}+\frac{1}{a}-\frac{4}{c+a}\right)+c\left(\frac{1}{a}+\frac{1}{b}-\frac{4}{a+b}\right)\ge 0,$$

$$\frac{a^2(b-c)^2}{b+c}+\frac{b^2(c-a)^2}{c+a}+\frac{c^2(a-b)^2}{a+b}\ge 0.$$

The equality holds for $a=b=c$.

Second Solution. The inequality is equivalent to

$$\frac{a(c-b)}{b(b+c)}-\frac{b(c-a)}{c(c+a)}+\frac{c(b-a)}{a(a+b)}\ge 0.$$

Taking account of

$$b(c-a) = c(b-a) + a(c-b),$$

we may rewrite the inequality as

$$c(b-a)\left[\frac{1}{a(a+b)} - \frac{1}{c(c+a)}\right] + a(c-b)\left[\frac{1}{b(b+c)} - \frac{1}{c(c+a)}\right] \ge 0.$$

Since

$$\frac{1}{a(a+b)} - \frac{1}{c(c+a)} = \frac{c^2-a^2+a(c-b)}{ac(a+b)(c+a)} \ge \frac{c-b}{c(a+b)(c+a)}$$

and

$$\frac{1}{b(b+c)} - \frac{1}{c(c+a)} = \frac{c^2-b^2+c(a-b)}{bc(b+c)(c+a)} \ge \frac{a-b}{b(b+c)(c+a)},$$

it suffices to show that

$$\frac{c(b-a)(c-b)}{c(a+b)(c+a)} + \frac{a(c-b)(a-b)}{b(b+c)(c+a)} \ge 0.$$

This inequality is true if

$$\frac{1}{a+b} - \frac{a}{b(b+c)} \ge 0.$$

Indeed,

$$\frac{1}{a+b} - \frac{a}{b(b+c)} \ge \frac{1}{a+b} - \frac{1}{b+c} = \frac{c-a}{(a+b)(b+c)} \ge 0.$$

□

P 1.67. *Let a, b, c be positive real numbers such that $abc = 1$.*

(a) If $a \le b \le c$, then

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge a^{3/2} + b^{3/2} + c^{3/2};$$

(b) If $a \le 1 \le b \le c$, then

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge a^{\sqrt{3}} + b^{\sqrt{3}} + c^{\sqrt{3}}.$$

(Vasile Cîrtoaje, 2008)

Solution. (a) Since

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)=\left(\frac{a}{b}-1\right)\left(\frac{b}{c}-1\right)\left(\frac{c}{a}-1\right)\geq 0,$$

it suffices to show that

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\geq 2(a^{3/2}+b^{3/2}+c^{3/2}).$$

Indeed, by the AM-GM inequality, we have

$$\sum\frac{a}{b}+\sum\frac{b}{a}=\sum a\left(\frac{1}{b}+\frac{1}{c}\right)\geq\sum\frac{2a}{\sqrt{bc}}=2\sum a^{3/2}.$$

The equality holds for $a=b=c=1$.

(b) Let $k=\sqrt{3}$ and

$$E(a,b,c)=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-a^k-b^k-c^k.$$

We will show that

$$E(a,b,c)\geq E(a,\sqrt{bc},\sqrt{bc})\geq 0;$$

that is,

$$E(\frac{1}{bc},b,c)\geq E(\frac{1}{bc},\sqrt{bc},\sqrt{bc})\geq 0.$$

Substituting

$$t=\sqrt{bc},\quad t\geq 1,$$

we rewrite the right inequality as $f(t)\geq 0$, where

$$f(t)=\frac{1}{t^3}+1+t^3-\frac{1}{t^{2k}}-2t^k.$$

We have the derivative

$$\frac{f'(t)}{t^2}=g(t),\quad g(t)=\frac{-3}{t^6}+3+\frac{2k}{t^{2k+3}}-\frac{2k}{t^{3-k}}.$$

Since

$$\frac{1}{2}t^{2k+4}g'(t)=9t^{2k-3}-k(2k+3)+k(3-k)t^{3k}$$
$$\geq 9-k(2k+3)+k(3-k)=9-3k^2=0,$$

$g(t)$ is increasing for $t\geq 1$. Therefore, $g(t)\geq g(1)=0$, $f'(t)\geq 0$, $f(t)$ is increasing for $t\geq 1$, hence $f(t)\geq f(1)=0$.

Substituting $b=x^2$ and $c=y^2$, where $1\leq x\leq y$, the left inequality becomes

$$E\left(\frac{1}{x^2y^2},x^2,y^2\right)\geq E\left(\frac{1}{x^2y^2},xy,xy\right),$$

or, equivalently,

$$\frac{1}{x^4y^2} + \frac{x^2}{y^2} + x^2y^4 - \frac{1}{x^3y^3} - 1 - x^3y^3 \geq (y^k - x^k)^2.$$

We write this inequality as

$$(y-x)\left(x^2y^3 + \frac{1}{x^4y^3} - \frac{x+y}{y^2}\right) \geq (y^k - x^k)^2,$$

and then show that

$$(y-x)\left(x^2y^3 + \frac{1}{x^4y^3} - \frac{x+y}{y^2}\right) \geq (y-x)(y^3-x^3) \geq (y^k - x^k)^2. \qquad (*)$$

The left inequality (*) is true if $f(x,y) \geq 0$, where

$$f(x,y) = x^2y^3 + \frac{1}{x^4y^3} - \frac{x+y}{y^2} - y^3 + x^3.$$

We will show that

$$f(x,y) \geq f(1,y) \geq 0.$$

Since $1 \leq x \leq y$, we have

$$\begin{aligned} f(x,y) - f(1,y) &= x^3 - 1 + y^3(x^2-1) - \frac{1}{y^2}(x-1) - \frac{1}{y^3}\left(1 - \frac{1}{x^4}\right) \\ &\geq x^3 - 1 + (x^2-1) - (x-1) - \left(1 - \frac{1}{x^4}\right) \\ &= (x^2-1)\left[\left(x - \frac{1}{x^2}\right) + \left(1 - \frac{1}{x^4}\right)\right] \geq 0 \end{aligned}$$

and

$$f(1,y) = \frac{1}{y^3} - \frac{1+y}{y^2} + 1 = \frac{(1+y)(1-y)^2}{y^3} \geq 0.$$

In order to prove the right inequality (*), we will prove that

$$(y-x)(y^3-x^3) \geq \frac{3}{4}(y^2-x^2)^2 \geq (y^k-x^k)^2.$$

We have

$$4(y-x)(y^3-x^3) - 3(y^2-x^2)^2 = (y-x)^4 \geq 0.$$

To complete the proof, we only need to show that

$$\frac{k}{2}(y^2-x^2) \geq y^k - x^k, \quad k = \sqrt{3}.$$

For fixed y, let

$$g(x) = x^k - y^k + \frac{k}{2}(y^2-x^2), \quad 1 \leq x \leq y.$$

Since

$$g'(x) = kx(x^{k-2} - 1) \le 0,$$

$g(x)$ is decreasing, hence $g(x) \ge g(y) = 0$. The equality in (b) is an equality if and only if $a = b = c = 1$.

□

P 1.68. *If k and a, b, c are positive real numbers, then*

$$\frac{1}{(k+1)a+b} + \frac{1}{(k+1)b+c} + \frac{1}{(k+1)c+a} \ge \frac{1}{ka+b+c} + \frac{1}{kb+c+a} + \frac{1}{kc+a+b}.$$

(Vasile Cîrtoaje, 2011)

First Solution. For $k = 1$, we need to show that

$$\frac{1}{2a+b} + \frac{1}{2b+c} + \frac{1}{2c+a} \ge \frac{3}{a+b+c}.$$

This follows immediately from the AM-HM inequality, as follows:

$$\frac{1}{2a+b} + \frac{1}{2b+c} + \frac{1}{2c+a} \ge \frac{9}{(2a+b)+(2b+c)+(2c+a)}$$
$$= \frac{3}{a+b+c}.$$

Further, consider two cases: $k > 1$ and $0 < k < 1$.

Case 1: $k > 1$. By the Cauchy-Schwarz inequality, we have

$$\frac{k-1}{(k+1)a+b} + \frac{1}{kc+a+b} \ge \frac{[(k-1)+1]^2}{(k-1)[(k+1)a+b]+(kc+a+b)}$$
$$= \frac{k}{ka+b+c}.$$

Adding this inequality and the similar ones yields the desired inequality.

Case 2: $0 < k < 1$. By the Cauchy-Schwarz inequality, we have

$$\frac{1-k}{(k+1)a+b} + \frac{k}{ka+b+c} \ge \frac{[(1-k)+k]^2}{(1-k)[(k+1)a+b]+k(ka+b+c)}$$
$$= \frac{1}{kc+a+b}.$$

Adding this inequality and the similar ones yields the desired inequality.
The equality holds for $a = b = c$.

Second Solution (by *Vo Quoc Ba Can*). By the Cauchy-Schwarz inequality, we have

$$\frac{1}{(k+1)a+b} + \frac{k}{(k+1)b+c} + \frac{k^2}{(k+1)c+a} \ge$$

$$\geq \frac{(1+k+k^2)^2}{[(k+1)a+b]+k[(k+1)b+c]+k^2[(k+1)c+a]}$$
$$= \frac{1+k+k^2}{kc+a+b}.$$

Therefore, we get in succession

$$\sum \frac{1}{(k+1)a+b} + \sum \frac{k}{(k+1)b+c} + \sum \frac{k^2}{(k+1)c+a} \geq \sum \frac{1+k+k^2}{kc+a+b},$$
$$(1+k+k^2)\sum \frac{1}{(k+1)a+b} \geq (1+k+k^2)\sum \frac{1}{ka+b+c},$$
$$\sum \frac{1}{(k+1)a+b} \geq \sum \frac{1}{ka+b+c}.$$

Third Solution. We have

$$\frac{1}{(k+1)a+b} - \frac{1}{ka+b+c} = \frac{c-a}{(ka+a+b)(ka+b+c)}$$
$$\geq \frac{c-a}{(kc+a+b)(ka+b+c)} = \frac{1}{k-1}\left(\frac{1}{ka+b+c} - \frac{1}{kc+a+b}\right),$$

hence

$$\sum \frac{1}{(k+1)a+b} - \sum \frac{1}{ka+b+c} \geq \frac{1}{k-1}\left(\sum \frac{1}{ka+b+c} - \sum \frac{1}{kc+a+b}\right) = 0.$$

□

P 1.69. *If a, b, c are positive real numbers, then*

(a) $$\frac{a}{\sqrt{2a+b}} + \frac{b}{\sqrt{2b+c}} + \frac{c}{\sqrt{2c+a}} \leq \sqrt{a+b+c};$$

(b) $$\frac{a}{\sqrt{a+2b}} + \frac{b}{\sqrt{b+2c}} + \frac{c}{\sqrt{c+2a}} \geq \sqrt{a+b+c}.$$

Solution. (a) By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a}{\sqrt{2a+b}} = \sum \left(\sqrt{a}\cdot\sqrt{\frac{a}{2a+b}}\right) \leq \sqrt{\left(\sum a\right)\left(\sum \frac{a}{2a+b}\right)}.$$

Therefore, it suffices to show that

$$\sum \frac{a}{2a+b} \leq 1.$$

This inequality is equivalent to

$$\sum \frac{b}{2a+b} \ge 1.$$

Applying the Cauchy-Schwarz inequality, we get

$$\sum \frac{b}{2a+b} \ge \frac{\left(\sum b\right)^2}{\sum b(2a+b)} = 1.$$

The equality holds for $a = b = c$.

(b) By Hölder's inequality, we have

$$\left(\sum \frac{a}{\sqrt{a+2b}}\right)^2 \ge \frac{\left(\sum a\right)^3}{\sum a(a+2b)} = \sum a.$$

From this, the desired inequality follows. The equality holds for $a = b = c$.

□

P 1.70. *Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that*

$$a\sqrt{\frac{a+2b}{3}} + b\sqrt{\frac{b+2c}{3}} + c\sqrt{\frac{c+2a}{3}} \le 3.$$

First Solution. By the Cauchy-Schwarz inequality, we have

$$\sum a\sqrt{\frac{a+2b}{3}} \le \sqrt{\left(\sum a\right)\left[\sum \frac{a(a+2b)}{3}\right]} = \sqrt{\frac{\left(\sum a\right)^3}{3}} = 3.$$

The equality holds for $a = b = c = 1$, and also for $a = 3$ and $b = c = 0$ (or any cyclic permutation).

Second Solution. Applying Jensen's inequality to the concave function $f(x) = \sqrt{x}$, $x \ge 0$, we have

$$\begin{aligned} a\sqrt{a+2b} + b\sqrt{b+2c} + c\sqrt{c+2a} &\le \\ \le (a+b+c)\sqrt{\frac{a(a+2b)+b(b+2c)+c(c+2a)}{a+b+c}} \\ = (a+b+c)\sqrt{a+b+c} = 3\sqrt{3}. \end{aligned}$$

□

P 1.71. *If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then*

$$a\sqrt{1+b^3} + b\sqrt{1+c^3} + c\sqrt{1+a^3} \le 5.$$

(Pham Kim Hung, 2007)

Solution. Using the AM-GM inequality yields

$$\sqrt{1+b^3} = \sqrt{(1+b)(1-b+b^2)} \le \frac{(1+b)+(1-b+b^2)}{2} = 1+\frac{b^2}{2}.$$

Therefore,

$$\sum a\sqrt{1+b^3} \le \sum a\left(1+\frac{b^2}{2}\right) = 3+\frac{ab^2+bc^2+ca^2}{2}.$$

To complete the proof, it remains to show that

$$ab^2+bc^2+ca^2 \le 4.$$

But this is just the inequality in P 1.1. The equality occurs for $a=0$, $b=1$ and $c=2$ (or any cyclic permutation).

□

P 1.72. *If a,b,c are positive real numbers such that $abc=1$, then*

(a)
$$\sqrt{\frac{a}{b+3}}+\sqrt{\frac{b}{c+3}}+\sqrt{\frac{c}{a+3}} \ge \frac{3}{2};$$

(b)
$$\sqrt[3]{\frac{a}{b+7}}+\sqrt[3]{\frac{b}{c+7}}+\sqrt[3]{\frac{c}{a+7}} \ge \frac{3}{2}.$$

Solution. (a) Putting

$$a=\frac{x}{y},\quad b=\frac{z}{x},\quad c=\frac{y}{z},$$

the inequality can be restated as

$$\frac{x}{\sqrt{y(3x+z)}}+\frac{y}{\sqrt{z(3y+x)}}+\frac{z}{\sqrt{x(3z+y)}} \ge \frac{3}{2}.$$

By Hölder's inequality, we have

$$\left[\sum \frac{x}{\sqrt{y(3x+z)}}\right]^2 \left[\sum xy(3x+z)\right] \ge \left(\sum x\right)^3.$$

Therefore, it suffices to show that

$$4(x+y+z)^3 \ge 27(x^2y+y^2z+z^2x+xyz).$$

This is just the inequality (a) in P 1.9. The equality holds for $a=b=c=1$.

(b) Putting

$$a=\frac{x^4}{y^4},\quad b=\frac{z^4}{x^4},\quad c=\frac{y^4}{z^4},$$

the inequality becomes

$$\sum \sqrt[3]{\frac{x^8}{y^4(7x^4+z^4)}} \ge \frac{3}{2}.$$

By Hölder's inequality, we have

$$\left[\sum \sqrt[3]{\frac{x^8}{y^4(7x^4+z^4)}}\right]^3 \left[\sum(7x^4+z^4)\right] \ge \left(\sum \frac{x^2}{y}\right)^4.$$

Since $\sum(7x^4+z^4) = 8\sum x^4$, it is enough to show that

$$\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)^4 \ge 27(x^4+y^4+z^4),$$

which is just the inequality in P 1.60-(a). The equality holds for $a=b=c=1$.

□

P 1.73. *If a,b,c are positive real numbers, then*

$$\left(1+\frac{4a}{a+b}\right)^2+\left(1+\frac{4b}{b+c}\right)^2+\left(1+\frac{4c}{c+a}\right)^2 \ge 27.$$

(Vasile Cîrtoaje, 2012)

Solution. Let

$$x=\frac{a-b}{a+b}, \quad y=\frac{b-c}{b+c}, \quad z=\frac{c-a}{c+a}.$$

We have

$$-1 < x,y,z < 1$$

and

$$x+y+z+xyz=0.$$

Since

$$\frac{2a}{a+b}=x+1, \quad \frac{2b}{b+c}=y+1, \quad \frac{2c}{c+a}=z+1,$$

we can write the inequality as follows:

$$(2x+3)^2+(2y+3)^2+(2z+3)^2 \ge 27,$$

$$x^2+y^2+z^2+3(x+y+z) \ge 0,$$

$$x^2+y^2+z^2 \ge 3xyz.$$

By the AM-GM inequality, we have

$$x^2+y^2+z^2 \ge 3\sqrt[3]{x^2y^2z^2}.$$

Thus, it suffices to show that $|xyz| \le 1$, which is clearly true. The equality holds for $a=b=c$.

□

P 1.74. *If a, b, c are positive real numbers, then*

$$\sqrt{\frac{2a}{a+b}}+\sqrt{\frac{2b}{b+c}}+\sqrt{\frac{2c}{c+a}}\le 3.$$

(Vasile Cîrtoaje, 1992)

First Solution. By the Cauchy-Schwarz inequality, we have

$$\sum\sqrt{\frac{2a}{a+b}}\le\sqrt{\left[\sum\frac{2a}{(a+b)(a+c)}\right]\left[\sum(a+c)\right]}.$$

Thus, it suffices to show that

$$\sum\frac{a}{(a+b)(a+c)}\le\frac{9}{4(a+b+c)},$$

which is equivalent to

$$a(b-c)^2+b(c-a)^2+c(a-b)^2\ge 0.$$

The equality occurs for $a=b=c$.

Second Solution. By the Cauchy-Schwarz inequality, we have

$$\sum\sqrt{\frac{2a}{a+b}}\le\sqrt{\left[\sum\frac{1}{(a+b)(b+c)}\right]\left[\sum 2a(b+c)\right]}.$$

Thus, it suffices to show that

$$\sum\frac{1}{(a+b)(b+c)}\le\frac{9}{4(ab+bc+ca)},$$

which is equivalent to

$$a(b-c)^2+b(c-a)^2+c(a-b)^2\ge 0.$$

□

P 1.75. *If a, b, c are nonnegative real numbers, then*

$$\sqrt{\frac{a}{4a+5b}}+\sqrt{\frac{b}{4b+5c}}+\sqrt{\frac{c}{4c+5a}}\le 1.$$

(Vasile Cîrtoaje, 2004)

Solution. If one of a, b, c is zero, then the inequality is clearly true. Otherwise, using the substitution

$$u = \frac{b}{a}, \quad v = \frac{c}{b}, \quad w = \frac{a}{c},$$

we need to show that $uvw = 1$ involves

$$\frac{1}{\sqrt{4+5u}} + \frac{1}{\sqrt{4+5v}} + \frac{1}{\sqrt{4+5w}} \le 1.$$

Using the contradiction method, it suffices to show that

$$\frac{1}{\sqrt{4+5u}} + \frac{1}{\sqrt{4+5v}} + \frac{1}{\sqrt{4+5w}} > 1$$

involves $uvw < 1$. Let

$$x = \frac{1}{\sqrt{4+5u}}, \quad y = \frac{1}{\sqrt{4+5v}}, \quad z = \frac{1}{\sqrt{4+5w}},$$

where $x, y, z \in \left(0, \frac{1}{2}\right)$. Since

$$u = \frac{1-4x^2}{5x^2}, \quad v = \frac{1-4y^2}{5y^2}, \quad w = \frac{1-4z^2}{5z^2},$$

we have to prove that $x + y + z > 1$ involves

$$(1-4x^2)(1-4y^2)(1-4z^2) < 125x^2y^2z^2.$$

Since

$$1 - 4x^2 < (x+y+z)^2 - 4x^2 = (-x+y+z)(3x+y+z),$$

it suffices to prove the homogeneous inequality

$$(3x+y+z)(3y+z+x)(3z+x+y)(-x+y+z)(-y+z+x)(-z+x+y) \le 125x^2y^2z^2.$$

By the AM-GM inequality, we have

$$(3x+y+z)(3y+z+x)(3z+x+y) \le 125\left(\frac{x+y+z}{3}\right)^3.$$

Therefore, it is enough to show that

$$\left(\frac{x+y+z}{3}\right)^3 (-x+y+z)(-y+z+x)(-z+x+y) \le x^2y^2z^2.$$

Using the substitution

$$a = -x+y+z, \quad b = -y+z+x, \quad c = -z+x+y,$$

where $a, b, c > 0$, the inequality can be restated as

$$64abc(a+b+c)^3 \le 27(b+c)^2(c+a)^2(a+b)^2.$$

The known inequality

$$9(b+c)(c+a)(a+b) \ge 8(a+b+c)(ab+bc+ca),$$

equivalent to

$$a(b-c)^2 + b(c-a)^2 + c(a-b)^2 \ge 0,$$

involves

$$81(b+c)^2(c+a)^2(a+b)^2 \ge 64(a+b+c)^2(ab+bc+ca)^2.$$

Thus, it suffices to show that

$$3abc(a+b+c) \le (ab+bc+ca)^2.$$

which is also a known inequality, equivalent to

$$a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2 \ge 0.$$

Thus, the proof is completed. The equality occurs for $a = b = c$.

□

P 1.76. *If a, b, c are positive real numbers, then*

$$\frac{a}{\sqrt{4a^2+ab+4b^2}} + \frac{b}{\sqrt{4b^2+bc+4c^2}} + \frac{c}{\sqrt{4c^2+ca+4a^2}} \le 1.$$

(Bin Zhao, 2006)

Solution. By the AM-GM inequality, we have

$$ab + 4b^2 \ge 5\sqrt[5]{ab \cdot b^8} = 5\sqrt[5]{ab^9},$$

$$\frac{a}{\sqrt{4a^2+ab+4b^2}} \le \frac{a}{\sqrt{4a^2+5\sqrt[5]{ab^9}}} = \sqrt{\frac{a^{9/5}}{4a^{9/5}+5b^{9/5}}}.$$

Therefore, it suffices to show that

$$\sqrt{\frac{a^{9/5}}{4a^{9/5}+5b^{9/5}}} + \sqrt{\frac{b^{9/5}}{4b^{9/5}+5c^{9/5}}} + \sqrt{\frac{c^{9/5}}{4c^{9/5}+5a^{9/5}}} \le 1.$$

Replacing $a^{9/5}, b^{9/5}, c^{9/5}$ by a, b, c, respectively, we get the inequality in P 1.75. The equality holds for $a = b = c$.

□

P 1.77. *If a, b, c are positive real numbers, then*

$$\sqrt{\frac{a}{a+b+7c}}+\sqrt{\frac{b}{b+c+7a}}+\sqrt{\frac{c}{c+a+7b}} \ge 1.$$

(Vasile Cîrtoaje, 2006)

Solution. Substituting

$$x=\sqrt{\frac{a}{a+b+7c}}, \quad y=\sqrt{\frac{b}{b+c+7a}}, \quad z=\sqrt{\frac{c}{c+a+7b}},$$

we have

$$\begin{cases} (x^2-1)a+x^2b+7x^2c=0 \\ (y^2-1)b+y^2c+7y^2a=0 \\ (z^2-1)c+z^2a+7z^2b=0 \end{cases},$$

which involves

$$\begin{vmatrix} x^2-1 & x^2 & 7x^2 \\ 7y^2 & y^2-1 & y^2 \\ z^2 & 7z^2 & z^2-1 \end{vmatrix} = 0 \ ;$$

that is,

$$F(x,y,z)=0,$$

where

$$F(x,y,z)=324x^2y^2z^2+6\sum x^2y^2+\sum x^2-1.$$

We need to show that $F(x,y,z)=0$ involves $x+y+z\ge 1$, where $x,y,z>0$. To do this, we use the contradiction method. Assume that $x+y+z<1$ and show that $F(x,y,z)<0$. Since $F(x,y,z)$ is strictly increasing in each of its arguments, it is enough to prove that $x+y+z=1$ involves $F(x,y,z)\le 0$. We have

$$\begin{aligned} F(x,y,z) &= 324x^2y^2z^2+6\left(\sum xy\right)^2-12xyz\sum x+\left(\sum x\right)^2-2\sum xy-1 \\ &= 324x^2y^2z^2+6\left(\sum xy\right)^2-12xyz-2\sum xy \\ &= 12xyz(27xyz-1)+2\left(\sum xy\right)\left(3\sum xy-1\right). \end{aligned}$$

Because

$$27xyz \le \left(\sum x\right)^3 = 1$$

and

$$3\sum xy \le \left(\sum x\right)^2 = 1,$$

the conclusion follows. The equality occurs for $a=b=c$.

□

P 1.78. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

(a)
$$\sqrt{\frac{a}{3b+c}}+\sqrt{\frac{b}{3c+a}}+\sqrt{\frac{c}{3a+b}} \ge \frac{3}{2};$$

(b)
$$\sqrt{\frac{a}{2b+c}}+\sqrt{\frac{b}{2c+a}}+\sqrt{\frac{c}{2a+b}} \ge \sqrt[4]{8}.$$

(Vasile Cîrtoaje and Pham Kim Hung, 2006)

Solution. Consider the inequality

$$\sqrt{\frac{(k+1)a}{kb+c}}+\sqrt{\frac{(k+1)b}{kc+a}}+\sqrt{\frac{(k+1)c}{ka+b}} \ge A_k, \qquad k>0,$$

and use the substitution

$$x=\sqrt{\frac{(k+1)a}{kb+c}}, \quad y=\sqrt{\frac{(k+1)b}{kc+a}}, \quad z=\sqrt{\frac{(k+1)c}{ka+b}}.$$

From the identity

$$(kb+c)(kc+a)(ka+b)=(k^3+1)abc+kbc(kb+c)+kca(kc+a)+kab(ka+b),$$

written as

$$\frac{kb+c}{(k+1)a}\cdot\frac{kc+a}{(k+1)b}\cdot\frac{ka+b}{(k+1)c}=\frac{k^2-k+1}{(k+1)^2}+\frac{k}{(k+1)^2}\left[\frac{kb+c}{(k+1)a}+\frac{kc+a}{(k+1)b}+\frac{ka+b}{(k+1)c}\right],$$

we get

$$\frac{1}{x^2y^2z^2}=\frac{k^2-k+1}{(k+1)^2}+\frac{k}{(k+1)^2}\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right),$$

which is equivalent to $F(x,y,z)=0$, where

$$F(x,y,z)=k(x^2y^2+y^2z^2+z^2x^2)+(k^2-k+1)x^2y^2z^2-(k+1)^2.$$

So, we need to show that $F(x,y,z)=0$ yields $x+y+z\ge A_k$. To do this, we use the contradiction method. Assume that $x+y+z<A_k$ and show that $F(x,y,z)<0$. Since $F(x,y,z)$ is strictly increasing in each of its variables, it suffices to prove that $x+y+z=A_k$ involves $F(x,y,z)\le 0$. Let

$$k_1=\frac{49+9\sqrt{17}}{32}\approx 2.691.$$

(a) We need to show that $F(x,y,z)\le 0$ for $x+y+z=A_k=3$ and $k=3$. We will show a more general inequality, namely $F(x,y,z)\le 0$ for $k\ge k_1$ and all nonnegative numbers x,y,z satisfying $x+y+z=3$. The AM-GM inequality $x+y+z\ge 3\sqrt[3]{xyz}$ involves $xyz\le 1$. On the other hand, by Schur's inequality

$$(x+y+z)^3+9xyz\ge 4(x+y+z)(xy+yz+zx)$$

we get

$$4(xy + yz + zx) \le 9 + 3xyz,$$

hence

$$(xy + yz + zx)^2 - 9 \le \frac{(9 + 3xyz)^2}{16} - 9 = \frac{9}{16}(xyz - 1)(xyz + 7).$$

Therefore,

$$\begin{aligned} F(x, y, z) &= k[(xy + yz + zx)^2 - 6xyz] + (k^2 - k + 1)x^2y^2z^2 - (k + 1)^2 \\ &= k[(xy + yz + zx)^2 - 9] + (k^2 - k + 1)(x^2y^2z^2 - 1) - 6k(xyz - 1) \\ &\le \frac{9k}{16}(xyz - 1)(xyz + 7) + (k^2 - k + 1)(x^2y^2z^2 - 1) - 6k(xyz - 1) \\ &= \frac{1}{16}(xyz - 1)\left[(16k^2 - 7k + 16)xyz + 16k^2 - 49k + 16\right] \le 0. \end{aligned}$$

Since $xyz - 1 \le 0$ and $16k^2 - 7k + 16 > 0$, it suffices to show that $16k^2 - 49k + 16 \ge 0$; indeed, this inequality is true for $k \ge k_1$.
The equality occurs for $a = b = c$. In addition, when $k = k_1$, the equality also occurs for $a = 0$ and $b/c = \sqrt{k}$ (or any cyclic permutation).

(b) We need to show that $F(x, y, z) \le 0$ for $A_k = \sqrt[4]{72}$ and $k = 2$. We will show a more general inequality, that $F(x, y, z) \le 0$ for $1 \le k \le k_1$ and all nonnegative numbers x, y, z satisfying

$$x + y + z = A_k = 2\sqrt[4]{\frac{(k + 1)^2}{k}}.$$

From

$$\begin{aligned} F(x, y, z) &= k(x^2y^2 + y^2z^2 + z^2x^2) + (k^2 - k + 1)x^2y^2z^2 - (k + 1)^2 \\ &= k(xy + yz + zx)^2 - 2kA_kxyz + (k^2 - k + 1)x^2y^2z^2 - (k + 1)^2, \end{aligned}$$

it follows that for fixed xyz, $F(x, y, z)$ is maximum when $xy + yz + zx$ is maximum; that is, according to P 3.58 in Volume 1, when two of x, y, z are equal. Due to symmetry, we only need to show that $F(x, y, z) \le 0$ for $y = z$. Write the inequality $F(x, y, z) \le 0$ as follows:

$$k(x^2y^2 + y^2z^2 + z^2x^2) + (k^2 - k + 1)x^2y^2z^2 - k\left(\frac{x + y + z}{2}\right)^4 \le 0,$$

$$k\left[\left(\frac{x + y + z}{2}\right)^4 - x^2y^2 - y^2z^2 - z^2x^2\right] \ge (k^2 - k + 1)x^2y^2z^2,$$

$$k\sqrt{k}\ (x + y + z)^2\left[(x + y + z)^4 - 16(x^2y^2 + y^2z^2 + z^2x^2\right] \ge 64(k^3 + 1)x^2y^2z^2.$$

Due to homogeneity, we may only consider the cases $y = z = 0$ and $y = z = 1$. In the non-trivial case $y = z = 1$, the inequality becomes

$$k\sqrt{k}\ x(x + 2)^2(x^3 + 8x^2 - 8x + 32) \ge 64(k^3 + 1)x^2.$$

This is true because

$$297k\sqrt{k} \ge 64(k^3+1)$$

for $1 \le k \le k_1$, and

$$x(x+2)^2(x^3+8x^2-8x+32) \ge 297x^2.$$

Notice that

$$x(x+2)^2(x^3+8x^2-8x+32) - 297x^2 = x(x-1)^2(x^3+14x^2+55x+128) \ge 0.$$

If $1 \le k < k_1$, then the equality occurs only for $a=0$ and $b/c=\sqrt{k}$ (or any cyclic permutation). Therefore, if $k=2$, then the equality holds for $a=0$ and $b/c=\sqrt{2}$ (or any cyclic permutation).

Remark. From the proof above, it follows that the following more general statement holds:

- *Let a,b,c be nonnegative real numbers, no two of which are zero. If $k>0$, then*

$$\sqrt{\frac{a}{kb+c}}+\sqrt{\frac{b}{kc+a}}+\sqrt{\frac{c}{ka+b}} \ge \min\left\{\frac{3}{\sqrt{k+1}}, \frac{2}{\sqrt[4]{k}}\right\}.$$

For $k=1$, we get the known inequality

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}} \ge 2,$$

with equality for $a=0$ and $b=c$ (or any cyclic permutation). We can get this inequality by summing the inequalities

$$\sqrt{\frac{a}{b+c}} \ge \frac{2a}{a+b+c}, \qquad \sqrt{\frac{b}{c+a}} \ge \frac{2b}{a+b+c}, \qquad \sqrt{\frac{c}{a+b}} \ge \frac{2c}{a+b+c}.$$

□

P 1.79. *If a,b,c are positive real numbers such that $ab+bc+ca=3$, then*

(a) $$\frac{1}{(a+b)(3a+b)}+\frac{1}{(b+c)(3b+c)}+\frac{1}{(c+a)(3c+a)} \ge \frac{3}{8};$$

(b) $$\frac{1}{(2a+b)^2}+\frac{1}{(2b+c)^2}+\frac{1}{(2c+a)^2} \ge \frac{1}{3}.$$

(Vasile Cîrtoaje and Pham Kim Hung, 2007)

Solution. (a) Using the Cauchy-Schwarz inequality and the inequality in P 1.78-(a) gives

$$\sum \frac{1}{(a+b)(3a+b)} = \sum \frac{1}{(b+c)(3b+c)} \geq \frac{\left(\sum \sqrt{\frac{a}{3b+c}}\right)^2}{\sum a(b+c)} \geq \frac{9}{8(ab+bc+ca)} = \frac{3}{8}.$$

The equality holds for $a = b = c$.

(b) We consider two cases (*Vo Quoc Ba Can*).

Case 1: $4(ab+bc+ca \geq a^2+b^2+c^2$. By the Cauchy-Schwarz inequality, we get

$$\sum \frac{1}{(2a+b)^2} \geq \frac{9\left(\sum a\right)^2}{\sum (2a+b)^2(b+2c)^2}.$$

Thus, it suffices to show that

$$9p^2q \geq \sum (2a+b)^2(b+2c)^2,$$

where $p = a+b+c$, $q = ab+bc+ca$. Since

$$(2a+b)(b+2c) = pb + q + 3ac,$$

we have

$$\sum (2a+b)^2(b+2c)^2 = p^2 \sum a^2 + 3q^2 + 9\sum a^2b^2 + 2p^2q + 18abcp + 6q^2$$

$$= p^2(p^2-2q) + 9q^2 + 9(q^2 - 2abcp) + 2p^2q + 18abcp = p^4 + 18q^2,$$

and the inequality becomes

$$9p^2q \geq p^4 + 18q^2,$$

$$(p^2-3q)(6q-p^2) \geq 0.$$

The latter inequality is true since $p^2 - 3q \geq 0$ and

$$6q - p^2 = 4(ab+bc+ca) - a^2 - b^2 - c^2 \geq 0.$$

Case 2: $4(ab+bc+ca < a^2+b^2+c^2$. Assume that $a = \max\{a,b,c\}$. From

$$a^2 - 4(b+c)a + (b+c)^2 > 6bc > 0,$$

we get

$$a > (2+\sqrt{3})(b+c) > 2(b+c).$$

Since

$$\frac{1}{(2a+b)^2}+\frac{1}{(2b+c)^2}+\frac{1}{(2c+a)^2}>\frac{1}{(2b+c)^2}+\frac{1}{(2c+a)^2}\ge\frac{2}{(2b+c)(2c+a)},$$

it suffices to show that

$$\frac{2}{(2b+c)(2c+a)}\ge\frac{1}{ab+bc+ca}.$$

This is equivalent to the obvious inequality

$$c(a-2b-2c)\ge 0.$$

The proof is completed. The equality holds for $a=b=c$.

Open problem. *Let a,b,c be nonnegative real numbers, no two of which are zero. If $k>0$, then*

(a) $$\frac{1}{(a+b)(ka+b)}+\frac{1}{(b+c)(kb+c)}+\frac{1}{(c+a)(kc+a)}\ge\frac{9}{2(k+1)(ab+bc+ca)};$$

(b) $$\frac{1}{(ka+b)^2}+\frac{1}{(kb+c)^2}+\frac{1}{(kc+a)^2}\ge\frac{9}{(k+1)^2(ab+bc+ca)}.$$

For $k=1$, from (a) and (b), we get the well-known inequality (Iran 96):

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\ge\frac{9}{4(ab+bc+ca)}.$$

□

P 1.80. *If a,b,c are nonnegative real numbers, then*

$$a^4+b^4+c^4+15(a^3b+b^3c+c^3a)\ge\frac{47}{4}(a^2b^2+b^2c^2+c^2a^2).$$

(Vasile Cîrtoaje, 2011)

Solution. Without loss of generality, assume that $a=\min\{a,b,c\}$. There are two cases to consider: $a\le b\le c$ and $a\le c\le b$.

Case 1: $a\le b\le c$. For $a=0$, the inequality is true because is equivalent to

$$b^4+c^4+15b^3c-\frac{47}{4}b^2c^2\ge 0,$$

$$\left(b-\frac{c}{2}\right)^2(b^2+16bc+4c^2)\ge 0.$$

Based on this result, it suffices to prove that

$$a^4+15(a^3b+c^3a)\ge\frac{47}{4}a^2(b^2+c^2).$$

This inequality is true if

$$a^3b + c^3a \geq a^2(b^2 + c^2).$$

Indeed,

$$\begin{aligned}a^2b + c^3 - a(b^2 + c^2) &= c^2(c - a) - ab(b - a) \geq c^2(b - a) - ab(b - a)\\ &= (c^2 - ab)(b - a) \geq 0.\end{aligned}$$

Case 2: $a \leq c \leq b$. It suffices to show that

$$a^3b + b^3c + c^3a \geq a^2b^2 + b^2c^2 + c^2a^2.$$

Since

$$ab^3 + bc^3 + ca^3 - (a^3b + b^3c + c^3a) = (a + b + c)(a - b)(b - c)(c - a) \leq 0,$$

we have

$$\sum a^3b \geq \frac{1}{2}(\sum a^3b + \sum ab^3) = \frac{1}{2}\sum ab(a^2 + b^2) \geq \sum a^2b^2.$$

The equality holds for $a = 0$ and $2b = c$ (or any cyclic permutation).

□

P 1.81. *If a, b, c are nonnegative real numbers such that $a + b + c = 4$, then*

$$a^3b + b^3c + c^3a \leq 27.$$

Solution. Assume that $a = \max\{a, b, c\}$. There are two possible cases: $a \geq b \geq c$ and $a \geq c \geq b$.

Case 1: $a \geq b \geq c$. Using the AM-GM inequality gives

$$\begin{aligned}3(a^3b + b^3c + c^3a) &\leq 3ab(a^2 + ac + c^2) \leq 3ab(a + c)^2\\ &= a \cdot 3b \cdot (a + c) \cdot (a + c) \leq \left[\frac{a + 3b + (a + c) + (a + c)}{4}\right]^4\\ &= \left(\frac{3a + 3b + 2c}{4}\right)^4 \leq \left(\frac{3a + 3b + 3c}{4}\right)^4 = 81.\end{aligned}$$

Case 2: $a \geq c \geq b$. Since

$$ab^3 + bc^3 + ca^3 - (a^3b + b^3c + c^3a) = (a + b + c)(a - b)(b - c)(c - a) \geq 0,$$

it suffices to prove that

$$a^3b + b^3c + c^3a + (ab^3 + bc^3 + ca^3) \leq 54.$$

Indeed,

$$\begin{aligned}\sum a^3b+\sum ab^3 &\le (a^2+b^2+c^2)(ab+bc+ca)\\ &\le \frac{1}{8}[a^2+b^2+c^2+2(ab+bc+ca)]^2\\ &= \frac{1}{8}(a+b+c)^4 = 32 < 54.\end{aligned}$$

The equality holds for $a=3$, $b=1$ and $c=0$ (or any cyclic permutation).

Remark. The following sharper inequality holds (*Michael Rozenberg*).

- *If a,b,c are nonnegative real numbers such that $a+b+c=4$, then*

$$a^3b+b^3c+c^3a+\frac{473}{64}\,abc\le 27,$$

with equality for $a=b=c=4/3$, and also for $a=3$, $b=1$ and $c=0$ (or any cyclic permutation).

Write the inequality in the homogeneous form

$$27(a+b+c)^4\ge 256(a^3b+b^3c+c^3a)+473abc(a+b+c).$$

Assuming that $c=\min\{a,b,c\}$ and using the substitution

$$a=c+p,\qquad b=c+q,\qquad p,q\ge 0,$$

this inequality can be restated as

$$Ac^2+Bc+C\ge 0,$$

where

$$A=217(p^2-pq+q^2)\ge 0,$$

$$B=68p^3-269p^2q+499pq^2+68q^3\ge 60p(p^2-5pq+8q^2)\ge 0,$$

$$C=(p-3q)^2(27p^2+14pq+3q^2)\ge 0.$$

□

P 1.82. *Let a,b,c be nonnegative real numbers such that*

$$a^2+b^2+c^2=\frac{10}{3}(ab+bc+ca).$$

Prove that

$$a^4+b^4+c^4\ge\frac{82}{27}(a^3b+b^3c+c^3a).$$

(Vasile Cîrtoaje, 2011)

Solution *(by Vo Quoc Ba Can).* We see that the equality holds for $a = 3$, $b = 1$, $c = 0$. From

$$a^4 + b^4 + c^4 + 2(ab + bc + ca)^2 = (a^2 + b^2 + c^2)^2 + 4abc(a + b + c),$$

we get

$$a^4 + b^4 + c^4 \geq (a^2 + b^2 + c^2)^2 - 2(ab + bc + ca)^2$$
$$= \frac{82}{9}(ab + bc + ca)^2.$$

Therefore, it suffices to show that

$$3(ab + bc + ca)^2 \geq a^3b + b^3c + c^3a.$$

In addition, since

$$ab + bc + ca = \frac{3(a^2 + b^2 + c^2) + 6(ab + bc + ca)}{16} = 3\left(\frac{a + b + c}{4}\right)^2,$$

it suffices to show that

$$27\left(\frac{a + b + c}{4}\right)^4 \geq a^3b + b^3c + c^3a,$$

which is the inequality from the previous P 1.81. The equality holds for $a = 3b$ and $c = 0$ (or any cyclic permutation).

□

P 1.83. *If a, b, c are positive real numbers, then*

$$\frac{a^3}{2a^2 + b^2} + \frac{b^3}{2b^2 + c^2} + \frac{c^3}{2c^2 + a^2} \geq \frac{a + b + c}{3}.$$

(Vasile Cîrtoaje, 2005)

Solution. We write the inequality as

$$\left(\frac{a^3}{2a^2 + b^2} - \frac{a}{3}\right) + \left(\frac{b^3}{2b^2 + c^2} - \frac{b}{3}\right) + \left(\frac{c^3}{2c^2 + a^2} - \frac{c}{3}\right) \geq 0,$$

$$\frac{a(a^2 - b^2)}{2a^2 + b^2} + \frac{b(b^2 - c^2)}{2b^2 + c^2} + \frac{c(c^2 - a^2)}{2c^2 + a^2} \geq 0.$$

Taking into account that

$$\frac{a(a^2 - b^2)}{2a^2 + b^2} - \frac{b(a^2 - b^2)}{2b^2 + a^2} = \frac{(a + b)(a - b)^2(a^2 - ab + b^2)}{(2a^2 + b^2)(2b^2 + a^2)} \geq 0,$$

it suffices to show that

$$\frac{b(a^2 - b^2)}{2b^2 + a^2} + \frac{b(b^2 - c^2)}{2b^2 + c^2} + \frac{c(c^2 - a^2)}{2c^2 + a^2} \geq 0.$$

Since

$$\frac{b(a^2-b^2)}{2b^2+a^2}+\frac{b(b^2-c^2)}{2b^2+c^2}=\frac{3b^2(a^2-c^2)}{(2b^2+a^2)(2b^2+c^2)},$$

the last inequality is equivalent to

$$(c^2-a^2)(c-b)[a^2(3b^2+bc+c^2)+2b^2c(c-2b)]\ge 0. \tag{*}$$

Similarly, the desired inequality is true if

$$(a^2-b^2)(a-c)[b^2(3c^2+ca+a^2)+2c^2a(a-2c)]\ge 0. \tag{**}$$

Without loss of generality, assume that

$$c=\max\{a,b,c\}.$$

According to (*), the desired inequality is true if

$$a^2(3b^2+bc+c^2)+2b^2c(c-2b)\ge 0.$$

We claim that this inequality holds for $a\ge b$, and also for $2ac\ge\sqrt{3}\,b^2$. If $a\ge b$, then

$$\begin{aligned}a^2(3b^2+bc+c^2)+2b^2c(c-2b)&\ge b^2(3b^2+bc+c^2)+2b^2c(c-2b)\\&=3b^2[b^2+c(c-b)]>0;\end{aligned}$$

also, if $2ac\ge\sqrt{3}\,b^2$, then

$$\begin{aligned}a^2(3b^2+bc+c^2)+2b^2c(c-2b)&\ge\frac{3b^4}{4c^2}(3b^2+bc+c^2)+2b^2c(c-2b)\\&=\frac{b^2}{4c^2}(8c^4-16bc^3+3b^2c^2+3b^3c+9b^4)\\&=\frac{b^2}{4c^2}[2c(c+b)(2c-3b)^2+9b^2(c-b)^2+3b^3c]>0.\end{aligned}$$

Consequently, we only need to consider that $a<b\le c$ and $\sqrt{3}\,b^2>2ac$. According to (**), the desired inequality is true if

$$b^2(3c^2+ca+a^2)+2c^2a(a-2c)\ge 0.$$

We have

$$\begin{aligned}b^2(3c^2+ca+a^2)+2c^2a(a-2c)&>\frac{4ac}{3}(3c^2+ca+a^2)+2c^2a(a-2c)\\&=\frac{2a^2c(2a+5c)}{3}>0.\end{aligned}$$

This completes the proof. The equality occurs for $a=b=c$.

□

P 1.84. *If a, b, c are positive real numbers, then*

$$\frac{a^4}{a^3+b^3}+\frac{b^4}{b^3+c^3}+\frac{c^4}{c^3+a^3} \ge \frac{a+b+c}{2}.$$

(Vasile Cîrtoaje, 2005)

Solution *(by Vo Quoc Ba Can).* Multiplying by $a^3+b^3+c^3$, the inequality becomes

$$\sum a^4 + \sum \frac{a^4c^3}{a^3+b^3} \ge \frac{1}{2}\left(\sum a\right)\left(\sum a^3\right).$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a^4c^3}{a^3+b^3} \ge \frac{(\sum a^2c^2)^2}{\sum c(a^3+b^3)} = \frac{(\sum a^2b^2)^2}{\sum a(b^3+c^3)}.$$

According to the inequality

$$\frac{x^2}{y} \ge x - \frac{y}{4}, \quad x, y > 0,$$

we have

$$\frac{(\sum a^2b^2)^2}{\sum a(b^3+c^3)} \ge \sum a^2b^2 - \frac{1}{4}\sum a(b^3+c^3).$$

Therefore, it suffices to show that

$$\sum a^4 + \sum a^2b^2 - \frac{1}{4}\sum a(b^3+c^3) \ge \frac{1}{2}\left(\sum a\right)\left(\sum a^3\right),$$

which is equivalent to

$$2\sum a^4 + 4\sum a^2b^2 \ge 3\sum ab(a^2+b^2),$$

$$\sum [a^4+b^4+4a^2b^2-3ab(a^2+b^2)] \ge 0,$$

$$\sum (a-b)^2(a^2-ab+b^2) \ge 0.$$

This completes the proof. The equality occurs for $a=b=c$.

□

P 1.85. *If a, b, c are positive real numbers such that $abc = 1$, then*

(a) $$3\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)+4\left(\frac{b}{a^2}+\frac{c}{b^2}+\frac{a}{c^2}\right) \ge 7(a^2+b^2+c^2);$$

(b) $$8\left(\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}\right)+5\left(\frac{b}{a^3}+\frac{c}{b^3}+\frac{a}{c^3}\right) \ge 13(a^3+b^3+c^3).$$

(Vasile Cîrtoaje, 1992)

Solution. (a) We use the AM-GM inequality, as follows:

$$3\sum \frac{a^2}{b} + 4\sum \frac{b}{a^2} = \sum \left(3\frac{a^2}{b} + \frac{c}{b^2} + 3\frac{a}{c^2}\right) \ge 7\sum \sqrt[7]{\left(\frac{a^2}{b}\right)^3 \cdot \frac{c}{b^2} \cdot \left(\frac{a}{c^2}\right)^3}$$

$$= 7\sum \sqrt[7]{\frac{a^9}{b^5c^5}} = 7\sum a^2.$$

The equality holds for $a = b = c = 1$.

(b) By the AM-GM inequality, we have

$$8\sum \frac{a^3}{b} + 5\sum \frac{b}{a^3} = \sum \left(8\frac{a^3}{b} + \frac{c}{b^3} + 4\frac{a}{c^3}\right) \ge 13\sum \sqrt[13]{\left(\frac{a^3}{b}\right)^8 \cdot \frac{c}{b^3} \cdot \left(\frac{a}{c^3}\right)^4}$$

$$= 13\sum \sqrt[13]{\frac{a^{28}}{b^{11}c^{11}}} = 13\sum a^3.$$

The equality holds for $a = b = c = 1$.

□

P 1.86. *If a, b, c are positive real numbers, then*

$$\frac{ab}{b^2 + bc + c^2} + \frac{bc}{c^2 + ca + a^2} + \frac{ca}{a^2 + ab + b^2} \le \frac{a^2 + b^2 + c^2}{ab + bc + ca}.$$

(Tran Quoc Anh, 2007)

Solution. Write the inequality as follows:

$$\sum \left(\frac{a^2}{ab + bc + ca} - \frac{ab}{b^2 + bc + c^2}\right) \ge 0,$$

$$\sum \frac{ac(ac - b^2)}{b^2 + bc + c^2} \ge 0,$$

$$\sum \left[\frac{ac(ac - b^2)}{b^2 + bc + c^2} + ac\right] \ge \sum ac,$$

$$\sum \frac{ac^2(a + b + c)}{b^2 + bc + c^2} \ge \sum ac,$$

$$\sum \frac{ac^2}{b^2 + bc + c^2} \ge \frac{ab + bc + ca}{a + b + c}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{ac^2}{b^2 + bc + c^2} \ge \frac{\left(\sum ac\right)^2}{\sum a(b^2 + bc + c^2)} = \frac{ab + bc + ca}{a + b + c}.$$

The equality holds for $a = b = c$.

□

P 1.87. *If a, b, c are positive real numbers, then*

$$\frac{a-b}{b(2b+c)}+\frac{b-c}{c(2c+a)}+\frac{c-a}{a(2a+b)} \ge 0.$$

Solution. Write the inequality as follows:

$$\sum \frac{ac(a-b)}{2b+c} \ge 0,$$

$$\sum \left[\frac{ac(a-b)}{2b+c}+ac\right] \ge ab+bc+ca,$$

$$\sum \frac{ac}{2b+c} \ge \frac{ab+bc+ca}{a+b+c}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{ac}{2b+c} \ge \frac{\left(\sum ac\right)^2}{\sum ac(2b+c)} = \frac{\left(\sum ab\right)^2}{6abc+\sum a^2b}.$$

Thus, it suffices to prove that

$$\frac{\sum ab}{6abc+\sum a^2b} \ge \frac{1}{\sum a},$$

which is equivalent to

$$\sum ab^2 \ge 3abc.$$

Clearly, the latter inequality follows immediately from the AM-GM inequality. The equality holds for $a=b=c$.

□

P 1.88. *If a, b, c are positive real numbers, then*

(a)
$$\frac{a^2+6bc}{ab+2bc}+\frac{b^2+6ca}{bc+2ca}+\frac{c^2+6ab}{ca+2ab} \ge 7;$$

(b)
$$\frac{a^2+7bc}{ab+bc}+\frac{b^2+7ca}{bc+ca}+\frac{c^2+7ab}{ca+ab} \ge 12.$$

(Vasile Cîrtoaje, 2012)

Solution. (a) Write the inequality as follows:

$$\sum ac(a^2+6bc)(b+2a)(c+2b) \ge 7abc(a+2c)(b+2a)(c+2b),$$

$$2\sum a^2b^4 + abc\left(72abc+4\sum a^3+26\sum a^2b+7\sum ab^2\right) \ge$$

$$\geq 7abc\left(9abc + 4\sum a^2b + 2\sum ab^2\right),$$

$$2\left(\sum a^2b^4 - abc\sum a^2b\right) + abc\left(4\sum a^3 + 9abc - 7\sum ab^2\right) \geq 0.$$

Since

$$2\left(\sum a^2b^4 - abc\sum a^2b\right) = \sum(ab^2 - bc^2)^2 \geq 0,$$

it suffices to show that

$$4\sum a^3 + 9abc - 7\sum ab^2 \geq 0.$$

Assume that $a = \min\{a, b, c\}$. Using the substitution

$$b = a + x, \quad c = a + y, \quad x, y \geq 0,$$

we have

$$4\sum a^3 + 9abc - 7\sum ab^2 = 5(x^2 - xy + y^2)a + 4x^3 + 4y^3 - 7xy^2 \geq 0,$$

since

$$4x^3 + 4y^3 = 4x^3 + 2y^3 + 2y^3 \geq 3\sqrt[3]{4x^3 \cdot 2y^3 \cdot 2y^3} = 6\sqrt[3]{2}\; xy^2 \geq 7xy^2.$$

The equality holds for $a = b = c$.

(b) Write the inequality as follows:

$$\sum ac(a^2 + 7bc)(b + a)(c + b) \geq 12abc(a + c)(b + a)(c + b),$$

$$\sum a^2b^4 + abc\left(21abc + \sum a^3 + 15\sum a^2b + 8\sum ab^2\right) \geq$$

$$\geq 12abc\left(2abc + \sum a^2b + \sum ab^2\right),$$

$$\left(\sum a^2b^4 - abc\sum a^2b\right) + abc\left(\sum a^3 - 3abc + 4\sum a^2b - 4\sum ab^2\right) \geq 0.$$

Since

$$\sum a^2b^4 - abc\sum a^2b = \frac{1}{2}\sum(ab^2 - bc^2)^2 \geq 0,$$

it suffices to show that

$$\sum a^3 - 3abc + 4\sum a^2b - 4\sum ab^2 \geq 0,$$

which is equivalent to

$$\frac{1}{2}(a + b + c)\sum(a - b)^2 - 4(a - b)(b - c)(c - a) \geq 0.$$

Assume that $a = \min\{a, b, c\}$. Making the substitution

$$b = a + x, \quad c = a + y, \quad x, y \geq 0,$$

we have

$$\frac{1}{2}(a+b+c)\sum(a-b)^2 - 4(a-b)(b-c)(c-a) =$$

$$= (x^2-xy+y^2)(3a+x+y) + 4xy(x-y)$$

$$= 3(x^2-xy+y^2)a + x^3 + y^3 + 4xy(x-y)$$

$$= 3(x^2-xy+y^2)a + x^3 + y(2x-y)^2 \ge 0.$$

The equality holds for $a = b = c$.

□

P 1.89. *If a, b, c are positive real numbers, then*

(a) $$\frac{ab}{2b+c} + \frac{bc}{2c+a} + \frac{ca}{2a+b} \le \frac{a^2+b^2+c^2}{a+b+c};$$

(b) $$\frac{ab}{b+c} + \frac{bc}{c+a} + \frac{ca}{a+b} \le \frac{3(a^2+b^2+c^2)}{2(a+b+c)};$$

(c) $$\frac{ab}{4b+5c} + \frac{bc}{4c+5a} + \frac{ca}{4a+5b} \le \frac{a^2+b^2+c^2}{3(a+b+c)}.$$

(Vasile Cîrtoaje, 2012)

Solution. (a) ***First Solution.*** Since

$$\frac{2ab}{2b+c} = a - \frac{ac}{2b+c},$$

we can write the inequality as

$$\sum \frac{ac}{2b+c} + \frac{2(a^2+b^2+c^2)}{a+b+c} \ge a+b+c.$$

By the Cauchy-Schwarz inequality,

$$\sum \frac{ac}{2b+c} \ge \frac{\left(\sum\sqrt{ac}\right)^2}{\sum(2b+c)} = \frac{(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})^2}{3(a+b+c)}.$$

Then, it suffices to show that

$$\frac{(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})^2 + 6(a^2+b^2+c^2)}{3(a+b+c)} \ge a+b+c,$$

which is equivalent to

$$3(a^2+b^2+c^2) + 2\sqrt{abc}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right) \ge 5(ab+bc+ca).$$

Using the substitution
$$x=\sqrt{a},\quad y=\sqrt{b},\quad z=\sqrt{c},$$
the inequality can be restated as
$$3(x^4+y^4+z^4)+2xyz(x+y+z)\ge 5(x^2y^2+y^2z^2+z^2x^2).$$
We can get it by summing Schur's inequality of degree four
$$2(x^4+y^4+z^4)+2xyz(x+y+z)\ge 2\sum xy(x^2+y^2)$$
and
$$x^4+y^4+z^4+2\sum xy(x^2+y^2)\ge 5(x^2y^2+y^2z^2+z^2x^2),$$
the latter being equivalent to the obvious inequality
$$(x^4+y^4+z^4-x^2y^2-y^2z^2-z^2x^2)+2\sum xy(x-y)^2\ge 0.$$
The equality holds for $a=b=c$.

Second Solution. By the Cauchy-Schwarz inequality, we have
$$\frac{1}{2b+c}=\frac{1}{b+b+c}\le\frac{a^2/b+b+c}{(a+b+c)^2}=\frac{a^2+b^2+bc}{b(a+b+c)^2},$$
$$\frac{ab}{2b+c}\le\frac{a(a^2+b^2+bc)}{(a+b+c)^2},$$
$$\sum\frac{ab}{2b+c}\le\frac{\sum a^3+\sum ab^2+3abc}{(a+b+c)^2}.$$
Since $3abc\le\sum a^2b$ (by the AM-GM inequality), we get
$$\sum\frac{ab}{2b+c}\le\frac{\sum a^3+\sum ab^2+\sum a^2b}{(a+b+c)^2}=\frac{a^2+b^2+c^2}{a+b+c}.$$

Third Solution. Write the inequality as
$$\sum\frac{ab(a+b+c)}{2b+c}\le a^2+b^2+c^2.$$
Since
$$2ab(a+b+c)=(a^2+2ab)(2b+c)-2ab^2-a^2c,$$
we can write the inequality as
$$\sum\frac{2ab^2}{2b+c}+\sum\frac{a^2c}{2b+c}+p\ge 2q,$$

where

$$p = a^2 + b^2 + c^2, \quad q = ab + bc + ca, \qquad p \geq q.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{ab^2}{2b+c} \geq \frac{\left(\sum ab\right)^2}{\sum a(2b+c)} = \frac{q}{3}$$

and

$$\sum \frac{a^2c}{2b+c} \geq \frac{\left(\sum ac\right)^2}{\sum c(2b+c)} = \frac{q^2}{p+2q}.$$

Thus, it suffices to show that

$$\frac{2q}{3} + \frac{q^2}{p+2q} + p \geq 2q,$$

which is equivalent to the obvious inequality

$$(p-q)(3p+5q) \geq 0.$$

(b) Write the inequality as

$$\frac{3}{2}(a^2+b^2+c^2) \geq \sum \frac{ab(a+b+c)}{b+c}.$$

Since

$$\frac{ab(a+b+c)}{b+c} = \frac{a^2b}{b+c} + ab = a^2 + ab - \frac{a^2c}{b+c},$$

the inequality can be written as

$$\sum \frac{a^2c}{b+c} + \frac{1}{2}(a^2+b^2+c^2) \geq ab+bc+ca.$$

By the Cauchy-Schwarz inequality,

$$\sum \frac{a^2c}{b+c} \geq \frac{\left(\sum ac\right)^2}{\sum c(b+c)} = \frac{q^2}{p+q},$$

where

$$p = a^2 + b^2 + c^2, \quad q = ab + bc + ca, \qquad p \geq q.$$

Therefore, we have

$$\sum \frac{a^2c}{b+c} + \frac{1}{2}(a^2+b^2+c^2) - (ab+bc+ca) \geq \frac{q^2}{p+q} + \frac{p}{2} - q = \frac{p(p-q)}{2(p+q)} \geq 0.$$

The equality holds for $a = b = c$.

(c) Since

$$\frac{4ab}{4b+5c} = a - \frac{5ac}{4b+5c},$$

we can write the inequality as

$$5\sum \frac{ac}{4b+5c} + \frac{4(a^2+b^2+c^2)}{3(a+b+c)} \ge a+b+c.$$

By the Cauchy-Schwarz inequality,

$$\sum \frac{ac}{4b+5c} \ge \frac{(\sum ac)^2}{\sum ac(4b+5c)} = \frac{(ab+bc+ca)^2}{12abc+5(a^2b+b^2c+c^2a)}.$$

Therefore, it suffices to show that

$$\frac{5(ab+bc+ca)^2}{12abc+5(a^2b+b^2c+c^2a)} + \frac{4(a^2+b^2+c^2)}{3(a+b+c)} \ge a+b+c.$$

Due to homogeneity, we may assume that $a+b+c=3$. Using the notation

$$q = ab+bc+ca, \qquad q \le 3,$$

this inequality becomes

$$\frac{5q^2}{5(a^2b+b^2c+c^2a+abc)+7abc} + \frac{4(9-2q)}{9} \ge 3.$$

According to the inequality (a) in P 1.9, we have

$$a^2b+b^2c+c^2a+abc \le 4.$$

On the other hand, from

$$(ab+bc+ca)^2 \ge 3abc(a+b+c),$$

we get

$$abc \le \frac{q^2}{9}.$$

Thus, it suffices to prove that

$$\frac{5q^2}{20+7q^2/9} + \frac{4(9-2q)}{9} \ge 3,$$

which is equivalent to

$$(q-3)(14q^2-75q+135) \le 0.$$

This is true since $q-3 \le 0$ and

$$14q^2-75q+135 > 3(4q^2-25q+39) = 3(3-q)(13-4q) \ge 0.$$

The equality holds for $a=b=c$.

□

P 1.90. *If a, b, c are positive real numbers, then*

(a) $$a\sqrt{b^2+8c^2}+b\sqrt{c^2+8a^2}+c\sqrt{a^2+8b^2}\le (a+b+c)^2;$$

(b) $$a\sqrt{b^2+3c^2}+b\sqrt{c^2+3a^2}+c\sqrt{a^2+3b^2}\le a^2+b^2+c^2+ab+bc+ca.$$

(Vo Quoc Ba Can, 2007)

Solution. (a) By the AM-GM inequality, we have

$$\sqrt{b^2+8c^2}=\frac{\sqrt{(b^2+8c^2)(b+2c)^2}}{b+2c}\le\frac{(b^2+8c^2)+(b+2c)^2}{2(b+2c)}$$

$$=\frac{b^2+2bc+6c^2}{b+2c}=b+3c-\frac{3bc}{b+2c},$$

hence

$$a\sqrt{b^2+8c^2}\le ab+3ac-\frac{3abc}{b+2c},$$

$$\sum a\sqrt{b^2+8c^2}\le 4\sum ab-3abc\sum\frac{1}{b+2c}.$$

Therefore, it suffices to show that

$$\left(\sum a\right)^2+3abc\sum\frac{1}{b+2c}\ge 4\sum ab.$$

Since

$$\sum\frac{1}{b+2c}\ge\frac{9}{\sum(b+2c)}=\frac{3}{\sum a},$$

it is enough to prove that

$$\left(\sum a\right)^3+9abc\ge 4\left(\sum a\right)\left(\sum ab\right).$$

This is Shur's inequality of degree three. The equality holds for $a=b=c$.

(b) Similarly, we have

$$\sqrt{b^2+3c^2}=\frac{\sqrt{(b^2+3c^2)(b+c)^2}}{b+c}\le\frac{(b^2+3c^2)+(b+c)^2}{2(b+c)}$$

$$=\frac{b^2+bc+2c^2}{b+c}=b+2c-\frac{2bc}{b+c},$$

hence

$$a\sqrt{b^2+3c^2}\le ab+2ac-\frac{2abc}{b+c},$$

$$\sum a\sqrt{b^2+3c^2}\le 3\sum ab-2abc\sum\frac{1}{b+c}.$$

Thus, it suffices to show that

$$\left(\sum a\right)^2 + 2abc\sum \frac{1}{b+c} \ge 4\sum ab.$$

Since

$$\sum \frac{1}{b+c} \ge \frac{9}{\sum(b+c)} = \frac{9}{2\sum a},$$

it is enough to prove that

$$\left(\sum a\right)^3 + 9abc \ge 4\left(\sum a\right)\left(\sum ab\right),$$

which is just Shur's inequality of degree three. The equality holds for $a = b = c$.

□

P 1.91. *If a, b, c are positive real numbers, then*

(a)
$$\frac{1}{a\sqrt{a+2b}} + \frac{1}{b\sqrt{b+2c}} + \frac{1}{c\sqrt{c+2a}} \ge \sqrt{\frac{3}{abc}};$$

(b)
$$\frac{1}{a\sqrt{a+8b}} + \frac{1}{b\sqrt{b+8c}} + \frac{1}{c\sqrt{c+8a}} \ge \sqrt{\frac{1}{abc}}.$$

(Vasile Cîrtoaje, 2007)

Solution. (a) Write the inequality as

$$\sum \sqrt{\frac{bc}{3a(a+2b)}} \ge 1.$$

Replacing a, b, c by $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, respectively, the inequality can be restated as

$$\sum \frac{x}{\sqrt{3z(2x+y)}} \ge 1.$$

Since

$$\sqrt{3z(2x+y)} \le \frac{3z+(2x+y)}{2},$$

it suffices to show that

$$\sum \frac{x}{2x+y+3z} \ge \frac{1}{2}.$$

Indeed, using the Cauchy-Schwarz inequality gives

$$\sum \frac{x}{2x+y+3z} \ge \sum \frac{(\sum x)^2}{\sum x(2x+y+3z)} = \frac{1}{2}.$$

The equality holds for $a = b = c$.

(b) Write the inequality as

$$\sum \sqrt{\frac{bc}{a(a+8b)}} \ge 1.$$

Replacing a, b, c by $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$, respectively, the inequality becomes

$$\sum \frac{x^2}{z\sqrt{8x^2+y^2}} \ge 1.$$

Applying the Cauchy-Schwarz inequality yields

$$\sum \frac{x^2}{z\sqrt{8x^2+y^2}} \ge \frac{(\sum x)^2}{\sum z\sqrt{8x^2+y^2}}.$$

Therefore, it suffices to show that

$$\sum z\sqrt{8x^2+y^2} \le (x+y+z)^2,$$

which is just the inequality in P 1.90-(a). The equality holds for $a = b = c$.

□

P 1.92. *If a, b, c are positive real numbers, then*

$$\frac{a}{\sqrt{5a+4b}} + \frac{b}{\sqrt{5b+4c}} + \frac{c}{\sqrt{5c+4a}} \le \sqrt{\frac{a+b+c}{3}}.$$

(Vasile Cîrtoaje, 2012)

Solution. By the Cauchy-Schwarz inequality, we have

$$\left(\sum \frac{a}{\sqrt{5a+4b}}\right)^2 \le \left(\sum \frac{a}{4a+4b+c}\right)\left(\sum \frac{a(4a+4b+c)}{5a+4b}\right).$$

It suffices to show that

$$\sum \frac{a}{4a+4b+c} \le \frac{1}{3}$$

and

$$\sum \frac{a(4a+4b+c)}{5a+4b} \le a+b+c.$$

The first is just the inequality in P 1.18, while the second is equivalent to

$$\sum a\left(1 - \frac{4a+4b+c}{5a+4b}\right) \ge 0,$$

$$\sum \frac{a(a-c)}{5a+4b} \ge 0,$$

$$\sum a(a-c)(5b+4c)(5c+4a) \ge 0,$$

$$\sum a^2b^2 + 4\sum ab^3 \ge 5abc\sum a.$$

The last inequality follows from the well-known inequality

$$\sum a^2b^2 \ge abc\sum a$$

and the known inequality

$$\sum ab^3 \ge abc\sum a,$$

which follows from the Cauchy-Schwarz inequality, as follows:

$$\left(\sum c\right)\left(\sum ab^3\right) \ge \left(\sum \sqrt{ab^3c}\right)^2 = abc\left(\sum b\right)^2.$$

The equality holds for $a = b = c$.

□

P 1.93. *If a, b, c are positive real numbers, then*

(a)
$$\frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}} \ge \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{2}};$$

(b)
$$\frac{a}{\sqrt{a+b}} + \frac{b}{\sqrt{b+c}} + \frac{c}{\sqrt{c+a}} \ge \sqrt[4]{\frac{27(ab+bc+ca)}{4}}.$$

(Lev Buchovsky - 1995, Pham Huu Duc - 2007)

Solution. (a) By squaring, the inequality becomes

$$\sum \frac{a^2}{a+b} + 2\sum \frac{ab}{\sqrt{(a+b)(b+c)}} \ge \frac{1}{2}\sum a + \sum \sqrt{ab}.$$

The sequences

$$\left\{\frac{1}{\sqrt{a+b}}, \quad \frac{1}{\sqrt{b+c}}, \quad \frac{1}{\sqrt{c+a}}\right\}$$

and

$$\left\{\frac{ab}{\sqrt{a+b}}, \quad \frac{bc}{\sqrt{b+c}}, \quad \frac{ca}{\sqrt{c+a}}\right\}$$

are always reversely ordered; therefore, according to the rearrangement inequality, we have

$$\frac{1}{\sqrt{a+b}}\cdot\frac{ab}{\sqrt{a+b}} + \frac{1}{\sqrt{b+c}}\cdot\frac{bc}{\sqrt{b+c}} + \frac{1}{\sqrt{c+a}}\cdot\frac{ca}{\sqrt{c+a}} \le$$

$$\leq \frac{1}{\sqrt{a+b}} \cdot \frac{ca}{\sqrt{c+a}} + \frac{1}{\sqrt{b+c}} \cdot \frac{ab}{\sqrt{a+b}} + \frac{1}{\sqrt{c+a}} \cdot \frac{bc}{\sqrt{b+c}},$$

$$\sum \frac{ab}{a+b} \leq \sum \frac{ab}{\sqrt{(a+b)(b+c)}}.$$

Thus, it suffices to show that

$$\sum \frac{a^2}{a+b} + 2\sum \frac{ab}{a+b} \geq \frac{1}{2}\sum a + \sum \sqrt{ab}.$$

Since

$$\sum \frac{a^2}{a+b} + \sum \frac{ab}{a+b} = \sum a,$$

the inequality becomes as follows:

$$\sum a + \sum \frac{ab}{a+b} \geq \frac{1}{2}\sum a + \sum \sqrt{ab},$$

$$\sum \frac{a+b}{2} + \sum \frac{2ab}{a+b} \geq 2\sum \sqrt{ab},$$

$$\sum \left(\sqrt{\frac{a+b}{2}} - \sqrt{\frac{2ab}{a+b}} \right)^2 \geq 0.$$

The equality holds for $a = b = c$.

(b) By Hölder's inequality, we have

$$\left(\sum \frac{a}{\sqrt{a+b}} \right)^2 \sum a(a+b) \geq \left(\sum a \right)^3.$$

Thus, it suffices to show that

$$\left(\sum a \right)^3 \geq \frac{3}{2} \left(\sum a^2 + \sum ab \right) \sqrt{3(ab+bc+ca)},$$

which is equivalent to

$$2p^3 + q^3 \geq 3p^2 q,$$

where $p = a+b+c$ and $q = \sqrt{3(ab+bc+ca)}$. By the AM-GM inequality, we have

$$2p^3 + q^3 \geq 3\sqrt[3]{p^6 q^3} = 3p^2 q.$$

The equality holds for $a = b = c$.

□

P 1.94. *If a, b, c are nonnegative real numbers such that $a+b+c = 3$, then*

$$\sqrt{3a+b^2} + \sqrt{3b+c^2} + \sqrt{3c+a^2} \geq 6.$$

First Solution. Assume that $a = \max\{a,b,c\}$. We can get the desired inequality by summing the inequalities

$$\sqrt{3b+c^2}+\sqrt{3c+a^2} \ge \sqrt{3a+c^2}+b+c$$

and

$$\sqrt{3a+b^2}+\sqrt{3a+c^2} \ge 2a+b+c.$$

By squaring two times, the first inequality becomes in succession

$$\sqrt{(3b+c^2)(3c+a^2)} \ge (b+c)\sqrt{3a+c^2},$$

$$[b(a+b+c)+c^2][c(a+b+c)+a^2] \ge (b+c)^2[a(a+b+c)+c^2],$$

$$b(a-b)(a-c)(a+b+c) \ge 0.$$

Similarly, the second inequality becomes

$$\sqrt{(3a+b^2)(3a+c^2)} \ge (a+b)(a+c),$$

$$[a(a+b+c)+b^2][a(a+b+c)+c^2] \ge (a+b)^2(a+c)^2,$$

$$a(a+b+c)(b-c)^2 \ge 0.$$

The original inequality becomes an equality when $a=b=c$, and also when two of a,b,c are zero.

Second Solution. Write the inequality as

$$\sqrt{X}+\sqrt{Y}+\sqrt{Z} \le \sqrt{A}+\sqrt{B}+\sqrt{C},$$

where

$$X=(b+c)^2, \quad Y=(c+a)^2, \quad Z=(a+b)^2,$$

$$A=3a+b^2, \quad B=3b+c^2, \quad C=3c+a^2.$$

According to Lemma from the proof of P 2.11 in Volume 2, since

$$X+Y+Z=A+B+C,$$

it suffices to show that

$$\max\{X,Y,Z\} \ge \max\{A,B,C\}, \qquad \min\{X,Y,Z\} \le \min\{A,B,C\}.$$

To show that $\max\{X,Y,Z\} \ge \max\{A,B,C\}$, we assume that

$$a=\min\{a,b,c\}, \qquad \max\{X,Y,Z\}=X.$$

From

$$X-A=(c^2-a^2)+b(c-a)+c(b-a) \ge 0,$$

$$X-B=b(c-a) \ge 0,$$

$$X - C = (b^2 - a^2) + c(b - a) \geq 0,$$

the conclusion follows. Similarly, to show that $\min\{X, Y, Z\} \leq \min\{A, B, C\}$, we assume that

$$a = \max\{a, b, c\}, \qquad \min\{X, Y, Z\} = X,$$

when

$$A - X = (a^2 - c^2) + b(a - c) + c(a - b) \geq 0,$$

$$B - X = b(a - c) \geq 0,$$

$$C - X = (a^2 - b^2) + c(a - b) \geq 0.$$

□

P 1.95. *If a, b, c are nonnegative real numbers, then*

$$\sqrt{a^2 + b^2 + 2bc} + \sqrt{b^2 + c^2 + 2ca} + \sqrt{c^2 + a^2 + 2ab} \geq 2(a + b + c).$$

(Vasile Cîrtoaje, 2012)

First Solution *(by Nguyen Van Quy).* Assume that $a = \max\{a, b, c\}$. We can get the desired inequality by summing the inequalities

$$\sqrt{a^2 + b^2 + 2bc} + \sqrt{b^2 + c^2 + 2ca} \geq \sqrt{a^2 + b^2 + 2ca} + b + c$$

and

$$\sqrt{c^2 + a^2 + 2ab} + \sqrt{a^2 + b^2 + 2ca} \geq 2a + b + c.$$

By squaring two times, the first inequality becomes

$$\sqrt{(a^2 + b^2 + 2bc)(b^2 + c^2 + 2ca)} \geq (b + c)\sqrt{a^2 + b^2 + 2ca},$$

$$c(a - b)(a^2 - c^2) \geq 0.$$

Similarly, the second inequality becomes

$$\sqrt{(c^2 + a^2 + 2ab)(a^2 + b^2 + 2ca)} \geq (a + b)(a + c),$$

$$a(b + c)(b - c)^2 \geq 0.$$

The original inequality becomes an equality when $a = b = c$, and also when two of a, b, c are zero.

Second Solution. Let $\{x, y, z\}$ be a permutation of $\{ab, bc, ca\}$. We will prove that

$$2(a + b + c) \leq \sqrt{b^2 + c^2 + 2x} + \sqrt{c^2 + a^2 + 2y} + \sqrt{a^2 + b^2 + 2z}.$$

Due to symmetry, assume that $a \geq b \geq c$. Using the substitution

$$X = a^2 + b^2 + 2ab, \quad Y = c^2 + a^2 + 2ca, \quad Z = b^2 + c^2 + 2bc,$$

$$A = b^2 + c^2 + 2x, \quad B = c^2 + a^2 + 2y, \quad C = a^2 + b^2 + 2z,$$

we can write the inequality as

$$\sqrt{X} + \sqrt{Y} + \sqrt{Z} \le \sqrt{A} + \sqrt{B} + \sqrt{C}.$$

Since $X + Y + Z = A + B + C$, $X \ge Y \ge Z$ and

$$X \ge \max\{A, B, C\}, \quad Z \le \min\{A, B, C\},$$

the conclusion follow by Lemma from the proof of P 2.11 in Volume 2.

□

P 1.96. *If a, b, c are nonnegative real numbers, then*

$$\sqrt{a^2 + b^2 + 7bc} + \sqrt{b^2 + c^2 + 7ca} + \sqrt{c^2 + a^2 + 7ab} \ge 3\sqrt{3(ab + bc + ca)}.$$

(Vasile Cîrtoaje, 2012)

Solution. Assume that $a = \max\{a, b, c\}$. We can get the desired inequality by summing the inequalities

$$\sqrt{a^2 + b^2 + 7bc} + \sqrt{b^2 + c^2 + 7ca} \ge \sqrt{a^2 + b^2 + 7ca} + \sqrt{b^2 + c^2 + 7bc}$$

and

$$\sqrt{a^2 + c^2 + 7ab} + \sqrt{a^2 + b^2 + 7ac} \ge 3\sqrt{3(ab + bc + ca)} - \sqrt{b^2 + c^2 + 7bc}.$$

By squaring, the first inequality becomes

$$(a^2 + b^2 + 7b)(b^2 + c^2 + 7ca) \ge (a^2 + b^2 + 7ca)(b^2 + c^2 + 7bc),$$

$$c(a - b)(a^2 - c^2) \ge 0.$$

Similarly, the second inequality becomes

$$a^2 + \sqrt{E} + 3\sqrt{3F} \ge 10a(b + c) + 17bc,$$

where

$$\begin{aligned} E &= (a^2 + c^2 + 7ab)(a^2 + b^2 + 7ac) \\ &= a^4 + 7(b + c)a^3 + (b^2 + c^2 + 49bc)a^2 + 7(b^3 + c^3)a + b^2c^2 \end{aligned}$$

and

$$F = (ab + bc + ca)(b^2 + c^2 + 7bc).$$

Due to homogeneity, we may assume that $b + c = 1$. Let us denote $x = bc$. We need to show that $f(x) \ge 0$ for $0 \le x \le \dfrac{1}{4}$ and $a \ge \dfrac{1}{2}$, where

$$f(x) = a^2 - 10a - 17x + \sqrt{g(x)} + 3\sqrt{3h(x)},$$

with

$$g(x) = a^4 + 7a^3 + (1 + 47x)a^2 + 7(1 - 3x)a + x^2$$
$$= x^2 + a(47a - 21)x + a^4 + 7a^3 + a^2 + 7a,$$

$$h(x) = (a + x)(1 + 5x) = 5x^2 + (5a + 1)x + a.$$

We have the derivatives

$$f'(x) = -17 + \frac{g'}{2\sqrt{g}} + \frac{3\sqrt{3}h'}{2\sqrt{h}}$$
$$= -17 + \frac{2x + a(47a - 21)}{2\sqrt{g}} + \frac{3\sqrt{3}(10x + 5a + 1)}{2\sqrt{h}},$$

$$f''(x) = \frac{2g''g - (g')^2}{4g\sqrt{g}} + \frac{3\sqrt{3}[2h''h - (h')^2]}{4h\sqrt{h}}$$
$$= \frac{a(28 - 45a)(7a - 1)^2}{4g\sqrt{g}} - \frac{3\sqrt{3}(5a - 1)^2}{4h\sqrt{h}}.$$

We will show that $g \geq 3h$. Since $0 \leq x \leq \frac{1}{4}$ and $a \geq \frac{1}{2}$, we have

$$g - 3h = -14x^2 + (47a^2 - 36a - 3)x + a^4 + 7a^3 + a^2 + 4a$$
$$\geq -\frac{7}{8} + (47a^2 - 36a - 3)x + a^4 + 7a^3 + a^2 + 4a.$$

For the non-trivial case $47a^2 - 36a - 3 < 0$, we get

$$g - 3h \geq -\frac{7}{8} + \frac{47a^2 - 36a - 3}{4} + a^4 + 7a^3 + a^2 + 4a$$
$$= \frac{(2a - 1)(4a^3 + 30a^2 + 66a + 13)}{8} \geq 0.$$

We will prove now that $f''(x) < 0$. This is clearly true for $a \geq \frac{28}{45}$. Otherwise, for $\frac{1}{2} \leq a \leq \frac{28}{45}$, we have

$$f''(x) \leq \frac{a(28 - 45a)(7a - 1)^2 - 27(5a - 1)^2}{4g\sqrt{g}} < 0,$$

since

$$a(28 - 45a)(7a - 1)^2 - 27(5a - 1)^2 < \left(28 - \frac{45}{2}\right)(7a - 1)^2 - 27(5a - 1)^2$$

$$< \frac{27}{4}(7a - 1)^2 - 27(5a - 1)^2 = \frac{27(1 - 3a)(17a - 3)}{4} < 0.$$

Since f is concave, it suffices to show that $f(0) \ge 0$ and $f\left(\frac{1}{4}\right) \ge 0$.

From
$$f(0) = \sqrt{a}\left(a\sqrt{a} - 10\sqrt{a} + 3\sqrt{3} + \sqrt{a^3 + 7a^2 + a + 7}\right),$$
it follows that $f(0) \ge 0$ for all $a \ge \frac{1}{2}$ if and only if
$$\sqrt{a^3 + 7a^2 + a + 7} \ge -a\sqrt{a} + 10\sqrt{a} - 3\sqrt{3}.$$
This is true if
$$a^3 + 7a^2 + a + 7 \ge (-a\sqrt{a} + 10\sqrt{a} - 3\sqrt{3})^2,$$
which is equivalent to
$$(\sqrt{3a} - 2)^2(9a + 10\sqrt{a} - 5) \ge 0.$$
Clearly, this inequality holds for $a \ge \frac{1}{2}$.

Since
$$g\left(\frac{1}{4}\right) = \left(\frac{4a^2 + 14a + 1}{4}\right)^2$$
and
$$h\left(\frac{1}{4}\right) = \frac{9(4a+1)}{16},$$
we get
$$f\left(\frac{1}{4}\right) = \frac{8a^2 - 26a - 16 + 9\sqrt{3(4a+1)}}{4}.$$
Using the substitution
$$x = \sqrt{\frac{4a+1}{3}}, \quad x \ge 1,$$
we find
$$f\left(\frac{1}{4}\right) = \frac{9x^4 - 45x^2 + 54x - 18}{8} = \frac{(x-1)^2(9x^2 + 18x - 18)}{8} \ge 0.$$
Thus, the proof is completed. The equality holds for $a = b = c$, and also for $3a = 4b$ and $c = 0$ (or any cyclic permutation).

□

P 1.97. *If a, b, c are positive real numbers, then*
$$\frac{a^2 + 3ab}{(b+c)^2} + \frac{b^2 + 3bc}{(c+a)^2} + \frac{c^2 + 3ca}{(a+b)^2} \ge 3.$$

Solution. Write the inequality as

$$\sum \frac{a(a+b)}{(b+c)^2} + 2\sum \frac{ab}{(b+c)^2} \geq 3.$$

The sequences

$$\{bc, \quad ca, \quad ab\}$$

and

$$\left\{ \frac{1}{(b+c)^2}, \quad \frac{1}{(c+a)^2}, \quad \frac{1}{(a+b)^2} \right\}$$

are reversely ordered. Thus, by the rearrangement inequality, we have

$$\sum \frac{bc}{(b+c)^2} \leq \sum \frac{ab}{(b+c)^2}.$$

Therefore, it suffices to show that

$$\sum \frac{a(a+b)}{(b+c)^2} + \sum \frac{b(c+a)}{(b+c)^2} \geq 3,$$

which is equivalent to

$$\sum a \left[\frac{a+b}{(b+c)^2} + \sum \frac{b+c}{(a+b)^2} \right] \geq 3.$$

By the AM-GM inequality, we have

$$\frac{a+b}{(b+c)^2} + \frac{b+c}{(a+b)^2} \geq \frac{2}{\sqrt{(a+b)(b+c)}} \geq \frac{4}{(a+b)+(b+c)}.$$

Thus, it is enough to prove that

$$\sum \frac{a}{a+2b+c} \geq \frac{3}{4}.$$

Indeed, by the Cauchy-Schwarz inequality, we get

$$\sum \frac{a}{a+2b+c} \geq \frac{\left(\sum a\right)^2}{\sum a(a+2b+c)} = \frac{\sum a^2 + 2\sum ab}{\sum a^2 + 3\sum ab} \geq \frac{3}{4}.$$

The equality holds for $a = b = c$.

□

P 1.98. *If a, b, c are positive real numbers, then*

$$\frac{a^2b+1}{a(b+1)} + \frac{b^2c+1}{b(c+1)} + \frac{c^2a+1}{c(a+1)} \geq 3.$$

Solution. By the Cauchy-Schwarz inequality, we have

$$(a^2b+1)\left(\frac{1}{b}+1\right) \ge (a+1)^2,$$

hence

$$\frac{a^2b+1}{a(b+1)} \ge \frac{b(a+1)^2}{a(b+1)^2}.$$

Therefore, it suffices to prove that

$$\sum \frac{b(a+1)^2}{a(b+1)^2} \ge 3.$$

This inequality follows immediately from the AM-GM inequality:

$$\sum \frac{b(a+1)^2}{a(b+1)^2} \ge 3\sqrt[3]{\prod \frac{b(a+1)^2}{a(b+1)^2}} = 3.$$

The equality holds for $a=b=c=1$.

□

P 1.99. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$\sqrt{a^3+3b}+\sqrt{b^3+3c}+\sqrt{c^3+3a} \ge 6.$$

Solution. By the Cauchy-Schwarz inequality, we have

$$(a^3+3b)(a+3b) \ge (a^2+3b)^2.$$

Thus, it suffices to show that

$$\sum \frac{a^2+3b}{\sqrt{a+3b}} \ge 6.$$

By Hölder's inequality, we have

$$\left(\sum \frac{a^2+3b}{\sqrt{a+3b}}\right)^2 \left[\sum (a^2+3b)(a+3b)\right] \ge \left[\sum (a^2+3b)\right]^3 = \left(\sum a^2+9\right)^3.$$

Therefore, it is enough to show that

$$\left(\sum a^2+9\right)^3 \ge 36\sum (a^2+3b)(a+3b).$$

Let

$$p=a+b+c=3, \qquad q=ab+bc+ca, \qquad q \le 3.$$

We have

$$\sum a^2+9=p^2-2q+9=2(9-q),$$

$$\begin{aligned}\sum(a^2+3b)(a+3b) &= \sum a^3+3\sum a^2b+9\sum a^2+3\sum ab\\ &= (p^3-3pq+3abc)+3\sum a^2b+9(p^2-2q)+3q\\ &= 108-24q+3\left(abc+\sum a^2b\right).\end{aligned}$$

Since $abc+\sum a^2b\le 4$ (see the inequality (a) in P 1.9), we get

$$\sum(a^2+3b)(a+3b)\le 24(5-q).$$

Thus, it suffices to show that

$$(9-q)^3\ge 108(5-q),$$

which is equivalent to

$$(3-q)^2(21-q)\ge 0.$$

The equality holds for $a=b=c=1$.

□

P 1.100. *If a,b,c are positive real numbers such that $abc=1$, then*

$$\sqrt{\frac{a}{a+6b+2bc}}+\sqrt{\frac{b}{b+6c+2ca}}+\sqrt{\frac{c}{c+6a+2ab}}\ge 1.$$

(Nguyen Van Quy and Vasile Cîrtoaje, 2013)

Solution. By Hölder's inequality, we have

$$\left(\sum\sqrt{\frac{a}{a+6b+2bc}}\right)^2\left[\sum a(a+6b+2bc)\right]\ge\left(\sum a^{2/3}\right)^3.$$

Therefore, it suffices to show that

$$\left(\sum a^{2/3}\right)^3\ge\sum a^2+6\sum ab+6,$$

which is equivalent to

$$3\sum(ab)^{2/3}(a^{2/3}+b^{2/3})\ge 6\sum ab.$$

Since

$$a^{2/3}+b^{2/3}\ge 2(ab)^{1/3},$$

the desired conclusion follows. The equality holds for $a=b=c=1$.

□

P 1.101. *If a, b, c are positive real numbers such that $abc = 1$, then*

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 \ge 6(a+b+c-1).$$

(Marius Stanean, 2014)

Solution *(by Michael Rozenberg).* By the AM-GM inequality, we have

$$\begin{aligned}\sum\left(a+\frac{1}{b}\right)^2+6 &= \sum(a+ac)^2+6\\ &= \sum(a^2+a^2c^2+2a^2c)+6\\ &= \sum(a^2+a^2b^2+2a^2c+2)\\ &\ge 6\sum\sqrt[6]{a^2\cdot a^2b^2\cdot a^2c\cdot a^2c\cdot 1\cdot 1} = 6\sum a.\end{aligned}$$

The equality holds for $a = b = c = 1$.

□

P 1.102. *If a, b, c are positive real numbers, then*

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a} \ge \frac{a+b+c}{a+b+c-\sqrt[3]{abc}}.$$

(Michael Rozenberg, 2014)

Solution. There are two cases to consider.

Case 1: $ab+bc+ca \ge \sqrt[3]{abc}\,(a+b+c)$. By the Cauchy-Schwarz inequality, we have

$$\sum\frac{a}{a+b} \ge \frac{\left(\sum a\right)^2}{\sum a(a+b)} = \frac{(a+b+c)^2}{(a+b+c)^2-(ab+bc+ca)}.$$

Therefore, it suffices to show that

$$\frac{(a+b+c)^2}{(a+b+c)^2-(ab+bc+ca)} \ge \frac{a+b+c}{a+b+c-\sqrt[3]{abc}},$$

which is equivalent to

$$ab+bc+ca-\sqrt[3]{abc}\,(a+b+c) \ge 0.$$

Case 2: $\sqrt[3]{abc}\,(a+b+c) \ge ab+bc+ca$. By the Cauchy-Schwarz inequality, we have

$$\sum\frac{a}{a+b} \ge \frac{\left(\sum ac\right)^2}{\sum ac^2(a+b)} = \frac{(ab+bc+ca)^2}{(ab+bc+ca)^2-abc(a+b+c)}.$$

Thus, it suffices to show that

$$\frac{(ab+bc+ca)^2}{(ab+bc+ca)^2-abc(a+b+c)} \geq \frac{a+b+c}{a+b+c-\sqrt[3]{abc}},$$

which is equivalent to

$$\left[\sqrt[3]{abc}\,(a+b+c)\right]^2 \geq (ab+bc+ca)^2,$$

$$\sqrt[3]{abc}\,(a+b+c) \geq ab+bc+ca.$$

The proof is completed. The equality does not hold.

□

P 1.103. *If a, b, c are positive real numbers such that $a+b+c=3$, then*

$$a\sqrt{b^2+b+1}+b\sqrt{c^2+c+1}+c\sqrt{a^2+a+1} \leq 3\sqrt{3}.$$

(Nguyen Van Quy, 2014)

Solution. From

$$4(b^2+b+1) = 2(b+1)^2+2(b^2+1) \geq 3(b+1)^2,$$

we get

$$\sqrt{b^2+b+1} \geq \frac{\sqrt{3}}{2}(b+1),$$

hence

$$\sum a\sqrt{b^2+b+1} = \sum \frac{a(b^2+b+1)}{\sqrt{b^2+b+1}} \leq \sum \frac{2a(b^2+b+1)}{\sqrt{3}(b+1)}.$$

Therefore, it suffices to prove that

$$\sum \frac{a(b^2+b+1)}{b+1} \leq \frac{9}{2},$$

which is equivalent to

$$\sum \frac{ab^2}{b+1} \leq \frac{3}{2}.$$

In addition, since $b+1 \geq 2\sqrt{b}$, it is enough to show that

$$\sum ab^{3/2} \leq 3.$$

Replacing a, b, c by a^2, b^2, c^2, respectively, we need to show that $a^2+b^2+c^2=3$ involves $a^2b^3+b^2c^3+c^2a^3 \leq 3$, which is just the inequality in P 1.7. The equality holds for $a=b=c$.

□

P 1.104. *If a, b, c are positive real numbers, then*

$$\frac{1}{b(a+2b+3c)^2}+\frac{1}{c(b+2c+3a)^2}+\frac{1}{a(c+2a+3b)^2} \le \frac{1}{12abc}.$$

(Vo Quoc Ba Can, 2012)

Solution. Assume that $a = \max\{a, b, c\}$, and write the inequality as

$$\frac{ca}{(a+2b+3c)^2}+\frac{ab}{(b+2c+3a)^2}+\frac{bc}{(c+2a+3b)^2} \le \frac{1}{12}.$$

Case 1: $a \ge b \ge c$. By the AM-GM inequality, we have

$$(a+2b+3c)^2 \ge 4(2b+c)(2c+a);$$

thus, it suffices to show that

$$\sum \frac{ca}{(2b+c)(2c+a)} \le \frac{1}{3},$$

which is equivalent to

$$3\sum ca(2a+b) \le (2a+b)(2b+c)(2c+a),$$

$$ab^2+bc^2+ca^2 \le a^2b+b^2c+c^2a,$$

$$(a-b)(b-c)(c-a) \le 0.$$

Clearly, the last inequality is true.

Case 2: $a \ge c \ge b$. Since, by the AM-GM inequality,

$$(a+2b+3c)^2 \ge 12c(a+2b),$$

$$(b+2c+3a)^2 \ge 4(2a+b)(2c+a),$$

$$(c+2a+3b)^2 \ge 4(a+2b)(a+b+c),$$

it suffices to show that

$$\frac{a}{3(a+2b)}+\frac{ab}{(2a+b)(2c+a)}+\frac{bc}{(a+2b)(a+b+c)} \le \frac{1}{3},$$

which is equivalent to

$$\frac{ab}{(2a+b)(2c+a)}+\frac{bc}{(a+2b)(a+b+c)} \le \frac{2b}{3(a+2b)},$$

$$\frac{a}{(2a+b)(2c+a)}+\frac{c}{(a+2b)(a+b+c)} \le \frac{2}{3(a+2b)},$$

$$\frac{a(a+2b)}{(2a+b)(2c+a)} + \frac{c}{a+b+c} \le \frac{2}{3},$$

$$\frac{a(a+2b)}{2a+b} + \frac{c(2c+a)}{a+b+c} \le \frac{2(2c+a)}{3},$$

$$\frac{c(2c+a)}{a+b+c} - \frac{2(2c+a)}{3} \le \frac{3a^2}{2a+b} - 2a,$$

$$f(c) \le f(a),$$

where

$$f(x) = \frac{x(2x+a)}{a+b+x} - \frac{2(2x+a)}{3}.$$

We have

$$\begin{aligned} f(a) - f(c) =& (a-c)\left[\frac{3a^2+4ac+b(3a+2c)}{(a+b+c)(2a+b)} - \frac{4}{3}\right] \\ =& \frac{(a-c)[a^2-3ab-4b^2+2c(2a+b)]}{3(a+b+c)(2a+b)} \ge 0, \end{aligned}$$

because

$$a^2 - 3ab - 4b^2 + 2c(2a+b) \ge a^2 - 3ab - 4b^2 + 2b(2a+b) = (a-b)(a+2b) \ge 0.$$

The equality holds for $a = b = c$.

□

P 1.105. *Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that*

(a) $$\frac{a^2+9b}{b+c} + \frac{b^2+9c}{c+a} + \frac{c^2+9a}{a+b} \ge 15;$$

(b) $$\frac{a^2+3b}{a+b} + \frac{b^2+3c}{b+c} + \frac{c^2+3a}{c+a} \ge 6.$$

Solution. (a) Write the inequality as follows:

$$\sum \frac{a^2+3b(a+b+c)}{b+c} \ge 5(a+b+c),$$

$$\sum \left[\frac{a^2+3b(a+b+c)}{b+c} - 3b\right] \ge 2(a+b+c),$$

$$\sum \frac{a^2+3ab}{b+c} \ge 2(a+b+c),$$

$$\sum \left(\frac{a^2+3ab}{b+c} - 2a\right) \ge 0,$$

$$\sum \frac{a(a+b-2c)}{b+c} \ge 0,$$

$$\sum \frac{a(a-c)}{b+c} + \sum \frac{a(b-c)}{b+c} \ge 0,$$

$$\sum \frac{a(a-c)}{b+c} + \sum \frac{b(c-a)}{c+a} \ge 0,$$

$$\sum (a-c)\left(\frac{a}{b+c} - \frac{b}{c+a}\right) \ge 0,$$

$$(a+b+c)\sum \frac{(a-b)(a-c)}{(b+c)(c+a)} \ge 0.$$

Therefore, we need to show that

$$\sum (a^2-b^2)(a-c) \ge 0,$$

which is equivalent to the obvious inequality

$$\sum a(a-c)^2 \ge 0.$$

The equality holds for $a=b=c$.

(b) Write the inequality as follows:

$$\sum \frac{a^2+b(a+b+c)}{a+b} \ge 2(a+b+c),$$

$$\sum \frac{a^2+bc}{a+b} \ge a+b+c,$$

$$\sum \left(\frac{a^2+bc}{a+b} - a\right) \ge 0,$$

$$\sum \frac{b(c-a)}{a+b} \ge 0,$$

$$\sum \frac{bc}{a+b} \ge \sum \frac{ab}{a+b}.$$

Since the sequences

$$\{ab, \quad bc, \quad ca\}$$

and

$$\left\{\frac{1}{a+b}, \quad \frac{1}{b+c}, \quad \frac{1}{c+a}\right\}$$

are reversely ordered, the inequality follows from the rearrangement inequality. The equality holds for $a=b=c$.

□

P 1.106. *If $a, b, c \in [0, 1]$, then*

(a)
$$\frac{bc}{2ab+1}+\frac{ca}{2bc+1}+\frac{ab}{2ca+1}\le 1.$$

(b)
$$\frac{a}{ab+1}+\frac{b}{bc+1}+\frac{c}{ca+1}\le \frac{3}{2}.$$

(Vasile Cîrtoaje, 2010)

Solution. (a) ***First Solution.*** It suffices to prove that

$$\frac{bc}{2abc+1}+\frac{ca}{2abc+1}+\frac{ab}{2abc+1}\le 1;$$

that is,

$$2abc+1\ge ab+bc+ca,$$

$$1-bc\ge a(b+c-2bc).$$

Since $a\le 1$ and

$$b+c-2bc=b(1-c)+c(1-b)\ge 0,$$

it suffices to show that

$$1-bc\ge b+c-2bc,$$

which is equivalent to

$$(1-b)(1-c)\ge 0.$$

The equality holds for $a=b=c=1$, and for $a=0$ and $b=c=1$ (or any cyclic permutation).

Second Solution. Assume that $a=\max\{a,b,c\}$. It suffices to show that

$$\frac{bc}{2bc+1}+\frac{ca}{2bc+1}+\frac{ab}{2bc+1}\le 1;$$

that is,

$$a(b+c)\le 1+bc.$$

We have

$$1+bc-a(b+c)\ge 1+bc-(b+c)=(1-b)(1-c)\ge 0.$$

(b) We will show that

$$E(a,b,c)\le E(1,b,c)\le E(1,1,c)=\frac{3}{2},$$

where

$$E(a,b,c)=\frac{a}{ab+1}+\frac{b}{bc+1}+\frac{c}{ca+1}.$$

Write the inequality $E(a,b,c) \le E(1,b,c)$ as follows:

$$\frac{a}{ab+1}+\frac{c}{ca+1} \le \frac{1}{b+1}+\frac{c}{c+1},$$

$$(1-a)\left[\frac{1}{(b+1)(ab+1)}-\frac{c^2}{(c+1)(ca+1)}\right] \ge 0,$$

$$(1-a)[(c+1)(ca+1)-(b+1)(ab+1)c^2] \ge 0.$$

Since $1-a \ge 0$ and $c \le 1$, it suffices to show that

$$(c+1)(ca+1)-(b+1)(ab+1)c \ge 0,$$

which is true because

$$\begin{aligned}(c+1)(ca+1)-(b+1)(ab+1)c &\ge (c+1)(ca+1)-2(a+1)c \\ &= (1-c)(1-ac) \ge 0.\end{aligned}$$

Setting $a=1$ in the similar inequality

$$E(a,b,c) \le E(a,1,c),$$

it follows that

$$E(1,b,c) \le E(1,1,c).$$

Finally,

$$E(1,1,c)=\frac{1}{2}+\frac{1}{c+1}+\frac{c}{c+1}=\frac{3}{2}.$$

The equality holds for $a=b=1$ (or any cyclic permutation).

□

P 1.107. *If a,b,c are nonnegative real numbers, then*

$$a^4+b^4+c^4+5(a^3b+b^3c+c^3a) \ge 6(a^2b^2+b^2c^2+c^2a^2).$$

Solution. Assume that $a=\min\{a,b,c\}$ and use the substitution

$$b=a+p, \quad c=a+q, \qquad p,q \ge 0.$$

The inequality becomes

$$9Aa^2+3Ba+C \ge 0,$$

where

$$A=p^2-pq+q^2, \quad B=3p^3+p^2q-4pq^2+3q^3,$$
$$C=p^4+5p^3q-6p^2q^2+q^4.$$

Since

$$A \geq 0,$$

$$B = 3p(p-q)^2 + q(7p^2 - 7pq + 3q^2) \geq 0,$$

$$C = (p-q)^4 + pq(3p-2q)^2 \geq 0,$$

the inequality is obviously true. The equality occurs for $a = b = c$.

□

P 1.108. *If a, b, c are positive real numbers, then*

$$a^5 + b^5 + c^5 - a^4b - b^4c - c^4a \geq 2abc(a^2 + b^2 + c^2 - ab - bc - ca).$$

(Vasile Cîrtoaje, 2006)

Solution. Since

$$5\left(\sum a^5 - \sum a^4b\right) = \sum(4a^5 + b^5 - 5a^4b) = \sum(a-b)^2(4a^3 + 3a^2b + 2ab^2 + b^3)$$

and

$$2\left(\sum a^2 - \sum ab\right) = \sum(a-b)^2,$$

we can write the inequality in the form

$$A(a-b)^2 + B(b-c)^2 + C(c-a)^2 \geq 0,$$

where

$$A = 4a^3 + 3a^2b + 2ab^2 + b^3 - 5abc,$$

$$B = 4b^3 + 3b^2c + 2bc^2 + c^3 - 5abc,$$

$$C = 4c^3 + 3c^2a + 2ca^2 + a^3 - 5abc.$$

Without loss of generality, assume that $a = \max\{a, b, c\}$. We have

$$A > a(4a^2 + 3ab - 5bc) > a(4c^2 + 3b^2 - 5bc) > 0,$$

$$C > a(3c^2 + 2ca + a^2 - 5bc) > a(3c^2 - 3ca + a^2) > 0,$$

$$\begin{aligned} A + B &> 4a^3 + 5b^3 + c^3 + 3a^2b + 2bc^2 - 10abc \\ &\geq 3\sqrt[3]{4a^3 \cdot 5b^3 \cdot c^3} + 2\sqrt{3a^2b \cdot 2bc^2} - 10abc \\ &= (3\sqrt[3]{20} + 2\sqrt{6} - 10)abc > 0, \end{aligned}$$

$$\begin{aligned} B + C &> a^3 + 4b^3 + 5c^3 + 3b^2c + 2ca^2 - 10abc \\ &\geq 3\sqrt[3]{a^3 \cdot 4b^3 \cdot 5c^3} + 2\sqrt{3b^2c \cdot 2ca^2} - 10abc \\ &= (3\sqrt[3]{20} + 2\sqrt{6} - 10)abc > 0. \end{aligned}$$

If $a \ge b \ge c$, then

$$\sum A(a-b)^2 \ge B(b-c)^2 + C(a-c)^2 \ge (B+C)(b-c)^2 \ge 0.$$

If $a \ge c \ge b$, then

$$\sum A(a-b)^2 \ge A(a-b)^2 + B(c-b)^2 \ge (A+B)(c-b)^2 \ge 0.$$

The equality holds for $a = b = c$.

□

P 1.109. *If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3$, then*

$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+a} \ge \frac{3}{2}.$$

(Vasile Cîrtoaje, 2005)

Solution. Let

$$p = a+b+c, \quad q = ab+bc+ca, \quad p^2 = 3+2q.$$

First Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a}{1+b} \ge \frac{\left(\sum a\right)^2}{\sum a(1+b)} = \frac{3+2q}{p+q}.$$

Thus, it suffices to prove that

$$6 + q \ge 3p.$$

Indeed,

$$2(6+q-3p) = 12 + (p^2-3) - 6p = (p-3)^2 \ge 0.$$

The equality holds for $a = b = c = 1$.

Second Solution. By the AM-GM inequality, we have

$$\sum \frac{a}{1+b} = \sum \frac{a(a+c)}{(1+b)(a+c)} \ge \sum \frac{4a(a+c)}{[(1+b)+(a+c)]^2}$$
$$= \frac{4\left(\sum a^2 + \sum ac\right)}{(1+p)^2} = \frac{4(3+q)}{(1+p)^2} = \frac{6+2p^2}{(1+p)^2}.$$

Therefore, it suffices to show that

$$\frac{6+2p^2}{(1+p)^2} \ge \frac{3}{2},$$

which is equivalent to $(p-3)^2 \ge 0$.

Open problem. *If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3$, then*

$$\frac{a}{5+4b} + \frac{b}{5+4c} + \frac{c}{5+4a} \ge \frac{1}{3}.$$

□

P 1.110. *If a, b, c are nonnegative real numbers such that $a+b+c=3$, then*

$$a\sqrt{a+b}+b\sqrt{b+c}+c\sqrt{c+a} \ge 3\sqrt{2}.$$

(Hong Ge Chen, 2011)

First Solution. Denote

$$q=\sqrt{\frac{ab+bc+ca}{3}}, \qquad q \le 1.$$

By squaring, the inequality turns into

$$\sum a^3+\sum a^2b+2\sum ac\sqrt{a^2+3q^2} \ge 18.$$

Since

$$2\sqrt{a^2+3q^2} \ge a+3q,$$

we have

$$2\sum ac\sqrt{a^2+3q^2} \ge \sum ac(a+3q)=\sum ab^2+9q^3.$$

Thus, it suffices to show that

$$\sum a^3+\sum ab(a+b)+9q^3 \ge 18,$$

which is equivalent to

$$(a+b+c)(a^2+b^2+c^2)+9q^3 \ge 18,$$

$$3(9-6q^2)+9q^3 \ge 0,$$

$$1-2q^2+q^3 \ge 0,$$

$$(1-q^2)^2+q^3(1-q) \ge 0.$$

Clearly, the last inequality is true. The equality holds for $a=b=c=1$.

Second Solution. Using the substitution

$$\sqrt{\frac{a+b}{2}}=\frac{x+y}{2}, \quad \sqrt{\frac{b+c}{2}}=\frac{y+z}{2}, \quad \sqrt{\frac{c+a}{2}}=\frac{z+x}{2}$$

gives

$$x=\sqrt{\frac{a+b}{2}}+\sqrt{\frac{a+c}{2}}-\sqrt{\frac{b+c}{2}} \ge 0,$$

$$a=\left(\frac{x+y}{2}\right)^2+\left(\frac{x+z}{2}\right)^2-\left(\frac{y+z}{2}\right)^2=\frac{x(x+y+z)-yz}{2}.$$

In addition, $a+b+c=3$ involves

$$x^2+y^2+z^2+xy+yz+zx=6,$$

which is equivalent to

$$p^2 - q = 6,$$

where

$$p = x + y + z, \qquad q = xy + yz + zx.$$

From

$$\begin{aligned} 18 - 2p^2 &= 3(x^2 + y^2 + z^2 + xy + yz + zx) - 2(x + y + z)^2 \\ &= x^2 + y^2 + z^2 - xy - yz - zx \ge 0, \end{aligned}$$

it follows that

$$p \le 3.$$

The desired inequality is equivalent to

$$\sum (xp - yz)(x + y) \ge 12,$$

$$p \sum (x^2 + xy) \ge 3xyz + \sum y^2 z + 12,$$

$$6p \ge 3xyz + \sum y^2 z + 12,$$

$$6p + \sum yz^2 \ge pq + 12.$$

Since

$$\left(\sum yz^2\right)\left(\sum y\right) \ge \left(\sum yz\right)^2$$

(by the Cauchy-Schwarz inequality), it suffices to show that

$$6p + \frac{q^2}{p} \ge pq + 12.$$

Indeed,

$$6p + \frac{q^2}{p} - pq = \frac{p^2(6 - q) + q^2}{p} = \frac{(6 + q)(6 - q) + q^2}{p} = \frac{36}{p} \ge 12.$$

Open problem. *If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then*

$$a\sqrt{4a + 5b} + b\sqrt{4b + 5c} + c\sqrt{4c + 5a} \ge 9.$$

□

P 1.111. *If a, b, c are positive real numbers such that $a + b + c = 3$, then*

$$\frac{a}{2b^2 + c} + \frac{b}{2c^2 + a} + \frac{c}{2a^2 + b} \ge 1.$$

(Vasile Cîrtoaje and Nguyen Van Quy, 2007)

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a}{2b^2+c} \ge \frac{\left(\sum a\sqrt{a+c}\right)^2}{\sum a(a+c)(2b^2+c)}.$$

Since $\sum a\sqrt{a+c} \ge 3\sqrt{2}$ (see the previous P 1.110), it suffices to prove that

$$\sum a(a+c)(2b^2+c) \le 18,$$

which is equivalent to

$$2\sum a^2b^2 + 6abc + \sum ac(a+c) \le 18,$$

$$2\sum a^2b^2 + 3abc + \left(\sum a\right)\left(\sum ab\right) \le 18.$$

Denoting

$$q = ab + bc + ca,$$

the inequality becomes

$$9abc + 18 \ge 2q^2 + 3q.$$

This inequality is true for $q < 2$ because $18 > 2q^2 + 3q$. Since $q \le p^2/3 = 3$, consider further the case $2 \le q \le 3$. By Schur's inequality of degree three, we have

$$9abc \ge 4pq - p^3 = 12q - 27.$$

Therefore,

$$\begin{aligned} 9abc + 18 - (2q^2 + 3q) &\ge (12q - 27) + 18 - (2q^2 + 3q) \\ &= -2q^2 + 9q - 9 = (3-q)(2q-3) \ge 0. \end{aligned}$$

This completes the proof. The equality holds for $a = b = c = 1$.

□

P 1.112. *If a, b, c are positive real numbers such that $a + b + c = ab + bc + ca$, then*

$$\frac{1}{a^2+b+1} + \frac{1}{b^2+c+1} + \frac{1}{c^2+a+1} \le 1.$$

Solution. By the Cauchy-Schwarz inequality, we have

$$\frac{1}{a^2+b+1} \le \frac{1+b+c^2}{(a+b+c)^2},$$

hence

$$\sum \frac{1}{a^2+b+1} \le \sum \frac{1+b+c^2}{(a+b+c)^2} = \frac{3+a+b+c+a^2+b^2+c^2}{(a+b+c)^2}.$$

It suffices to show that

$$3 + a + b + c \le 2(ab + bc + ca),$$

which is equivalent to

$$a + b + c \ge 3.$$

We can get this inequality from the known inequality

$$(a + b + c)^2 \ge 3(ab + bc + ca).$$

The equality holds for $a = b = c = 1$.

□

P 1.113. *If a, b, c are positive real numbers, then*

$$\frac{1}{(a+2b+3c)^2} + \frac{1}{(b+2c+3a)^2} + \frac{1}{(c+2a+3b)^2} \le \frac{1}{4(ab+bc+ca)}.$$

Solution. By the AM-GM inequality, we have

$$(a + 2b + 3c)^2 = [(a + c) + 2(b + c)]^2 = (a + c)^2 + 4(b + c)^2 + 4(a + c)(b + c)$$
$$\ge 3(b + c)^2 + 6(a + c)(b + c) = 3(b + c)(2a + b + 3c).$$

Thus, it suffices to show that

$$\sum \frac{1}{(b+c)(2a+b+3c)} \le \frac{3}{4(ab+bc+ca)}.$$

Write this inequality as follows:

$$\frac{3}{4} - \sum \frac{ab+bc+ca}{(b+c)(2a+b+3c)} \ge 0,$$

$$\sum \left[1 - \frac{2(ab+bc+ca)}{(b+c)(2a+b+3c)}\right] \ge \frac{3}{2},$$

$$\sum \frac{(b+c)^2 + 2c^2}{(b+c)(2a+b+3c)} \ge \frac{3}{2},$$

$$\sum \frac{b+c}{2a+b+3c} + \sum \frac{2c^2}{(b+c)(2a+b+3c)} \ge \frac{3}{2}.$$

Applying the Cauchy-Schwarz inequality, we get

$$\sum \frac{b+c}{2a+b+3c} \ge \frac{[\sum(b+c)]^2}{\sum (b+c)(2a+b+3c)} = \frac{4\left(\sum a\right)^2}{4\left(\sum a\right)^2} = 1$$

and

$$\sum \frac{c^2}{(b+c)(2a+b+3c)} \ge \frac{\left(\sum c\right)^2}{\sum (b+c)(2a+b+3c)} = \frac{1}{4},$$

from where the conclusion follows. The equality holds for $a = b = c$.

□

P 1.114. *If a, b, c are positive real numbers, then*

$$\sqrt{\frac{a}{a+b+2c}}+\sqrt{\frac{b}{b+c+2a}}+\sqrt{\frac{c}{c+a+2b}} \le \frac{3}{2}.$$

Solution. Apply the Cauchy-Schwarz inequality as follows:

$$\left(\sum\sqrt{\frac{a}{a+b+2c}}\right)^2 \le \left[\sum(b+c+2a)\right]\left[\sum\frac{a}{(b+c+2a)(a+b+2c)}\right]$$
$$= \frac{4\left(\sum a\right)\left[\sum a(c+a+2b)\right]}{(b+c+2a)(c+a+2b)(a+b+2c)}.$$

Thus, it suffices to show that

$$16\left(\sum a\right)\left[\sum a(c+a+2b)\right] \le 9(b+c+2a)(c+a+2b)(a+b+2c).$$

Denoting

$$p=a+b+c, \qquad q=ab+bc+ca,$$

the inequality becomes

$$16p(p^2+q) \le 9(p+a)(p+b)(p+c),$$

$$16p(p^2+q) \le 9(2p^3+pq+abc),$$

$$2p^3-7pq+9abc \ge 0.$$

Using Schur's inequality of degree three

$$p^3+9abc \ge 4pq,$$

we have

$$2p^3-7pq+9abc = (p^3+9abc-4pq)+p(p^2-3q) \ge 0.$$

The equality holds for $a=b=c$.

□

P 1.115. *If a, b, c are positive real numbers, then*

$$\sqrt{\frac{5a}{a+b+3c}}+\sqrt{\frac{5b}{b+c+3a}}+\sqrt{\frac{5c}{c+a+3b}} \le 3.$$

Solution. Substituting

$$x=\sqrt{\frac{5a}{a+b+3c}},\quad y=\sqrt{\frac{5b}{b+c+3a}},\quad z=\sqrt{\frac{5c}{c+a+3b}},$$

we have

$$\begin{cases}(x^2-5)a+x^2b+3x^2c=0\\ 3y^2a+(y^2-5)b+y^2c=0\\ z^2a+3z^2b+(z^2-5)c=0\end{cases},$$

which involves

$$\begin{vmatrix} x^2-5 & x^2 & 3x^2\\ 3y^2 & y^2-5 & y^2\\ z^2 & 3z^2 & z^2-5\end{vmatrix}=0\ ;$$

that is,

$$F(x,y,z)=0,$$

where

$$F(x,y,z)=4x^2y^2z^2+2\sum x^2y^2+5\sum x^2-25.$$

We need to show that $F(x,y,z)=0$ involves $x+y+z\le 3$, where $x,y,z>0$. According to the contradiction method, assume that $x+y+z>3$ and show that $F(x,y,z)>0$. Since $F(x,y,z)$ is strictly increasing in each of its arguments, it is enough to prove that

$$x+y+z=3$$

involves

$$F(x,y,z)\ge 0.$$

Denote

$$q=xy+yz+zx,\qquad r=xyz.$$

Since

$$\sum x^2y^2=q^2-6r,\qquad \sum x^2=9-2q,$$

we have

$$F(x,y,z)=4r^2+2(q^2-6r)+5(9-2q)-25=2(2r^2-6r+q^2-5q+10),$$

$$\frac{1}{2}F(x,y,z)=2(r-1)^2+q^2-5q+8-2r.$$

It suffices to show that

$$q^2-5q+8\ge 2r.$$

From the known inequality

$$(xy+yz+zx)^2\ge 3xyz(x+y+z),$$

it follows that $q^2 \geq 9r$. Therefore, it suffices to prove that

$$q^2 - 5q + 8 \geq \frac{2q^2}{9},$$

which is equivalent to

$$(3-q)(24-7q) \geq 0.$$

Since

$$q \leq \frac{1}{3}(x+y+z)^2 = 3,$$

the conclusion follows. The original inequality is an equality for $a = b = c$.

□

P 1.116. *If* $a,\ b,\ c \in [0,\ 1]$, *then*

$$ab^2 + bc^2 + ca^2 + \frac{5}{4} \geq a + b + c.$$

(Ji Chen, 2007)

Solution. We use the substitution

$$a = 1 - x, \quad b = 1 - y, \quad c = 1 - z,$$

where $x, y, z \in [0,\ 1]$. Since

$$\sum a(1-b^2) = \sum y(1-x)(2-y) = \sum y(2 - 2x - y + xy)$$
$$= 2\sum x - \left(\sum x\right)^2 + \sum xy^2,$$

the inequality can be written as

$$\frac{5}{4} \geq 2\sum x - \left(\sum x\right)^2 + \sum xy^2.$$

According to the known inequality in P 1.1, we have

$$\sum xy^2 \leq \frac{4}{27}\left(\sum x\right)^3.$$

Thus, it suffices to prove the following inequality

$$\frac{5}{4} \geq 2t - t^2 + \frac{4}{27}t^3,$$

where

$$t = x + y + z \leq 3.$$

This inequality is equivalent to

$$(15 - 4t)(3 - 2t)^2 \geq 0,$$

which is obviously true for $t \leq 3$. The proof is completed. The equality occurs for $a = 0$, $b = 1$ and $c = \frac{1}{2}$ (or any cyclic permutation thereof).

□

P 1.117. *If a, b, c are nonnegative real numbers such that*

$$a+b+c=3, \qquad a \le b \le 1 \le c,$$

then

$$a^2b+b^2c+c^2a \le 3.$$

Solution. Since

$$ab^2+bc^2+ca^2-(a^2b+b^2c+c^2)=(a-b)(b-c)(c-a) \ge 0,$$

it suffices to prove that

$$a^2b+b^2c+c^2+(ab^2+bc^2+ca^2) \le 6;$$

that is,

$$(a+b+c)(ab+bc+ca)-3abc \le 6,$$

$$ab+bc+ca-abc \le 2,$$

$$1-(a+b+c)+ab+bc+ca-abc \le 0,$$

$$(1-a)(1-b)(1-c) \le 0.$$

The equality occurs for $a=b=c=1$.

□

P 1.118. *Let a, b, c be nonnegative real numbers such that*

$$a+b+c=3, \qquad a \le 1 \le b \le c.$$

Prove that

(a)
$$a^2b+b^2c+c^2a \ge ab+bc+ca;$$

(b)
$$a^2b+b^2c+c^2a \ge abc+2;$$

(c)
$$\frac{1}{abc}+2 \ge \frac{9}{a^2b+b^2c+c^2a};$$

(d)
$$ab^2+bc^2+ca^2 \ge 3.$$

(Vasile Cîrtoaje, 2008)

Solution. (a) We have

$$\begin{aligned}a^2b+b^2c+c^2a-ab-bc-ca &= ab(a-1)+bc(b-1)+ca(c-1)\\ &= -ab[(b-1)+(c-1)]+bc(b-1)+ca(c-1)\\ &= b(b-1)(c-a)+a(c-1)(c-b)\geq 0.\end{aligned}$$

The equality holds for $a=b=c=1$, and also for $a=0$, $b=1$ and $c=2$.

(b) Since

$$a(b-a)(b-c)\leq 0,$$

we have

$$\begin{aligned}a^2b+b^2c+c^2a &\geq a^2b+b^2c+c^2a+a(b-a)(b-c)\\ &= b^2(a+c)+ac(a+c-b).\end{aligned}$$

Thus, it suffices to prove that

$$b^2(a+c)+ac(a+c-b)\geq abc+2.$$

This inequality is equivalent to

$$b^2(a+c)-2\geq ac(2b-a-c),$$

$$b^2(3-b)-2\geq ac(3b-3).$$

From $(b-a)(b-c)\leq 0$, it follows that

$$ac\leq b(a+c-b)=b(3-2b).$$

Thus, it suffices to show that

$$b^2(3-b)-2\geq b(3-2b)(3b-3),$$

which is equivalent to the obvious inequality

$$(5b-2)(b-1)^2\geq 0.$$

The equality holds for $a=b=c=1$, and also for $a=0$, $b=1$ and $c=2$.

(c) According to the inequality in (a), it suffices to show that

$$\frac{1}{abc}+2\geq\frac{9}{abc+2},$$

which is equivalent to

$$(abc-1)^2\geq 0.$$

The equality holds for $a=b=c=1$.

(d) Since

$$ab^2 + bc^2 + ca^2 - (a^2b + b^2c + c^2) = (a - b)(b - c)(c - a) \geq 0,$$

it suffices to prove that

$$ab^2 + bc^2 + ca^2 + (a^2b + b^2c + c^2) \geq 6;$$

that is,

$$(a + b + c)(ab + bc + ca) - 3abc \geq 6,$$

$$ab + bc + ca - abc \geq 2,$$

$$1 - (a + b + c) + ab + bc + ca - abc \geq 0,$$

$$(1 - a)(1 - b)(1 - c) \geq 0.$$

The equality holds for $a = b = c = 1$.

Remark 1. For

$$a + b + c = 3, \qquad 0 < a \leq 1 \leq b \leq c,$$

the following *open inequality* holds

$$\frac{1}{abc} + 6 \geq \frac{21}{a^2b + b^2c + c^2a},$$

which is sharper than the inequality in (c).

Remark 2. From the proof of the inequality in (d), the following identity follows for $a + b + c = 3$:

$$2(ab^2 + bc^2 + ca^2 - 3) = 3(1 - a)(1 - b)(1 - c) + (a - b)(b - c)(c - a).$$

□

P 1.119. *If a, b, c are nonnegative real numbers such that*

$$a + b + c = 3, \qquad a \leq 1 \leq b \leq c,$$

then

(a)
$$\frac{5 - 2a}{1 + b} + \frac{5 - 2b}{1 + c} + \frac{5 - 2c}{1 + a} \geq \frac{9}{2};$$

(b)
$$\frac{3 - 2b}{1 + a} + \frac{3 - 2c}{1 + b} + \frac{3 - 2a}{1 + c} \leq \frac{3}{2}.$$

(Vasile Cîrtoaje, 2008)

Solution. (a) Write the inequality as follows:

$$2\sum(5-2a)(1+c)(1+a) \ge 9(1+a)(1+b)(1+c),$$

$$2\left(21+7\sum ab-2\sum ab^2\right) \ge 9\left(4+\sum ab+abc\right),$$

$$6+5\sum ab \ge 9abc+4\sum ab^2.$$

By P 1.9-(a), we have

$$\sum ab^2 \le 4-abc.$$

Therefore, it suffices to prove that

$$6+5\sum ab \ge 9abc+4(4-abc),$$

which is equivalent to

$$\sum ab \ge 2+abc,$$

$$(1-a)(1-b)(1-c) \ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=0$, $b=1$, $c=2$.

(b) Write the inequality as follows:

$$2\sum(3-2b)(1+b)(1+c) \le 3(1+a)(1+b)(1+c),$$

$$2\left(3+5\sum ab-2\sum a^2b\right) \le 3\left(4+\sum ab+abc\right),$$

$$6+3abc+4\sum a^2b \ge 7\sum ab,$$

$$6+3abc+4\sum ab(a+b) \ge 7\sum ab+4\sum ab^2,$$

$$6+3abc+4\left(\sum a\right)\left(\sum ab\right)-12abc \ge 7\sum ab+4\sum ab^2,$$

$$6+5\sum ab \ge 9abc+4\sum ab^2.$$

By P 1.9-(a), we have

$$\sum ab^2 \le 4-abc.$$

Therefore, it suffices to prove that

$$6+5\sum ab \ge 9abc+4(4-abc),$$

which is equivalent to

$$\sum ab \ge 2+abc,$$

$$(1-a)(1-b)(1-c) \ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=0$, $b=1$, $c=2$.

□

P 1.120. *If a, b, c are nonnegative real numbers such that*

$$ab+bc+ca=3, \qquad a\le 1\le b\le c,$$

then

(a)
$$a^2b+b^2c+c^2a\ge 3;$$

(b)
$$ab^2+bc^2+ca^2+3(\sqrt{3}-1)abc\ge 3\sqrt{3}.$$

(Vasile Cîrtoaje, 2008)

Solution. (a) Since

$$a(b-a)(b-c)\le 0,$$

we have

$$\begin{aligned}a^2b+b^2c+c^2a&\ge a^2b+b^2c+c^2a+a(b-a)(b-c)\\&=b^2(a+c)+ac(a+c-b).\end{aligned}$$

Thus, it suffices to prove that

$$b^2(a+c)+ac(a+c-b)\ge 3.$$

Denote

$$x=a+c.$$

From $ab+bc+ca=3$, we get

$$ac=3-bx$$

and

$$x=\frac{3-ac}{b}\le\frac{3}{b}\le 3.$$

Thus, we need to show that

$$b^2x+(3-bx)(x-b)\ge 3,$$

$$2b^2x-(x^2+3)b+3x-3\ge 0.$$

Since

$$\begin{aligned}2b^2x-(x^2+3)b+3x-3&=2(b^2-2b+1)x+2(2b-1)x-(x^2+3)b+3x-3\\&=2(b-1)^2x+(3-x)(bx-b-1)\\&\ge(3-x)(bx-b-1),\end{aligned}$$

it is enough to prove that

$$bx-b-1\ge 0.$$

From the inequality $(b-a)(b-c)\le 0$, we get

$$bx\ge b^2+ac=b^2+3-bx, \quad bx\ge\frac{b^2+3}{2}.$$

Therefore,

$$bx - b - 1 \geq \frac{b^2+3}{2} - b - 1 = \frac{(b-1)^2}{2} \geq 0.$$

The proof is completed. The equality holds for $a = b = c = 1$, and for $a = 0$, $b = 1$ and $c = 3$.

(b) Since

$$ab^2 + bc^2 + ca^2 - (a^2b + b^2c + c^2) = (a-b)(b-c)(c-a) \geq 0,$$

it suffices to prove that

$$ab^2 + bc^2 + ca^2 + (a^2b + b^2c + c^2) + 6(\sqrt{3}-1)abc \geq 6\sqrt{3};$$

that is,

$$(a+b+c)(ab+bc+ca) + 3(2\sqrt{3}-3)abc \geq 6\sqrt{3},$$

$$a + b + c + (2\sqrt{3}-3)abc \geq 2\sqrt{3},$$

$$a[1 + (2\sqrt{3}-3)bc] + b + c \geq 2\sqrt{3},$$

$$a[1 + (2\sqrt{3}-3)p] + 2(s - \sqrt{3}) \geq 0,$$

where

$$s = \frac{b+c}{2}, \qquad p = bc, \qquad s^2 \geq p \geq 1.$$

From $ab + bc + ca = 3$, we get

$$a = \frac{3-p}{2s}, \qquad p \leq 3.$$

Therefore, we need to show that $F(s,p) \geq 0$, where

$$F(s,p) = (3-p)[1 + (2\sqrt{3}-3)p] + 4s(s - \sqrt{3}).$$

Since the inequality $F(s,p) \geq 0$ is true for $s - \sqrt{3} \geq 0$, consider further the case

$$s \leq \sqrt{3}.$$

We will show that

$$F(s,p) \geq F(s,s^2) \geq 0.$$

We have

$$\begin{aligned} F(s,p) - F(s,s^2) &= (2\sqrt{3}-3)(s^4 - p^2) - (6\sqrt{3}-10)(s^2 - p) \\ &= (s^2 - p)[(2\sqrt{3}-3)(s^2+p) - 6\sqrt{3} + 10]. \end{aligned}$$

Since $s^2 - p \geq 0$ and

$$(2\sqrt{3}-3)(s^2+p) - 6\sqrt{3} + 10 \geq (2\sqrt{3}-3)(1+1) - 6\sqrt{3} + 10 = 4 - 2\sqrt{3} > 0,$$

the left inequality is true. The right inequality is also true because

$$\begin{aligned} F(s,s^2) &= (3-s^2)[1+(2\sqrt{3}-3)s^2]+4s(s-\sqrt{3}) \\ &= (\sqrt{3}-s)[(\sqrt{3}+s)(1+(2\sqrt{3}-3)s^2)-4s] \\ &= (\sqrt{3}-s)[\sqrt{3}(1-s)^2(1+2s)-3s(1-s)^2] \\ &= (\sqrt{3}-s)(1-s)^2[\sqrt{3}+(2\sqrt{3}-3)s] \ge 0. \end{aligned}$$

The equality holds for $a=b=c=1$, and also for $a=0$ and $b=c=\sqrt{3}$.

□

P 1.121. *If a, b, c are nonnegative real numbers such that*

$$a^2+b^2+c^2=3, \qquad a \le 1 \le b \le c,$$

then

(a)
$$a^2b+b^2c+c^2a \ge 2abc+1;$$

(b)
$$2(ab^2+bc^2+ca^2) \ge 3abc+3.$$

(Vasile Cîrtoaje, 2008)

Solution. (a) Let

$$x=a+c, \qquad x \ge b.$$

From $a^2+b^2+c^2=3$, we get

$$ac=\frac{b^2+x^2-3}{2},$$

and from $(b-a)(b-c)\le 0$, we get

$$bx \ge b^2+ac,$$

$$bx \ge b^2+\frac{x^2+b^2-3}{2},$$

$$(x-b)^2 \le 3-2b^2, \qquad b \le \sqrt{\frac{3}{2}},$$

$$x \le b+d, \qquad d=\sqrt{3-2b^2}.$$

Since

$$a(b-a)(b-c) \le 0,$$

we have

$$\begin{aligned} a^2b+b^2c+c^2a &\ge a^2b+b^2c+c^2a+a(b-a)(b-c) \\ &= b^2x-ac(b-x). \end{aligned}$$

Thus, it suffices to prove that

$$b^2x - ac(3b - x) \geq 1,$$

which is equivalent to $f(x, b) \geq 0$, where

$$\begin{aligned} f(x, b) &= 2b^2x - (x^2 + b^2 - 3)(3b - x) - 2 \\ &= x^3 - 3bx^2 + 3(b^2 - 1)x - 3b^3 + 9b - 2. \end{aligned}$$

We will show that

$$f(x, b) \geq f(b + d, b) \geq 0.$$

Since $x \leq b + d$ and

$$\begin{aligned} f(x, b) - f(b + d, b) &= (x - b - d)[x^2 + x(b + d) + (b + d)^2 - 3b(x + b + d) + 3b^2 - 3] \\ &= (x - b - d)[x^2 - (2b - d)x - b^2 - bd], \end{aligned}$$

we need to show that $g(x) \leq 0$, where

$$g(x) = x^2 - (2b - d)x - b^2 - bd = (x - 2b)(x + d) + b(d - b).$$

Since $d - b \leq 0$, it suffices to show that $x - 2b \leq 0$. Indeed, we have

$$x^2 = (a + c)^2 \leq 2(a^2 + c^2) = 2(3 - b^2) \leq 4,$$

hence

$$x \leq 2 \leq 2b.$$

To prove the right inequality $f(b + d, b) \geq 0$, we have

$$f(b + d, b) = 2b^2(b + d) - 2bd(2b - d) - 2 = 2(3b - b^3 - 1 - b^2d).$$

We need to show that

$$3b - b^3 - 1 \geq b^2\sqrt{3 - 2b^2}$$

for

$$1 \leq b \leq \sqrt{\frac{3}{2}}.$$

We have

$$3b - b^3 - 1 \geq 3b - \frac{3b}{2} - 1 = \frac{3b - 2}{2} \geq 0.$$

By squaring, the inequality becomes

$$(3b - b^3 - 1)^2 \geq b^4(3 - 2b^2),$$

$$3b^6 - 9b^4 + 2b^3 + 9b^2 - 6b + 1 \geq 0,$$

$$(b - 1)^2(3b^4 + 6b^3 - 4b + 1) \geq 0.$$

The original inequality is an equality for $a = b = c = 1$.

(b) Denote

$$p = a + b + c, \qquad q = ab + bc + ca.$$

Since

$$ab^2 + bc^2 + ca^2 - (a^2b + b^2c + c^2) = (a - b)(b - c)(c - a) \geq 0,$$

it suffices to prove that

$$ab^2 + bc^2 + ca^2 + (a^2b + b^2c + c^2) \geq 3abc + 3;$$

that is,

$$pq \geq 6abc + 3.$$

From

$$(a - 1)(b - 1)(c - 1) \geq 0,$$

we get

$$abc \geq 1 - p + q,$$

therefore

$$\begin{aligned} pq - 6abc - 3 &\geq pq - 6(1 - p + q) - 3 \\ &= (p - 6)q + 6p - 9 \\ &= \frac{(p - 6)(p^2 - 3)}{2} + 6p - 9 \\ &= \frac{p(p - 3)^2}{2} \geq 0. \end{aligned}$$

The equality holds for $a = b = c = 1$.

□

P 1.122. *If a, b, c are nonnegative real numbers such that*

$$ab + bc + ca = 3, \qquad a \leq b \leq 1 \leq c,$$

then

$$ab^2 + bc^2 + ca^2 + 3abc \geq 6.$$

(Vasile Cîrtoaje, 2008)

Solution. Denote

$$p = a + b + c.$$

Since

$$ab^2 + bc^2 + ca^2 - (a^2b + b^2c + c^2) = (a - b)(b - c)(c - a) \geq 0,$$

it suffices to prove that

$$ab^2 + bc^2 + ca^2 + (a^2b + b^2c + c^2) + 6abc \geq 12;$$

that is,

$$(a+b+c)(ab+bc+ca)+3abc \ge 12,$$

$$a+b+c+abc \ge 4,$$

which is equivalent to

$$(a-1)(b-1)(c-1) \ge 0.$$

The equality holds for $a=b=c=1$.

□

P 1.123. *If a, b, c are nonnegative real numbers such that*

$$a^2+b^2+c^2=3, \qquad a \le b \le 1 \le c,$$

then

$$2(a^2b+b^2c+c^2a) \le 3abc+3.$$

(Vasile Cîrtoaje, 2008)

Solution. Consider two cases.

Case 1: $a+c \ge 2b$. Denote

$$x=a+c, \qquad x \ge 2b.$$

From $a^2+b^2+c^2=3$ and $(b-a)(b-c) \le 0$, we get in succession

$$ac=\frac{b^2+x^2-3}{2},$$

$$bx \ge b^2+ac,$$

$$bx \ge b^2+\frac{x^2+b^2-3}{2},$$

$$(x-b)^2 \le 3-2b^2,$$

$$x \le b+d, \qquad d=\sqrt{3-2b^2}.$$

Since

$$ab^2+bc^2+ca^2-(a^2b+b^2c+c^2)=(a-b)(b-c)(c-a) \ge 0,$$

it suffices to prove that

$$a^2b+b^2c+c^2a+(ab^2+bc^2+ca^2) \le 3abc+3;$$

that is,

$$(a+b+c)(ab+bc+ca) \le 6abc+3,$$

$$(x+b)(bx+ac) \le 6abc+3,$$

$$ac(x-5b)+bx(x+b)-3 \le 0.$$

Thus, we need to show that $f(x,b) \le 0$, where

$$\begin{aligned} f(x,b) &= (x^2+b^2-3)(x-5b)+2bx(x+b)-6 \\ &= x^3-3bx^2+3(b^2-1)x-5b^3+15b-6. \end{aligned}$$

We will show that

$$f(x,b) \le f(b+d,b) \le 0.$$

Since $x \le b+d$ and

$$\begin{aligned} f(x,b)-f(b+d,b) &= (x-b-d)[x^2+x(b+d)+(b+d)^2-3b(x+b+d)+3b^2-3] \\ &= (x-b-d)[x^2-(2b-d)x-b^2-bd], \end{aligned}$$

we need to show that $g(x) \ge 0$, where

$$g(x) = x^2-(2b-d)x-b^2-bd.$$

Since $x-2b \ge 0$ and $d-b \ge 0$, we have

$$g(x) = (x-2b)(x+d)+b(d-b) \ge 0.$$

To prove the right inequality $f(b+d,b) \le 0$, from

$$f(b+d,b) = 2bd(d-4b)+2b(b+d)(2b+d)-6 = 2(6b-2b^3-3-b^2d),$$

it follows that we need to show that

$$6b-2b^3-3 \le b^2\sqrt{3-2b^2}$$

for $0 \le b \le 1$. This inequality is true for $b \le \dfrac{1}{2}$ because

$$6b-2b^3-3 \le 3(2b-1) \le 0.$$

So, it suffices to prove the inequality for $1/2 < b \le 1$. By squaring, the inequality becomes

$$(6b-2b^3-3)^2 \le b^4(3-2b^2),$$

$$2b^6-9b^4+4b^3+12b^2-12b+3 \le 0,$$

$$(b-1)^3(2b^3+6b^2+3b-3) \le 0.$$

We only need to show that

$$2b^3+6b^2+3b-3 \ge 0.$$

Indeed,

$$2b^3+6b^2+3b-3 > 3(2b^2+b-1) = 3(2b-1)(b+1) > 0.$$

Case 2: $a+c \le 2b$. Consider the nontrivial case $a<c$, denote

$$b_1 = \frac{a+c}{2}, \qquad b_2 = \sqrt{\frac{a^2+c^2}{2}} \qquad (b_1 < b_2),$$

and write the inequality in the homogeneous form $E(a,b,c) \le 0$, where

$$E(a,b,c) = 2(a^2b + b^2c + c^2a) - 3abc - 3\left(\frac{a^2+b^2+c^2}{3}\right)^{3/2}.$$

From $a^2 + b^2 + c^2 = 3$ and $b \le 1$, it follows that $b \le b_2$. For fixed a and c, consider the function

$$f(b) = E(a,b,c), \qquad b \in [b_1, b_2].$$

We will show that

$$f(b) \le f(b_2)) \le 0.$$

The left inequality is true if $f'(b) \ge 0$ for $b \in [b_1, b_2]$. Since

$$\begin{aligned}
f'(b) &= 2a^2 + 4bc - 3ac - 3b\left(\frac{a^2+b^2+c^2}{3}\right)^{1/2} \\
&= 2a^2 + 4bc - 3ac - 3b = 2a^2 - 3ac + b(4c-3) \\
&\ge 2a^2 - 3ac + \frac{(a+c)(4c-3)}{2} \\
&= \frac{(a-c)^2 + 3(a^2+c^2-a-c)}{2} \\
&\ge \frac{3(a^2+c^2-a-c)}{2},
\end{aligned}$$

it suffices to show that

$$a^2 + c^2 \ge a + c.$$

From $a^2 + b^2 + c^2 = 3$ and $b \le 1$, it follows that $a^2 + c^2 \ge 2$. If $a + c \le 2$, then

$$a^2 + b^2 \ge 2 \ge a + c.$$

Also, if $a + c \ge 2$, then

$$a^2 + b^2 \ge \frac{1}{2}(a+c)^2 \ge a + c.$$

To prove the right inequality $f(b_2) \le 0$, we see that

$$\begin{aligned}
f(b_2) &= 2a^2b_2 + (a^2+c^2)c + 2c^2a - 3ab_2c - 3b_2\frac{a^2+c^2}{2} \\
&= c(a+c)^2 - \frac{(3c^2+6ac-a^2)}{2}b_2 \\
&= c(a+c)^2 - \frac{(3c^2+6ac-a^2)}{2}\sqrt{\frac{a^2+c^2}{2}}.
\end{aligned}$$

Thus, we need to show that

$$c^2(c+a)^4 \le \frac{(3c^2+6ac-a^2)^2(c^2+a^2)}{8},$$

which is equivalent to

$$c^6 + 4ac^5 - 9a^2c^4 - 8a^3c^3 + 23a^4c^2 - 12a^5c + a^6 \geq 0,$$

$$(c-a)^3(c^3 + 7c^2a + 9ca^2 - a^3) \leq 0.$$

The proof is completed. The equality holds for $a = b = c = 1$.

□

P 1.124. *If a, b, c are nonnegative real numbers such that*

$$a^2 + b^2 + c^2 = 3, \qquad a \leq b \leq 1 \leq c,$$

then

$$2(a^3b + b^3c + c^3a) \leq abc + 5.$$

(Vasile Cîrtoaje, 2008)

Solution. Let

$$p = a + b + c, \qquad q = ab + bc + ca.$$

Since

$$ab^3 + bc^3 + ca^3 - (a^3b + b^3c + c^3) = (a+b+c)(a-b)(b-c)(c-a) \geq 0,$$

it suffices to prove that

$$(a^3b + b^3c + c^3a) + (ab^3 + bc^3 + ca^3) \leq abc + 5,$$

which is equivalent to

$$(a^2 + b^2 + c^2)(ab + bc + ca) \leq abc(a + b + c + 1) + 5,$$

$$3q \leq abc(p+1) + 5.$$

From

$$(a-1)(b-1)(c-1) \geq 0,$$

we get

$$abc \geq q - p + 1.$$

Therefore, it suffices to show that

$$3q \leq (q - p + 1)(p + 1) + 5,$$

which is equivalent to

$$6 - p^2 \geq q(2 - p),$$

$$12 - 2p^2 \geq (p^2 - 3)(2 - p),$$

$$p^3 - 4p^2 - 3p + 18 \geq 0,$$

$$(p-3)^2(p+2) \geq 0.$$

The proof is completed. The equality holds for $a = b = c = 1$.

□

P 1.125. *If a, b, c are real numbers, then*

$$(a^2+b^2+c^2)^2 \ge 3(a^3b+b^3c+c^3a).$$

(Vasile Cîrtoaje, 1992)

First Solution. Write the inequality as

$$E_1 - 2E_2 \ge 0,$$

where

$$E_1 = a^3(a-b)+b^3(b-c)+c^3(c-a),$$
$$E_2 = a^2b(a-b)+b^2c(b-c)+c^2a(c-a).$$

Using the substitution

$$b = a+p, \quad c = a+q,$$

we have

$$\begin{aligned} E_1 &= a^3(a-b)+b^3[(b-a)+(a-c)]+c^3(c-a) \\ &= (a-b)^2(a^2+ab+b^2)+(a-c)(b-c)(b^2+bc+c^2) \\ &= p^2(a^2+ab+b^2)-q(p-q)(b^2+bc+c^2) \\ &= 3(p^2-pq+q^2)a^2+3(p^3-p^2q+q^3)a+p^4-p^3q+q^4 \end{aligned}$$

and

$$\begin{aligned} E_2 &= a^2b(a-b)+b^2c[(b-a)+(a-c)]+c^2a(c-a) \\ &= (a-b)b(a^2-bc)+(a-c)c(b^2-ca) \\ &= pb(bc-a^2)+qc(ca-b^2) \\ &= (p^2-pq+q^2)a^2+(p^3+p^2q-2pq^2+q^3)a+p^3q-p^2q^2. \end{aligned}$$

Thus, the inequality can be rewritten as

$$Aa^2+Ba+C \ge 0,$$

where

$$A = p^2-pq+q^2,$$
$$B = p^3-5p^2q+4pq^2+q^3,$$
$$C = p^4-3p^3q+2p^2q^2+q^4.$$

For the non-trivial case $A > 0$, it is enough to show that $\delta \le 0$, where $\delta = B^2-4AC$ is the discriminant of the quadratic function Aa^2+Ba+C. Indeed, we have

$$\begin{aligned} \delta &= -3(p^6-2p^5q-3p^4q^2+6p^3q^3+2p^2q^4-4pq^5+q^6) \\ &= -3(p^3-p^2q-2pq^2+q^3)^2 \le 0. \end{aligned}$$

The equality holds for $a = b = c$, and also for

$$\frac{a}{\sin^2 \frac{4\pi}{7}} = \frac{b}{\sin^2 \frac{2\pi}{7}} = \frac{c}{\sin^2 \frac{\pi}{7}}$$

(or any cyclic permutation).

Second Solution. Let us denote

$$x = a^2 - ab + bc,$$

$$y = b^2 - bc + ca,$$

$$z = c^2 - ca + ab.$$

We have

$$x^2 + y^2 + z^2 = \sum a^4 + 2\sum a^2b^2 - 2\sum a^3b$$

and

$$xy + yz + zx = \sum a^3b.$$

From the known inequality

$$x^2 + y^2 + z^2 \geq xy + yz + zx,$$

the desired inequality follows.

Third Solution. Let us denote

$$x = a(a - 2b - c),$$

$$y = b(b - 2c - a),$$

$$z = c(c - 2a - b).$$

We have

$$x^2 + y^2 + z^2 = \sum a^4 + 5\sum a^2b^2 + 4abc\sum a - 4\sum a^3b - 2\sum ab^3$$

and

$$xy + yz + zx = 3\sum a^2b^2 + 4abc\sum a - \sum a^3b - 2\sum ab^3.$$

The known inequality

$$x^2 + y^2 + z^2 \geq xy + yz + zx$$

leads to the desired inequality.

Remark 1. Let

$$E = (a^2 + b^2 + c^2)^2 - 3(a^3b + b^3c + c^3a).$$

Using the notations from the first solution, the formula

$$4A(Aa^2 + Ba + C) = (2Aa + B)^2 - \delta,$$

leads to the following identity

$$4E_1E = (A_1 - 5B_1 + 4C_1)^2 + 3(A_1 - B_1 - 2C_1 + 2D_1)^2,$$

where

$$A_1 = a^3 + b^3 + c^3, \quad B_1 = a^2b + b^2c + c^2a, \quad C_1 = ab^2 + bc^2 + ca^2, \quad D_1 = 3abc,$$

$$E_1 = a^2 + b^2 + c^2 - ab - bc - ca.$$

Remark 2. Let

$$E = (a^2 + b^2 + c^2)^2 - 3(a^3b + b^3c + c^3a),$$

The identity

$$x^2 + y^2 + z^2 - xy - yz - zx = \frac{1}{2}\sum(x - y)^2,$$

where x, y, z are defined in the second or third solution, leads to the identity

$$2E = \sum(a^2 - b^2 - ab + 2bc - ca)^2.$$

In addition, the following similar identities hold:

$$6E = \sum(2a^2 - b^2 - c^2 - 3ab + 3bc)^2,$$

$$4E = (2a^2 - b^2 - c^2 - 3ab + 3bc)^2 + 3(b^2 - c^2 - ab - bc + 2ca)^2.$$

Remark 3. The inequality in P 1.125 is known as *Vasc's inequality*, after the author's username on the Art of Problem Solving website.

□

P 1.126. *If a, b, c are real numbers, then*

$$a^4 + b^4 + c^4 + ab^3 + bc^3 + ca^3 \geq 2(a^3b + b^3c + c^3a).$$

(Vasile Cîrtoaje, 1992)

First Solution. Making the substitution

$$b = a + p, \quad c = a + q,$$

the inequality turns into

$$Aa^2 + Ba + C \geq 0,$$

where

$$A = 3(p^2 - pq + q^2), \quad B = 3(p^3 - 2p^2q + pq^2 + q^3), \quad C = p^4 - 2p^3q + pq^3 + q^4.$$

Since the discriminant of the quadratic trinomial $Aa^2 + Ba + C$ is nonpositive,

$$\delta = B^2 - 4AC = -3(p^6 - 6p^4q + 2p^3q^3 + 9p^2q^4 - 6pq^5 + q^6)$$
$$= -3(p^3 - 3pq^2 + q^3)^2 \le 0,$$

the conclusion follows. The equality holds for $a = b = c$, and also for

$$\frac{a}{\sin\frac{\pi}{9}} = \frac{b}{\sin\frac{7\pi}{9}} = \frac{c}{\sin\frac{13\pi}{9}}$$

(or any cyclic permutation).

Second Solution. Let us denote

$$x = a(a-b),$$
$$y = b(b-c),$$
$$z = c(c-a).$$

We have

$$x^2 + y^2 + z^2 = \sum a^4 + \sum a^2b^2 - 2\sum a^3b$$

and

$$xy + yz + zx = \sum a^2b^2 - \sum ab^3.$$

Applying the known inequality

$$x^2 + y^2 + z^2 \ge xy + yz + zx,$$

the desired inequality follows.

Third Solution. Let

$$x = a^2 + bc + ca,$$
$$y = b^2 + ca + ab,$$
$$z = c^2 + ab + bc.$$

We have

$$x^2 + y^2 + z^2 = \sum a^4 + 2\sum a^2b^2 + 4abc\sum a + 2\sum ab^3$$

and

$$xy + yz + zx = 2\sum a^2b^2 + 4abc\sum a + 2\sum a^3b + \sum ab^3.$$

The known inequality

$$x^2 + y^2 + z^2 \ge xy + yz + zx$$

leads to the desired inequality.

Remark 1. The inequality is more interesting in the case $abc < 0$. If a, b, c are positive, then the inequality is less sharp than Vasc's inequality in P 1.125 because it can be obtained by adding Vasc's inequality and

$$ab(a-b)^2 + bc(b-c)^2 + ca(c-a)^2 \ge 0.$$

On the other hand, if a, b, c are positive, then the inequality

$$3(a^4+b^4+c^4)+4(ab^3+bc^3+ca^3) \ge 7(a^3b+b^3c+c^3a)$$

is a refinement of the inequality in P 1.126. To prove this inequality, we write it as

$$3(a^4+b^4+c^4-a^3b-b^3c-c^3a)+4(ab^3+bc^3+ca^3-a^3b-b^3c-c^3a) \ge 0,$$

consider $a = \min\{a, b, c\}$ and use the substitution

$$b = a+p, \qquad c = a+q, \qquad a > 0,\ p \ge 0,\ q \ge 0.$$

Since

$$\sum a^4 - \sum a^3b = \sum a^3(a-b)$$
$$= 3(p^2-pq+q^2)a^2 + 3(p^3-p^2q+q^3)a + p^4 - p^3q + q^4$$

and

$$\sum ab^3 - \sum a^3b = (a+b+c)(a-b)(b-c)(c-a)$$
$$= pq(q-p)(3a+p+q),$$

the inequality becomes

$$Aa^2 + Ba + C \ge 0,$$

where

$$A = 9(p^2-pq+q^2), \quad B = 3(3p^3-7p^2q+4pq^2+3q^3),$$
$$C = 3p^4-7p^3q+4pq^3+3q^4.$$

The inequality $Aa^2+Ba+C \ge 0$ is true for $a > 0$ and $p, q \ge 0$ because

$$A \ge 0,$$

$$B = p(3p-4q)^2 + q(p-3q)^2 + 2pq(p+q) \ge 0,$$

$$3C = p(p+q)(3p-5q)^2 + 5q^2\left(p - \frac{13q}{10}\right)^2 + \frac{11}{20}q^4 \ge 0.$$

Remark 2. Let

$$E = a^4+b^4+c^4+ab^3+bc^3+ca^3-2(a^3b+b^3c+c^3a).$$

Using the notations from the first solution, the formula

$$4A(Aa^2+Ba+C) = (2Aa+B)^2 - \delta$$

leads to the following identity

$$4E_1E = (A_1-3C_1+2D_1)^2 + 3(A_1-2B_1+C_1)^2,$$

where

$$A_1 = a^3 + b^3 + c^3, \quad B_1 = a^2b + b^2c + c^2a, \quad C_1 = ab^2 + bc^2 + ca^2, \quad D_1 = 3abc,$$

$$E_1 = a^2 + b^2 + c^2 - ab - bc - ca.$$

Remark 3. Let

$$E = a^4 + b^4 + c^4 + ab^3 + bc^3 + ca^3 - 2(a^3b + b^3c + c^3a).$$

The identity

$$x^2 + y^2 + z^2 - xy - yz - zx = \frac{1}{2}\sum(x-y)^2,$$

where x, y, z are defined in the second or third solution, leads to the identity

$$2E = \sum(a^2 - b^2 - ab + bc)^2.$$

In addition, the following similar identities hold:

$$6E = \sum(2a^2 - b^2 - c^2 - 2ab + bc + ca)^2,$$

$$4E = (2a^2 - b^2 - c^2 - 2ab + bc + ca)^2 + 3(b^2 - c^2 - bc + ca)^2.$$

Remark 4. The inequalities in P 1.125 and P 1.126 are particular cases of the following more general statement (*Vasile Cîrtoaje*, 2007).

- *Let*

$$f_4(a,b,c) = \sum a^4 + A\sum a^2b^2 + Babc\sum a + C\sum a^3b + D\sum ab^3,$$

where A, B, C, D are real constants such that

$$1 + A + B + C + D = 0, \qquad 3(1 + A) \ge C^2 + CD + D^2.$$

If a, b, c are real numbers, then

$$f_4(a,b,c) \ge 0.$$

Note that the following identity holds:

$$4Sf_4(a,b,c) = [U+V+(C+D)S]^2 + 3\left(U - V + \frac{C-D}{3}S\right)^2 + \frac{4}{3}(3+3A-C^2-CD-D^2)S^2,$$

where

$$S = \sum a^2b^2 - \sum a^2bc,$$

$$U = \sum a^3b - \sum a^2bc,$$

$$V = \sum ab^3 - \sum a^2bc.$$

For the main case

$$3(1+A)=C^2+CD+D^2,$$

the inequality $f_4(a,b,c)\geq 0$ is equivalent to each of the following two inequalities

$$\sum[2a^2-b^2-c^2+Cab-(C+D)bc+Dca]^2\geq 0,$$

$$\sum[3b^2-3c^2+(C+2D)ab+(C-D)bc-(2C+D)ca]^2\geq 0.$$

□

P 1.127. *If a,b,c are positive real numbers, then*

(a)
$$\frac{a^2}{ab+2c^2}+\frac{b^2}{bc+2a^2}+\frac{c^2}{ca+2b^2}\geq 1;$$

(b)
$$\frac{a^3}{a^2b+2c^3}+\frac{b^3}{b^2c+2a^3}+\frac{c^3}{c^2a+2b^3}\geq 1.$$

Solution. (a) By the Cauchy-Schwarz inequality, we have

$$\sum\frac{a^2}{ab+2c^2}\geq\frac{\left(\sum a^2\right)^2}{\sum a^2(ab+2c^2)}=\frac{\left(\sum a^2\right)^2}{\sum a^3b+2\sum a^2b^2}.$$

Therefore, it suffices to show that

$$\left(\sum a^2\right)^2\geq 2\sum a^2b^2+\sum a^3b.$$

We get this inequality by summing the known inequality

$$\frac{2}{3}\left(\sum a^2\right)^2\geq 2\sum a^2b^2$$

and Vasc's inequality

$$\frac{1}{3}\left(\sum a^2\right)^2\geq\sum a^3b.$$

The equality holds for $a=b=c=1$.

(b) By the Cauchy-Schwarz inequality, we have

$$\sum\frac{a^3}{a^2b+2c^3}\geq\frac{\left(\sum a^2\right)^2}{\sum a(a^2b+2c^3)}=\frac{\left(\sum a^2\right)^2}{\sum a^3b+2\sum ac^3}=\frac{\left(\sum a^2\right)^2}{3\sum a^3b}.$$

Therefore, it suffices to show that

$$\left(\sum a^2\right)^2\geq 3\sum a^3b,$$

which is just Vasc's inequality. The equality holds for $a=b=c=1$.

□

P 1.128. *If a, b, c are positive real numbers such that $a+b+c=3$, then*

$$\frac{a}{ab+1}+\frac{b}{bc+1}+\frac{c}{ca+1}\ge\frac{3}{2}.$$

Solution. We use the following hint

$$\frac{a}{ab+1}=a-\frac{a^2b}{ab+1},\quad \frac{b}{bc+1}=b-\frac{b^2c}{bc+1},\quad \frac{c}{ca+1}=c-\frac{c^2a}{ca+1},$$

which transforms the desired inequality into

$$\frac{a^2b}{ab+1}+\frac{b^2c}{bc+1}+\frac{c^2a}{ca+1}\le\frac{3}{2}.$$

By the AM-GM inequality, we have

$$ab+1\ge 2\sqrt{ab},\quad bc+1\ge 2\sqrt{bc},\quad ca+1\ge 2\sqrt{ca}.$$

Consequently, it suffices to show that

$$\frac{a^2b}{2\sqrt{ab}}+\frac{b^2c}{2\sqrt{bc}}+\frac{c^2a}{2\sqrt{ca}}\le\frac{3}{2},$$

which is equivalent to

$$a\sqrt{ab}+b\sqrt{bc}+c\sqrt{ca}\le 3,$$

$$3(a\sqrt{ab}+b\sqrt{bc}+c\sqrt{ca})\le(a+b+c)^2.$$

Replacing $\sqrt{a}, \sqrt{b}, \sqrt{c}$ by a, b, c, respectively, we get Vasc's inequality in P 1.125. The equality holds for $a=b=c=1$.

□

P 1.129. *If a, b, c are positive real numbers such that $a+b+c=3$, then*

$$\frac{a}{3a+b^2}+\frac{b}{3b+c^2}+\frac{c}{3c+a^2}\le\frac{3}{2}.$$

(Vasile Cîrtoaje, 2007)

Solution. Since

$$\frac{a}{3a+b^2}=\frac{1}{3}-\frac{b^2}{3(3a+b^2)},\quad \frac{b}{3b+c^2}=\frac{1}{3}-\frac{c^2}{3(3b+c^2)},\quad \frac{c}{3c+a^2}=\frac{1}{3}-\frac{a^2}{3(3c+a)},$$

the desired inequality can be rewritten as

$$\frac{b^2}{3a+b^2}+\frac{c^2}{3b+c^2}+\frac{a^2}{3c+a^2}\ge\frac{3}{2}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{b^2}{3a+b^2} \geq \frac{\left(\sum b^2\right)^2}{\sum b^2(3a+b^2)} = \frac{\left(\sum a^2\right)^2}{\sum a^4 + \left(\sum a\right)\left(\sum ab^2\right)}$$

$$= \frac{\left(\sum a^2\right)^2}{\sum a^4 + \sum a^2b^2 + abc\sum a + \sum ab^3} \geq \frac{\left(\sum a^2\right)^2}{\left(\sum a^2\right)^2 + \sum ab^3}.$$

Thus, it is enough to show that

$$\left(\sum a^2\right)^2 \geq 3\sum ab^3,$$

which is Vasc's inequality. The equality holds for $a = b = c = 1$.

□

P 1.130. *If a, b, c are positive real numbers such that $a + b + c = 3$, then*

$$\frac{a}{b^2+c} + \frac{b}{c^2+a} + \frac{c}{a^2+b} \geq \frac{3}{2}.$$

(Pham Kim Hung, 2007)

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a}{b^2+c} \geq \frac{\left(\sum a^{3/2}\right)^2}{\sum a^2(b^2+c)} = \frac{\sum a^3 + 2\sum a^{3/2}b^{3/2}}{\sum a^2b^2 + \sum ab^2}.$$

Thus, it is enough to show that

$$2\sum a^3 + 4\sum a^{3/2}b^{3/2} \geq 3\sum a^2b^2 + 3\sum ab^2,$$

which is equivalent to the homogeneous inequality

$$2\left(\sum a\right)\left(\sum a^3\right) + 4\left(\sum a\right)\left(\sum a^{3/2}b^{3/2}\right) \geq 9\sum a^2b^2 + 3\left(\sum a\right)\left(\sum ab^2\right).$$

In order to get a symmetric inequality, we use Vasc's inequality. We have

$$\begin{aligned} 3\left(\sum a\right)\left(\sum ab^2\right) &= 3\sum a^2b^2 + 3abc\sum a + 3\sum ab^3 \\ &\leq 3\sum a^2b^2 + 3abc\sum a + \left(\sum a^2\right)^2 \\ &= \sum a^4 + 5\sum a^2b^2 + 3abc\sum a. \end{aligned}$$

Therefore, it suffices to prove the symmetric inequality

$$2\left(\sum a\right)\left(\sum a^3\right) + 4\left(\sum a\right)\left(\sum a^{3/2}b^{3/2}\right) \geq 9\sum a^2b^2 + \sum a^4 + 5\sum a^2b^2 + 3abc\sum a,$$

which is equivalent to

$$\sum a^4 + 2\sum ab(a^2+b^2) + 4abc\sum \sqrt{ab} + 4A \ge 14\sum a^2b^2 + 3abc\sum a,$$

where

$$A = \sum (ab)^{3/2}(a+b).$$

Since

$$A \ge 2\sum a^2b^2,$$

it suffices to prove that

$$\sum a^4 + 2\sum ab(a^2+b^2) + 4abc\sum \sqrt{ab} \ge 6\sum a^2b^2 + 3abc\sum a.$$

According to Schur's inequality of degree four

$$\sum a^4 \ge \sum ab(a^2+b^2) - abc\sum a,$$

it is enough to show that

$$3\sum ab(a^2+b^2) + 4abc\sum \sqrt{ab} \ge 6\sum a^2b^2 + 4abc\sum a.$$

Write this inequality as

$$3\sum ab(a-b)^2 \ge 2abc\sum \left(\sqrt{a}-\sqrt{b}\right)^2,$$

$$\sum ab\left(\sqrt{a}-\sqrt{b}\right)^2\left[3\left(\sqrt{a}+\sqrt{b}\right)^2 - 2c\right] \ge 0.$$

We will prove the stronger inequality

$$\sum ab\left(\sqrt{a}-\sqrt{b}\right)^2\left[\left(\sqrt{a}+\sqrt{b}\right)^2 - c\right] \ge 0,$$

which is equivalent to

$$\sum \left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{c}}\right)^2\left(\sqrt{a}+\sqrt{b}-\sqrt{c}\right) \ge 0.$$

Substituting $x=\sqrt{a}$, $y=\sqrt{b}$, $z=\sqrt{c}$, the inequality becomes

$$\sum \left(\frac{x-y}{z}\right)^2(x+y-z) \ge 0.$$

Without loss of generality, assume that $x \ge y \ge z$. It suffices to show that

$$\left(\frac{y-z}{x}\right)^2(y+z-x) + \left(\frac{x-z}{y}\right)^2(z+x-y) \ge 0.$$

Since

$$\left(\frac{x-z}{y}\right)^2 \geq \left(\frac{y-z}{x}\right)^2,$$

we have

$$\left(\frac{y-z}{x}\right)^2 (y+z-x) + \left(\frac{x-z}{y}\right)^2 (z+x-y) \geq$$

$$\geq \left(\frac{y-z}{x}\right)^2 (y+z-x) + \left(\frac{y-z}{x}\right)^2 (z+x-y)$$

$$= 2z\left(\frac{y-z}{x}\right)^2 \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 1.131. *If a, b, c are positive real numbers such that $abc = 1$, then*

$$\frac{a}{b^3+2} + \frac{b}{c^3+2} + \frac{c}{a^3+2} \geq 1.$$

Solution. Using the substitution

$$a = \frac{x}{y}, \quad b = \frac{z}{x}, \quad c = \frac{y}{z}, \qquad x, y, z > 0,$$

the inequality turns into

$$\sum \frac{x^4}{y(2x^3+z^3)} \geq 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{x^4}{y(2x^3+z^3)} \geq \frac{\left(\sum x^2\right)^2}{\sum y(2x^3+z^3)} = \frac{\left(\sum x^2\right)^2}{2\sum x^3y + \sum xy^3}.$$

Thus, it is enough to show that

$$\left(\sum x^2\right)^2 \geq 2\sum x^3y + \sum xy^3.$$

According to Vasc's inequality, we have

$$\left(\sum x^2\right)^2 \geq 3\sum x^3y$$

and

$$\left(\sum x^2\right)^2 \geq 3\sum xy^3.$$

Thus, the conclusion follows. The equality holds for $a = b = c = 1$.

□

P 1.132. *Let a, b, c be positive real numbers such that*

$$a^m + b^m + c^m = 3,$$

where $m > 0$. Prove that

$$\frac{a^{m-1}}{b} + \frac{b^{m-1}}{c} + \frac{c^{m-1}}{a} \ge 3.$$

Solution. Making the substitution

$$x = a^{\frac{1}{k}}, \quad y = b^{\frac{1}{k}}, \quad z = c^{\frac{1}{k}},$$

where

$$k = \frac{2}{m}, \qquad k > 0,$$

we need to show that $x^2 + y^2 + z^2 = 3$ yields

$$\frac{x^{2-k}}{y^k} + \frac{y^{2-k}}{z^k} + \frac{z^{2-k}}{x^k} \ge 3,$$

which is equivalent to

$$\frac{x^2}{(xy)^k} + \frac{y^2}{(yz)^k} + \frac{z^2}{(zx)^k} \ge 3.$$

Applying Jensen's inequality to the convex function $f(u) = \dfrac{1}{u^k}$, we get

$$\frac{x^2}{(xy)^k} + \frac{y^2}{(yz)^k} + \frac{z^2}{(zx)^k} \ge \frac{x^2 + y^2 + z^2}{\left(\dfrac{x^2 \cdot xy + y^2 \cdot yz + z^2 \cdot zx}{x^2 + y^2 + z^2}\right)^k}$$
$$= \frac{3^{k+1}}{(x^3y + y^3z + z^3x)^k}.$$

Thus, it suffices to show that $x^3y + y^3z + z^3x \le 3$. This is just Vasc's inequality in P 1.125. The equality holds for $a = b = c = 1$.

□

P 1.133. *If a, b, c are positive real numbers, then*

(a) $$\frac{1}{4a} + \frac{1}{4b} + \frac{1}{4c} + \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \ge 3\left(\frac{1}{3a+b} + \frac{1}{3b+c} + \frac{1}{3c+a}\right);$$

(b) $$\frac{1}{4a} + \frac{1}{4b} + \frac{1}{4c} + \frac{1}{a+3b} + \frac{1}{b+3c} + \frac{1}{c+3a} \ge 2\left(\frac{1}{3a+b} + \frac{1}{3b+c} + \frac{1}{3c+a}\right).$$

(Gabriel Dospinescu and Vasile Cîrtoaje, 2004)

Solution. We will prove that the following more general inequalities hold for $t \ge 0$:

$$\frac{t^{4a}}{4a}+\frac{t^{4b}}{4b}+\frac{t^{4c}}{4c}+\frac{t^{2a+2b}}{a+b}+\frac{t^{2b+2c}}{b+c}+\frac{t^{2c+2a}}{c+a}-3\left(\frac{t^{3a+b}}{3a+b}+\frac{t^{3b+c}}{3b+c}+\frac{t^{3c+a}}{3c+a}\right)\ge 0,$$

$$\frac{t^{4a}}{4a}+\frac{t^{4b}}{4b}+\frac{t^{4c}}{4c}+\frac{t^{a+3b}}{a+3b}+\frac{t^{b+3c}}{b+3c}+\frac{t^{c+3a}}{c+3a}-2\left(\frac{t^{3a+b}}{3a+b}+\frac{t^{3b+c}}{3b+c}+\frac{t^{3c+a}}{3c+a}\right)\ge 0.$$

For $t = 1$, we get the desired inequalities.

(a) Denoting the left hand side of the former inequality by $f(t)$, the inequality becomes $f(t) \ge f(0)$. This is true if $f'(t) \ge 0$ for $t > 0$. We have the derivative

$$tf'(t) = t^{4a}+t^{4b}+t^{4c}+2(t^{2a+2b}+t^{2b+2c}+t^{2c+2a})-3(t^{3a+b}+t^{3b+c}+t^{3c+a}).$$

Using the substitution $x = t^a$, $y = t^b$, $z = t^c$, the inequality $f'(t) \ge 0$ turns into

$$x^4+y^4+z^4+2(x^2y^2+y^2z^2+z^2x^2)\ge 3(x^3y+y^3z+z^3x),$$

which is Vasc's inequality in P 1.125. The equality holds for $a = b = c$.

(b) Similarly, we have the derivative

$$tf'(t) = t^{4a}+t^{4b}+t^{4c}+t^{a+3b}+t^{b+3c}+t^{c+3a}-2(t^{3a+b}+t^{3b+c}+t^{3c+a}).$$

Denoting $x = t^a$, $y = t^b$, $z = t^c$, the inequality $f'(t) \ge 0$ turns into

$$x^4+y^4+z^4+xy^3+yz^3+zx^3\ge 2(x^3y+y^3z+z^3x),$$

which is the the inequality in P 1.126. The equality holds for $a = b = c$.

□

P 1.134. *If a, b, c are positive real numbers such that $a^6 + b^6 + c^6 = 3$, then*

$$\frac{a^5}{b}+\frac{b^5}{c}+\frac{c^5}{a}\ge 3.$$

(Tran Quoc Anh, 2007)

Solution. By Hölder's inequality, we have

$$\left(\frac{a^5}{b}+\frac{b^5}{c}+\frac{c^5}{a}\right)^3 \ge \frac{(a^6+b^6+c^6)^4}{a^9b^3+b^9c^3+c^9a^3}=\frac{81}{a^9b^3+b^9c^3+c^9a^3}.$$

Therefore, it suffices to show that

$$a^9b^3+b^9c^3+c^9a^3\le 3.$$

This is equivalent to

$$3(a^9b^3+b^9c^3+c^9a^3)\le (a^6+b^6+c^6)^2,$$

which is Vasc's inequality (see P 1.125). The equality holds for $a = b = c$.

□

P 1.135. *If a, b, c are positive real numbers such that $a^2+b^2+c^2=3$, then*

$$\frac{a^3}{a+b^5}+\frac{b^3}{b+c^5}+\frac{c^3}{c+a^5}\ge\frac{3}{2}.$$

(Marin Bancos, 2010)

Solution. Write the inequality as

$$\sum\left(\frac{a^3}{a+b^5}-a^2\right)+\frac{3}{2}\ge 0,$$

$$\sum\frac{a^2b^5}{a+b^5}\le\frac{3}{2}.$$

Since

$$a+b^5\ge 2\sqrt{ab^5},$$

it suffices to show that

$$\sum ab^2\sqrt{ab}\le 3.$$

In addition, since $2\sqrt{ab}\le a+b$, it suffices to prove that

$$\sum a^2b^2+\sum ab^3\le 6.$$

This is true since

$$\sum a^2b^2\le\frac{1}{3}(a^2+b^2+c^2)^2=3,$$

and, according to Vasc's inequality,

$$\sum ab^3\le\frac{1}{3}(a^2+b^2+c^2)^2=3.$$

The equality holds for $a=b=c=1$.

□

P 1.136. *If a, b, c are real numbers such that $a^2+b^2+c^2=3$, then*

$$a^2b+b^2c+c^2a+3\ge a+b+c+ab+bc+ca.$$

(Vasile Cîrtoaje, 2007)

Solution. Write the inequality as follows:

$$\sum(1-ab)-\sum a(1-ab)\ge 0,$$

$$\sum(a^2+b^2+c^2-3ab)-\sum a(a^2+b^2+c^2-3ab)\ge 0,$$

$$3\left(\sum a^2 - \sum ab\right) - \sum a(a-b)^2 - \sum a(c^2 - ab) \geq 0,$$

$$\frac{3}{2}\sum(a-b)^2 - \sum a(a-b)^2 \geq 0,$$

$$\sum(a-b)^2(3-2a) \geq 0.$$

Assume that

$$a = \max\{a, b, c\}.$$

For $3 - 2a \geq 0$, the inequality is clearly true. Consider now that $3 - 2a < 0$. Since

$$(a-b)^2 = [(a-c) + (c-b)]^2 \leq 2[(a-c)^2 + (c-b)^2],$$

it suffices to show that

$$2[(a-c)^2 + (c-b)^2](3-2a) + (b-c)^2(3-2b) + (c-a)^2(3-2c) \geq 0,$$

which can be rewritten as

$$(a-c)^2(9-4a-2c) + (b-c)^2(9-4a-2b) \geq 0.$$

This inequality is true because $9 > 4a+2c$ and $9 > 4a+2b$. For instance, the last inequality is true if $81 > 4(2a+b)^2$; indeed, we have

$$\frac{81}{4} - (2a+b)^2 > 15 - (2a+b)^2 = 5(a^2+b^2+c^2) - (2a+b)^2 = (a-2b)^2 + 5c^2 \geq 0.$$

The equality holds for $a = b = c = 1$.

Remark. From $(a+b+c-3)^2 \geq 0$, we get

$$ab + bc + ca + 6 \geq 3(a+b+c),$$

hence

$$a + b + c + ab + bc + ca - 3 \geq 4(a+b+c) - 9.$$

So, the following statement is true:

- *If a, b, c are real numbers such that $a^2 + b^2 + c^2 = 3$, then*

$$a^2b + b^2c + c^2a + 9 \geq 4(a+b+c).$$

□

P 1.137. *If a, b, c are positive real numbers such that $a + b + c = 3$, then*

$$\frac{12}{a^2b + b^2c + c^2a} \leq 3 + \frac{1}{abc}.$$

(Vasile Cîrtoaje and Sheng Li Chen, 2009)

Solution. Let

$$p = a+b+c = 3, \quad q = ab+bc+ca, \quad r = abc \le 1.$$

Write the inequality as

$$2(a^2b+b^2c+c^2a) \ge \frac{24r}{3r+1}.$$

From

$$\begin{aligned}(a-b)^2(b-c)^2(c-a)^2 &= -27r^2 + 2(9pq-2p^3)r + p^2q^2 - 4q^3 \\ &= -27r^2 + 54(q-2)r + 9q^2 - 4q^3,\end{aligned}$$

we get

$$(a-b)(b-c)(c-a) \le \sqrt{-27r^2 + 54(q-2)r + 9q^2 - 4q^3},$$

hence

$$\begin{aligned}2(a^2b+b^2c+c^2a) &= \sum ab(a+b) - (a-b)(b-c)(c-a) \\ &= pq - 3r - (a-b)(b-c)(c-a) \\ &\ge 3q - 3r - \sqrt{-27r^2 + 54(q-2)r + 9q^2 - 4q^3}.\end{aligned}$$

Therefore, it suffices to show that

$$3q - 3r - \sqrt{-27r^2 + 54(q-2)r + 9q^2 - 4q^3} \ge \frac{24r}{3r+1}.$$

which is equivalent to

$$3[(3r+1)q - 3r^2 - 9r] \ge (3r+1)\sqrt{-27r^2 + 54(q-2)r + 9q^2 - 4q^3}.$$

Before squaring this inequality, we need to show that $(3r+1)q - 3r^2 - 9r \ge 0$. Using the known inequality $q^2 \ge 3pr$, we have

$$\begin{aligned}(3r+1)q - 3r^2 - 9r &\ge 3(3r+1)\sqrt{r} - 3r^2 - 9r \\ &= 3\sqrt{r}\ \left(1-\sqrt{r}\right)^3 \ge 0.\end{aligned}$$

By squaring, the desired inequality can be restated as

$$Aq^3 + C \ge 3Bq,$$

where

$$A = 4(3r+1)^2, \quad B = 72r(r+1)(3r+1), \quad C = 108r(r+1)(3r^2+12r+1).$$

By the AM-GM inequality,

$$Aq^3 + C = Aq^3 + \frac{C}{2} + \frac{C}{2} \ge 3\sqrt[3]{Aq^3\left(\frac{C}{2}\right)^2};$$

so, it is enough to show that

$$AC^2 \ge 4B^3,$$

which is equivalent to

$$(3r^2 + 12r + 1)^2 \ge 32r(r + 1)(3r + 1)).$$

Indeed,

$$(3r^2 + 12r + 1)^2 - 32r(r + 1)(3r + 1) = (r - 1)^2(3r - 1)^2 \ge 0,$$

or, by the AM-GM inequality,

$$3r^2 + 12r + 1 = 8r + (r + 1)(3r + 1) \ge 2\sqrt{8r(r + 1)(3r + 1)}.$$

The equality holds for $a = b = c = 1$, and also for $r = \dfrac{1}{3}$ and $q = \sqrt[3]{\dfrac{C}{2A}} = 2$; that is, when a, b, c are the roots of the equation

$$x^3 - 3x^2 + 2x - \frac{1}{3} = 0$$

such that $a \le b \le c$ or $b \le c \le a$ or $c \le a \le b$.

□

P 1.138. *If a, b, c are positive real numbers such that $ab + bc + ca = 3$, then*

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \ge a^2 + b^2 + c^2.$$

(Nguyen Viet Hung, 2024)

First Solution *(by Le Thu).* By the Cauchy-Schwarz inequality, we have

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} = \frac{a^4}{a^2b} + \frac{b^4}{b^2c} + \frac{c^4}{c^2a} \ge \frac{(a^2 + b^2 + c^2)^2}{a^2b + b^2c + c^2a}.$$

So, it suffices to show that

$$a^2 + b^2 + c^2 \ge a^2b + b^2c + c^2a.$$

According to Vasc's inequality (P 1.125), we only need to show that

$$\sqrt{3(a^3b + b^3c + c^3a)} \ge a^2b + b^2c + c^2a,$$

which is equivalent to

$$(ab + bc + ca)(a^3b + b^3c + c^3a) \ge (a^2b + b^2c + c^2a)^2.$$

Clearly, this inequality follows from the Cauchy-Schwarz inequality.

The equality occurs for $a = b = c = 1$.

Second Solution (by *Hai Duong*). From

$$(a+b+c)^2 \geq 3(ab+bc+ca) = (ab+bc+ca)^2,$$

we get

$$a+b+c \geq ab+bc+ca.$$

So, it suffices to prove the homogeneous inequality

$$(ab+bc+ca)\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right) \geq (a+b+c)(a^2+b^2+c^2),$$

which is equivalent to

$$\frac{a^3c}{b}+\frac{b^3a}{c}+\frac{c^3b}{a} \geq a^2b+b^2c+c^2a,$$

$$\sum\left(\frac{a^3c}{b}+\frac{b^3a}{c}-2a^2b\right) \geq 0,$$

$$\sum\frac{a(b^2-ac)^2}{bc} \geq 0.$$

□

P 1.139. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$\frac{24}{a^2b+b^2c+c^2a}+\frac{1}{abc} \geq 9.$$

(Vasile Cîrtoaje, 2009)

Solution *(by Vo Quoc Ba Can).* Let us denote

$$p = a+b+c = 3, \qquad q = ab+bc+ca, \qquad r = abc.$$

Write the inequality as

$$24r \geq (9r-1)(a^2b+b^2c+c^2a),$$

and consider further the nontrivial case

$$r \geq \frac{1}{9}.$$

From

$$\begin{aligned}(a-b)^2(b-c)^2(c-a)^2 &= -27r^2+2(9pq-2p^3)r+p^2q^2-4q^3\\ &= -27r^2+54(q-2)r+9q^2-4q^3,\end{aligned}$$

we get

$$-(a-b)(b-c)(c-a) \le \sqrt{-27r^2+54(q-2)r+9q^2-4q^3},$$

hence

$$\begin{aligned} 2(a^2b+b^2c+c^2a) &= \sum ab(a+b)-(a-b)(b-c)(c-a) \\ &= pq-3r-(a-b)(b-c)(c-a) \\ &\le 3q-3r+\sqrt{-27r^2+54(q-2)r+9q^2-4q^3}. \end{aligned}$$

Therefore, it suffices to show that

$$48r \ge (9r-1)\left[3q-3r+\sqrt{-27r^2+54(q-2)r+9q^2-4q^3}\right],$$

which is true if

$$3[9r^2+15r-(9r-1)q] \ge (9r-1)\sqrt{-27r^2+54(q-2)r+9q^2-4q^3}.$$

We need first to show that $9r^2+15r-(9r-1)q \ge 0$. From Schur's inequality

$$p^3+9r \ge 4pq,$$

we get

$$q \le \frac{3(r+3)}{4},$$

hence

$$9r^2+15r-(9r-1)q \ge 9r^2+15r-\frac{3(r+3)(9r-1)}{4} = \frac{9(r-1)^2}{4} \ge 0.$$

By squaring the desired inequality, we get

$$Aq^3+C \ge 3Bq,$$

where

$$A=(9r-1)^2, \quad B=18r(9r-1)(3r+1), \quad C=27r(27r^3+99r^2+r+1).$$

Using the AM-GM inequality, we have

$$Aq^3+C = Aq^3+\frac{C}{2}+\frac{C}{2} \ge 3\sqrt[3]{Aq^3\left(\frac{C}{2}\right)^2};$$

thus, it is enough to show that

$$AC^2 \ge 4B^3,$$

which is equivalent to

$$(27r^3+99r^2+r+1)^2 \ge 32r(9r-1)(3r+1)^3,$$

$$729r^6 - 2430r^5 + 2943r^4 - 1476r^3 + 199r^2 + 34r + 1 \ge 0,$$
$$(r-1)^2(27r^2 - 18r - 1)^2 \ge 0.$$

The equality holds for $a = b = c = 1$, and also for $r = \frac{3+2\sqrt{3}}{9}$ and $q = 1 + \sqrt{3}$; that is, when a, b, c are the roots of the equation

$$x^3 - 3x^2 + (1+\sqrt{3})x - \frac{3+2\sqrt{3}}{9} = 0$$

such that $a \ge b \ge c$ or $b \ge c \ge a$ or $c \ge a \ge b$.

□

P 1.140. *Let a, b, c be nonnegative real numbers such that*

$$2(a^2 + b^2 + c^2) = 5(ab + bc + ca).$$

Prove that

(a) $$8(a^4 + b^4 + c^4) \ge 17(a^3b + b^3c + c^3a);$$

(b) $$16(a^4 + b^4 + c^4) \ge 34(a^3b + b^3c + c^3a) + 81abc(a + b + c).$$

(Vasile Cîrtoaje, 2011)

Solution. (a) Let

$$x = a^2 + b^2 + c^2, \quad y = ab + bc + ca, \qquad 2x = 5y.$$

Since the equality holds for $a = 2$, $b = 1$, $c = 0$ (when $abc = 0$), we will use the inequality

$$a^2b^2 + b^2c^2 + c^2a^2 \le y^2$$

to get

$$a^4 + b^4 + c^4 = x^2 - 2(a^2b^2 + b^2c^2 + c^2a^2) \ge x^2 - 2y^2,$$

hence

$$a^4 + b^4 + c^4 \ge x^2 - 2y^2 = \frac{17}{144}(2x + y)^2.$$

Therefore, it suffices to prove that

$$(2x + y)^2 \ge 18(a^3b + b^3c + c^3a).$$

We will show that this inequality holds for all nonnegative real numbers a, b, c. Assume that $a = \max\{a, b, c\}$. There are two possible cases: $a \ge b \ge c$ and $a \ge c \ge b$.

Case 1: $a \ge b \ge c$. Using the AM-GM inequality gives

$$2(a^3b + b^3c + c^3a) \le 2ab(a^2 + bc + c^2) \le \left[\frac{2ab + (a^2 + bc + c^2)}{2}\right]^2.$$

Therefore, it suffices to show that

$$2x + y \geq \frac{3}{2}(2ab + a^2 + bc + c^2),$$

which is equivalent to the obvious inequality

$$(a - 2b)^2 + c(2a - b + c) \geq 0.$$

Case 2: $a \geq c \geq b$. Since

$$ab^3 + bc^3 + ca^3 - (a^3b + b^3c + c^3a) = (a + b + c)(a - b)(b - c)(c - a) \geq 0,$$

we have

$$2(a^3b + b^3c + c^3a) \leq (a^3b + b^3c + c^3a) + (ab^3 + bc^3 + ca^3) \leq xy.$$

Thus, it suffices to prove that

$$(2x + y)^2 \geq 9xy.$$

Since $x \geq y$, we get

$$(2x + y)^2 - 9xy = (x - y)(4x - y) \geq 0.$$

Thus, the proof is completed. The equality holds for $a = 2b$ and $c = 0$ (or any cyclic permutation).

(b) For $a = b = c = 0$, the inequality is trivial. Otherwise, let us denote

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc,$$

and write the inequality as

$$16\sum a^4 \geq 17\sum ab(a^2 + b^2) + 17\left(\sum a^3b - \sum ab^3\right) + 81abc\sum a.$$

Due to homogeneity, we may assume that $p = 3$, which involves $q = 2$. Since

$$abc\sum a = 3r,$$

$$\begin{aligned}\sum a^4 &= \left(\sum a^2\right)^2 - 2\sum a^2b^2\\ &= (p^2 - 2q)^2 - 2q^2 + 4pr = 17 + 12r,\end{aligned}$$

$$\begin{aligned}\sum ab(a^2 + b^2) &= \left(\sum ab\right)\left(\sum a^2\right) - abc\sum a\\ &= q(p^2 - 2q) - pr = 10 - 3r,\end{aligned}$$

$$\begin{aligned}\sum a^3b - \sum ab^3 &= -p(a - b)(b - c)(c - a)\\ &\leq p\sqrt{(a - b)^2(b - c)^2(c - a)^2}\\ &= p\sqrt{p^2q^2 - 4q^3 + 2p(9q - 2p^2)r - 27r^2}\\ &= 3\sqrt{4 - 27r^2},\end{aligned}$$

it suffices to prove that

$$16(17+12r) \ge 17(10-3r) + 51\sqrt{4-27r^2} + 243r,$$

which is equivalent to the obvious inequality

$$2 \ge \sqrt{4-27r^2}.$$

The equality holds for $a = 2b$ and $c = 0$ (or any cyclic permutation).

□

P 1.141. *Let a, b, c be nonnegative real numbers such that*

$$2(a^2+b^2+c^2) = 5(ab+bc+ca).$$

Prove that

(a) $$2(a^3b+b^3c+c^3a) \ge a^2b^2+b^2c^2+c^2a^2+abc(a+b+c);$$

(b) $$11(a^4+b^4+c^4) \ge 17(a^3b+b^3c+c^3a)+129abc(a+b+c);$$

(c) $$a^3b+b^3c+c^3a \le \frac{14+\sqrt{102}}{8}(a^2b^2+b^2c^2+c^2a^2).$$

Solution. For $a = b = c = 0$, the inequalities are trivial. Otherwise, let us denote

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc.$$

Due to homogeneity, we may assume that $p = 3$, which involves $q = 2$. From

$$\begin{aligned} \left|\sum a^3b - \sum ab^3\right| &= |-p(a-b)(b-c)(c-a)| \\ &= p\sqrt{(a-b)^2(b-c)^2(c-a)^2} \\ &= p\sqrt{p^2q^2-4q^3+2p(9q-2p^2)r-27r^2} \\ &= 3\sqrt{4-27r^2}, \end{aligned}$$

it follows that

$$-3\sqrt{4-27r^2} \le \sum a^3b - \sum ab^3 \le 3\sqrt{4-27r^2}.$$

In addition, we have

$$abc\sum a = 3r,$$

$$\sum a^2b^2 = q^2 - 2pr = 4 - 6r,$$

$$\sum ab(a^2+b^2) = q(p^2-2q) - pr = 10 - 3r,$$

$$\sum a^4 = p^4 - 4p^2q + 2q^2 + 4pr = 17 + 12r.$$

(a) Write the inequality as

$$\sum ab(a^2+b^2) + \left(\sum a^3b - \sum ab^3\right) \ge \sum a^2b^2 + abc\sum a.$$

It suffices to prove that

$$10 - 3r - 3\sqrt{4-27r^2} \ge 4 - 6r + 3r,$$

which is equivalent to the obvious inequality

$$2 \ge \sqrt{4-27r^2}.$$

The equality holds for $a = 0$ and $2b = c$ (or any cyclic permutation).

(b) Write the inequality as

$$22\sum a^4 \ge 17\sum ab(a^2+b^2) + 17\left(\sum a^3b - \sum ab^3\right) + 258abc\sum a.$$

It suffices to prove that

$$22(17+12r) \ge 17(10-3r) + 51\sqrt{4-27r^2} + 774r$$

for $0 \le r \le \dfrac{2}{3\sqrt{3}}$. Write this inequality as

$$4 - 9r \ge \sqrt{4-27r^2}.$$

We have $4 - 9r \ge 4 - 2\sqrt{3} > 0$. By squaring, the inequality becomes

$$(4-9r)^2 \ge 4 - 27r^2,$$

$$(3r-1)^2 \ge 0.$$

For $p = 3$, the equality holds when $q = 2$, $r = \dfrac{1}{3}$ and $(a-b)(b-c)(c-a) \le 0$. In general, the equality holds when a, b, c are proportional to the roots of the equation

$$3x^3 - 9x^2 + 6x - 1 = 0$$

and satisfy

$$(a-b)(b-c)(c-a) \le 0.$$

This occurs when (*Wolfgang Berndt*)

$$a\sin^2\frac{\pi}{9} = b\sin^2\frac{2\pi}{9} = c\sin^2\frac{4\pi}{9}.$$

(c) Write the inequality as

$$\sum ab(a^2+b^2)+\left(\sum a^3b-\sum ab^3\right)\le k(a^2b^2+b^2c^2+c^2a^2),$$

where

$$k=\frac{14+\sqrt{102}}{4}.$$

It suffices to prove that

$$10-3r+3\sqrt{4-27r^2}\le k(4-6r),$$

where $r\le \dfrac{2}{3\sqrt{3}}$. Write this inequality as

$$3\sqrt{4-27r^2}\le 4k-10-3(2k-1)r.$$

We have

$$4k-10-3(2k-1)r\ge 4k-10-\frac{2(2k-1)}{\sqrt{3}}=4\left(1-\frac{1}{\sqrt{3}}\right)k-10+\frac{2}{\sqrt{3}}>0.$$

By squaring, the inequality becomes

$$9(4-27r^2)\le[4k-10-3(2k-1)r]^2,$$

which is equivalent to

$$(r-k_1)^2\ge 0,$$

where

$$k_1=\frac{2}{129}\sqrt{\frac{787+72\sqrt{102}}{3}}\approx 0.3483.$$

For $p=3$, the equality holds when $q=2$, $r=k_1$ and $(a-b)(b-c)(c-a)\le 0$. In general, the equality holds when a,b,c are proportional to the roots of the equation

$$x^3-3x^2+2x-k_1=0$$

and satisfy

$$(a-b)(b-c)(c-a)\le 0.$$

□

P 1.142. *If a,b,c are real numbers such that*

$$a^3b+b^3c+c^3a\le 0,$$

then

$$a^2+b^2+c^2\ge k(ab+bc+ca),$$

where

$$k=\frac{1+\sqrt{21+8\sqrt{7}}}{2}\approx 3.7468.$$

(Vasile Cîrtoaje, 2012)

Solution. Let us denote

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

If $p = 0$, then

$$3(ab + bc + ca) \leq (a + b + c)^2 = 0,$$

hence

$$a^2 + b^2 + c^2 \geq 0 \geq k(ab + bc + ca).$$

Consider now that $p \neq 0$ and use the contradiction method. It suffices to prove that

$$a^2 + b^2 + c^2 < k(ab + bc + ca)$$

involves

$$a^3b + b^3c + c^3a > 0.$$

Since the statement remains unchanged by replacing a, b, c with $-a, -b, -c$, respectively, we may consider that $p > 0$. In addition, due to homogeneity, we may assume that $p = 1$. From the hypothesis $a^2 + b^2 + c^2 < k(ab + bc + ca)$, we get

$$q > \frac{1}{k+2}.$$

Write the desired inequality as

$$\sum ab(a^2 + b^2) + \sum a^3b - \sum ab^3 > 0.$$

Since

$$\sum ab(a^2 + b^2) = q(p^2 - 2q) - pr = q - 2q^2 - r$$

and

$$\sum a^3b - \sum ab^3 = -p(a-b)(b-c)(c-a) \geq -p\sqrt{(a-b)^2(b-c)^2(c-a)^2}$$

$$= -p\sqrt{p^2q^2 - 4q^3 + 2p(9q - 2p^2)r - 27r^2} = -\sqrt{q^2 - 4q^3 + 2(9q-2)r - 27r^2},$$

it suffices to prove that

$$q - 2q^2 - r > \sqrt{q^2 - 4q^3 + 2(9q-2)r - 27r^2}.$$

From $p^2 \geq 3q$, we get

$$\frac{1}{k+2} < q \leq \frac{1}{3},$$

and from $q^2 \geq 3pr$, we get $r \leq q^2/3$; therefore,

$$q - 2q^2 - r \geq q - 2q^2 - \frac{q^2}{3} = q\left(1 - \frac{7q}{3}\right) > 0.$$

By squaring, the desired inequality can be restated as

$$(q - 2q^2 - r)^2 > q^2 - 4q^3 + 2(9q-2)r - 27r^2,$$

$$7r^2 + (1-5q+q^2)r + q^4 > 0.$$

This is true if the discriminant

$$D = (1-5q+q^2)^2 - 28q^4 = [1-5q+(1+2\sqrt{7})q^2][1-5q+(1-2\sqrt{7})q^2]$$

is negative. Since

$$1-5q+(1+2\sqrt{7})q^2 = \left(1-\frac{5q}{2}\right)^2 + \frac{8\sqrt{7}-21}{4}q^2 > 0,$$

we only need to show that $f(q) > 0$, where

$$f(q) = (2\sqrt{7}-1)q^2 + 5q - 1.$$

Since $q > \dfrac{1}{k+2}$, we have

$$f(q) > \frac{2\sqrt{7}-1}{(k+2)^2} + \frac{5}{k+2} - 1 = 0.$$

For $p = 1$, the equality holds when $(a-b)(b-c)(c-a) > 0$ and

$$q = \frac{1}{k+2}, \quad r = \frac{-q^2}{\sqrt{7}} = -\frac{1}{\sqrt{7}(k+2)^2}.$$

In general, the equality holds when a, b, c are proportional to the roots of the equation

$$w^3 - w^2 + \frac{1}{k+2}w + \frac{1}{\sqrt{7}(k+2)^2} = 0$$

and satisfy $(a-b)(b-c)(c-a) > 0$.

□

P 1.143. *If a, b, c are real numbers such that*

$$a^3b + b^3c + c^3a \geq 0,$$

then

$$a^2 + b^2 + c^2 + k(ab + bc + ca) \geq 0,$$

where

$$k = \frac{-1+\sqrt{21+8\sqrt{7}}}{2} \approx 2.7468.$$

(Vasile Cîrtoaje, 2012)

Solution. Let us denote

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

At least two of a, b, c have the same sign; let b and c be these numbers. If $p = 0$, then the hypothesis $a^3b + b^3c + c^3a \geq 0$ can be written as

$$-(b + c)^3 b + b^3 c - c^3(b + c) \geq 0.$$

Clearly, this inequality is satisfied only for $a = b = c = 0$, when the desired inequality is trivial. Consider further that $p \neq 0$ and use the contradiction method. It suffices to prove that

$$a^2 + b^2 + c^2 + k(ab + bc + ca) < 0$$

involves

$$a^3b + b^3c + c^3a < 0.$$

Since the statement remains unchanged by replacing a, b, c with $-a, -b, -c$, respectively, we may consider $p > 0$. In addition, due to homogeneity, we may assume $p = 1$. From the hypothesis $a^2 + b^2 + c^2 + k(ab + bc + ca) < 0$, we get

$$q < \frac{-1}{k-2} \approx -1.339.$$

Write the desired inequality as

$$\sum ab(a^2 + b^2) + \sum a^3b - \sum ab^3 < 0,$$

Since

$$\sum ab(a^2 + b^2) = q(p^2 - 2q) - pr = q - 2q^2 - r$$

and

$$\sum a^3b - \sum ab^3 = -p(a - b)(b - c)(c - a) \leq p\sqrt{(a - b)^2(b - c)^2(c - a)^2}$$
$$= p\sqrt{p^2q^2 - 4q^3 + 2p(9q - 2p^2)r - 27r^2} = \sqrt{q^2 - 4q^3 + 2(9q - 2)r - 27r^2},$$

it suffices to prove that

$$\sqrt{q^2 - 4q^3 + 2(9q - 2)r - 27r^2} < r + 2q^2 - q.$$

Since $q < -1$, we have

$$\frac{1 - 2q}{3} > 1,$$

hence

$$r^2 = a^2b^2c^2 \leq \left(\frac{a^2 + b^2 + c^2}{3}\right)^3 = \left(\frac{1 - 2q}{3}\right)^3 < \left(\frac{1 - 2q}{3}\right)^4,$$

which implies

$$r > -\left(\frac{1 - 2q}{3}\right)^2.$$

Therefore,

$$r + 2q^2 - q > -\left(\frac{1-2q}{3}\right)^2 + 2q^2 - q = \frac{(2q-1)(7q+1)}{9} > 0.$$

By squaring, the desired inequality becomes

$$q^2 - 4q^3 + 2(9q-2)r - 27r^2 < (r + 2q^2 - q)^2,$$

$$7r^2 + (1 - 5q + q^2)r + q^4 > 0.$$

This is true if the discriminant

$$D = (1 - 5q + q^2)^2 - 28q^4 = [1 - 5q + (1 + 2\sqrt{7})q^2][1 - 5q + (1 - 2\sqrt{7})q^2]$$

is negative. Since

$$1 - 5q + (1 + 2\sqrt{7})q^2 > 0,$$

we only need to show that $f(q) > 0$, where

$$f(q) = (2\sqrt{7} - 1)q^2 + 5q - 1.$$

Since the derivative

$$f'(q) = 2(2\sqrt{7} - 1)q + 5 < 2(2\sqrt{7} - 1)(-1) + 5 = 7 - 4\sqrt{7} < 0,$$

$f(q)$ is strictly decreasing, hence

$$f(q) > f\left(\frac{-1}{k-2}\right) = 0.$$

For $p = 1$, the equality holds when $(a-b)(b-c)(c-a) < 0$ and

$$q = \frac{-1}{k-2}, \quad r = \frac{-q^2}{\sqrt{7}} = \frac{-1}{\sqrt{7}(k-2)^2}.$$

In general, the equality holds when a, b, c are proportional to the roots of the equation

$$w^3 - w^2 - \frac{1}{k-2}w + \frac{1}{\sqrt{7}(k-2)^2} = 0$$

and satisfy $(a-b)(b-c)(c-a) < 0$.

□

P 1.144. *If a, b, c are real numbers such that*

$$k(a^2 + b^2 + c^2) = ab + bc + ca, \qquad k \in \left(\frac{-1}{2}, 1\right),$$

then

$$\alpha_k \le \frac{a^3b+b^3c+c^3}{(a^2+b^2+c^2)^2} \le \beta_k,$$

where

$$27\alpha_k = 1+13k-5k^2-2(1-k)(1+2k)\sqrt{\frac{7(1-k)}{1+2k}},$$

$$27\beta_k = 1+13k-5k^2+2(1-k)(1+2k)\sqrt{\frac{7(1-k)}{1+2k}}.$$

(Vasile Cîrtoaje, 2012)

Solution. Let us denote

$$p=a+b+c, \quad q=ab+bc+ca, \quad r=abc.$$

The case $p=0$ is not possible because $p=0$ and $k(a^2+b^2+c^2)=ab+bc+ca$ lead to

$$ab+bc+ca=0,$$

$$a(b+c)+bc=0,$$

$$-(b+c)^2+bc=0,$$

$$b^2+bc+c^2=0,$$

which involves $a=b=c=0$. Consider further that $p\neq 0$. Since the statement remains unchanged by replacing a,b,c with $-a,-b,-c$, respectively, it suffices to consider the case $p>0$. In addition, due to homogeneity, we may assume $p=1$, which implies

$$q=\frac{k}{1+2k}.$$

(a) Write the desired left inequality as

$$2\alpha_k(a^2+b^2+c^2)^2 \le \sum ab(a^2+b^2)+\left(\sum a^3b-\sum ab^3\right).$$

Since

$$\sum a^2 = p^2-2q = 1-2q,$$

$$\sum ab(a^2+b^2) = q(p^2-2q)-pr = q-2q^2-r,$$

$$\sum a^3b-\sum ab^3 = -p(a-b)(b-c)(c-a) \ge -p\sqrt{(a-b)^2(b-c)^2(c-a)^2}$$

$$= -p\sqrt{\frac{4(p^2-3q)^3-(2p^3-9pq+27r)^2}{27}} = -\sqrt{\frac{4(1-3q)^3-(2-9q+27r)^2}{27}},$$

it suffices to prove that

$$2\alpha_k(1-2q)^2 \le q-2q^2-r-\sqrt{\frac{4(1-3q)^3-(2-9q+27r)^2}{27}}.$$

Applying Lemma below for

$$\alpha = \frac{1}{\sqrt{27}}, \quad \beta = \frac{-1}{27}, \quad x = 2(1-3q)\sqrt{1-3q}, \quad y = 2-9q+27r,$$

we get

$$\sqrt{\frac{4(1-3q)^3-(2-9q+27r)^2}{27}} + r + \frac{2-9q}{27} \le \frac{4(1-3q)\sqrt{7(1-3q)}}{27},$$

with equality for

$$(1-3q)\sqrt{\frac{1-3q}{7}} - 2 + 9q - 27r = 0.$$

Thus, it suffices to show that

$$2\alpha_k(1-2q)^2 \le q - 2q^2 + \frac{2-9q}{27} - \frac{4(1-3q)\sqrt{7(1-3q)}}{27},$$

which is equivalent to

$$27\alpha_k \le 1 + 13k - 5k^2 - 2(1-k)(1+2k)\sqrt{\frac{7(1-k)}{1+2k}}.$$

For $p = 1$, the equality holds when $(a-b)(b-c)(c-a) \ge 0$, $q = k/(1+2k)$ and

$$27r = (1-3q)\sqrt{\frac{1-3q}{7}} - 2 + 9q = \frac{r_1}{1+2k},$$

where

$$r_1 = 5k - 2 + (1-k)\sqrt{\frac{1-k}{7(1+2k)}}.$$

Therefore, the equality holds when a, b, c are proportional to the roots of the equation

$$w^3 - w^2 + \frac{k}{1+2k}w - \frac{r_1}{27(1+2k)} = 0$$

and satisfy $(a-b)(b-c)(c-a) \ge 0$.

(b) Write the desired right inequality as

$$2\beta_k(a^2+b^2+c^2)^2 \ge \sum ab(a^2+b^2) + \left(\sum a^3b - \sum ab^3\right).$$

Since

$$\sum a^2 = p^2 - 2q = 1 - 2q,$$

$$\sum ab(a^2+b^2) = q(p^2-2q) - pr = q - 2q^2 - r,$$

$$\sum a^3b - \sum ab^3 = -p(a-b)(b-c)(c-a) \le p\sqrt{(a-b)^2(b-c)^2(c-a)^2}$$

$$= p\sqrt{\frac{4(p^2-3q)^3-(2p^3-9pq+27r)^2}{27}} = \sqrt{\frac{4(1-3q)^3-(2-9q+27r)^2}{27}},$$

it suffices to prove that

$$2\beta_k(1-2q)^2 \ge q-2q^2-r+\sqrt{\frac{4(1-3q)^3-(2-9q+27r)^2}{27}}.$$

Applying Lemma below for

$$\alpha = \frac{1}{\sqrt{27}}, \quad \beta = \frac{1}{27}, \quad x = 2(1-3q)\sqrt{1-3q}, \quad y = 2-9q+27r,$$

we get

$$\sqrt{\frac{4(1-3q)^3-(2-9q+27r)^2}{27}} - r - \frac{2-9q}{27} \le \frac{4(1-3q)\sqrt{7(1-3q)}}{27},$$

with equality for

$$(1-3q)\sqrt{\frac{1-3q}{7}} + 2 - 9q + 27r = 0.$$

Thus, it suffices to show that

$$2\beta_k(1-2q)^2 \ge q-2q^2+\frac{2-9q}{27}+\frac{4(1-3q)\sqrt{7(1-3q)}}{27},$$

which is equivalent to

$$27\beta_k \ge 1+13k-5k^2+2(1-k)(1+2k)\sqrt{\frac{7(1-k)}{1+2k}}.$$

For $p=1$, the equality holds when $(a-b)(b-c)(c-a) \le 0$, $q = k/(1+2k)$ and

$$27r = 9q-2-(1-3q)\sqrt{\frac{1-3q}{7}} = \frac{r_0}{1+2k},$$

where

$$r_0 = 5k-2-(1-k)\sqrt{\frac{1-k}{7(1+2k)}}.$$

Therefore, the equality holds when a, b, c are proportional to the roots of the equation

$$w^3 - w^2 + \frac{k}{1+2k}w - \frac{r_0}{27(1+2k)} = 0$$

and satisfy $(a-b)(b-c)(c-a) \le 0$.

Lemma. *If α, β, x, y are real numbers such that*

$$\alpha \ge 0, \qquad x \ge 0, \qquad x^2 \ge y^2,$$

then

$$\alpha\sqrt{x^2-y^2} \le x\sqrt{\alpha^2+\beta^2}+\beta y,$$

with equality if and only if

$$\beta x + y\sqrt{\alpha^2+\beta^2} = 0.$$

Proof. Since

$$x\sqrt{\alpha^2+\beta^2}+\beta y \ge |\beta|x+\beta y \ge |\beta||y|+\beta y \ge 0,$$

we can write the inequality as

$$\alpha^2(x^2-y^2) \le \left(x\sqrt{\alpha^2+\beta^2}+\beta y\right)^2,$$

which is equivalent to

$$\left(\beta x + y\sqrt{\alpha^2+\beta^2}\right)^2 \ge 0.$$

□

P 1.145. *If a, b, c are positive real numbers such that $a+b+c=3$, then*

$$\frac{a^2}{4a+b^2}+\frac{b^2}{4b+c^2}+\frac{c^2}{4c+a^2} \ge \frac{3}{5}.$$

(Michael Rozenberg, 2008)

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a^2}{4a+b^2} \ge \frac{\left[\sum a(2a+c)\right]^2}{\sum(4a+b^2)(2a+c)^2} = \frac{(2\sum a^2+\sum ab)^2}{\sum(4a+b^2)(2a+c)^2}.$$

Therefore, it suffices to show that

$$5\left(2\sum a^2+\sum ab\right)^2 \ge 3\sum(4a+b^2)(2a+c)^2,$$

which is equivalent to the homogeneous inequalities

$$5\left(2\sum a^2+\sum ab\right)^2 \ge \sum[4a(a+b+c)+3b^2](2a+c)^2,$$

$$5\left(2\sum a^2+\sum ab\right)^2 \ge \sum(4a^2+3b^2+4ab+4ac)(4a^2+c^2+4ac),$$

$$2\sum a^4+5\sum a^2b^2 \ge abc\sum a+6\sum ab^3.$$

Using Vasc's inequality

$$3\sum ab^3 \le \left(\sum a^2\right)^2,$$

it is enough to prove the symmetric inequality

$$2\sum a^4 + 5\sum a^2b^2 \ge abc\sum a + 2\left(\sum a^2\right)^2,$$

which is equivalent to the well-known inequality

$$\sum a^2b^2 \ge abc\sum a.$$

The equality holds for $a = b = c = 1$.

□

P 1.146. *If a, b, c are positive real numbers, then*

$$\frac{a^2+bc}{a+b} + \frac{b^2+ca}{b+c} + \frac{c^2+ab}{c+a} \le \frac{(a+b+c)^3}{3(ab+bc+ca)}.$$

(Michael Rozenberg, 2013)

Solution *(by Manlio Marangelli).* Write the inequality as

$$\sum\left(\frac{a^2+bc}{a+b} - a\right) \le \frac{(a+b+c)^3}{3(ab+bc+ca)} - (a+b+c),$$

$$\sum\frac{b(c-a)}{a+b} \le \frac{(a+b+c)^3}{3(ab+bc+ca)} - (a+b+c),$$

$$\frac{\sum b(c^2-a^2)(b+c)}{(a+b)(b+c)(c+a)} \le \frac{(a+b+c)^3}{3(ab+bc+ca)} - (a+b+c),$$

$$\frac{3\sum ab^3 - 3abc\sum a}{(a+b)(b+c)(c+a)} \le \frac{(a+b+c)^3}{ab+bc+ca} - 3(a+b+c).$$

By the known Vasc's inequality

$$3\sum ab^3 \le \left(\sum a^2\right)^2,$$

it suffices to prove the symmetric inequality

$$\frac{(\sum a^2)^2 - 3abc\sum a}{(a+b)(b+c)(c+a)} \le \frac{(a+b+c)^3}{ab+bc+ca} - 3(a+b+c).$$

Using the notation

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc,$$

this inequality can be written as

$$\frac{(p^2-2q)^2 - 3pr}{pq - r} \le \frac{p^3}{q} - 3p,$$

which is equivalent to

$$q^2(p^2-4q)-(p^2-6q)pr \ge 0.$$

Case 1: $p^2-6q \ge 0$. Since $3pr \le q^2$, we have

$$q^2(p^2-4q)-(p^2-6q)pr \ge q^2(p^2-4q)-\frac{q^2(p^2-6q)}{3}=\frac{2q^2(p^2-3q)}{3} \ge 0.$$

Case 2: $p^2-6q \le 0$. Using Schur's inequality of fourth degree

$$6pr \ge (p^2-q)(4q-p^2),$$

we get

$$q^2(p^2-4q)-(p^2-6q)pr \ge q^2(p^2-4q)-\frac{(p^2-6q)(p^2-q)(4q-p^2)}{6}$$
$$=\frac{(p^2-3q)(p^2-4q)^2}{6} \ge 0.$$

The equality holds for $a=b=c=1$.

□

P 1.147. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$\sqrt{ab^2+bc^2}+\sqrt{bc^2+ca^2}+\sqrt{ca^2+ab^2} \le 3\sqrt{2}.$$

(Nguyen Van Quy, 2013)

Solution *(by Michael Rozenberg).* By the Cauchy-Schwarz inequality, we have

$$\left(\sum\sqrt{ab^2+bc^2}\right)^2 \le \sum\frac{ab+c^2}{a+c}\cdot\sum b(a+c).$$

Therefore, it suffices to show that

$$\sum\frac{ab+c^2}{a+c} \le \frac{9}{ab+bc+ca},$$

which is equivalent to the homogeneous inequality

$$\sum\frac{ab+c^2}{a+c} \le \frac{(a+b+c)^3}{3(ab+bc+ca)},$$

which is the inequality from the previous P 1.146. The equality holds for $a=b=c=1$.

□

P 1.148. *If a, b, c are positive real numbers such that $a^5 + b^5 + c^5 = 3$, then*

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq 3.$$

Solution. We will prove the inequality under the more general condition $a^m + b^m + c^m = 3$, where $0 < m \leq 21/4$. First, write the inequality in the homogeneous form

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq 3\left(\frac{a^m + b^m + c^m}{3}\right)^{1/m}.$$

By the Power Mean inequality, we have

$$\left(\frac{a^m + b^m + c^m}{3}\right)^{1/m} \leq \left(\frac{a^{21/4} + b^{21/4} + c^{21/4}}{3}\right)^{4/21}.$$

Thus, it suffices to show that

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq 3\left(\frac{a^{21/4} + b^{21/4} + c^{21/4}}{3}\right)^{4/21}.$$

By the known Vasc's inequality in P 1.125, namely

$$(x^2 + y^2 + z^2)^2 \geq 3(x^3y + y^3z + z^3x), \qquad x, y, z \in \mathbb{R},$$

we have

$$\left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}\right)^2 \geq 3\left(\frac{a^3}{\sqrt{bc}} + \frac{b^3}{\sqrt{ca}} + \frac{c^3}{\sqrt{ab}}\right).$$

Therefore, it suffices to prove the symmetric inequality

$$\frac{a^3}{\sqrt{bc}} + \frac{b^3}{\sqrt{ca}} + \frac{c^3}{\sqrt{ab}} \geq 3\left(\frac{a^{21/4} + b^{21/4} + c^{21/4}}{3}\right)^{8/21},$$

which is equivalent to

$$\left(\frac{\frac{a^3}{\sqrt{bc}} + \frac{b^3}{\sqrt{ca}} + \frac{c^3}{\sqrt{ab}}}{3}\right)^{21/4} \geq 3\left(\frac{a^{21/4} + b^{21/4} + c^{21/4}}{3}\right)^2,$$

Setting

$$a = x^{2/7}, \qquad b = y^{2/7}, \qquad c = z^{2/7}, \qquad x, y, z > 0,$$

the inequality becomes

$$\left(\frac{x + y + z}{3}\right)^{21/4} \geq 3(xyz)^{3/4}\left(\frac{x^{3/2} + y^{3/2} + z^{3/2}}{3}\right)^2.$$

By the Cauchy-Schwarz inequality, we have

$$(x+y+z)(x^2+y^2+z^2) \ge (x^{3/2}+y^{3/2}+z^{3/2})^2.$$

Thus, it is enough to prove that

$$\left(\frac{x+y+z}{3}\right)^{17/4} \ge \frac{1}{3}(xyz)^{3/4}(x^2+y^2+z^2).$$

Due to homogeneity, we may assume that $x+y+z=3$, when the inequality becomes

$$(xyz)^{3/4}(x^2+y^2+z^2) \le 3.$$

Since

$$\frac{3}{4} > \frac{1}{\sqrt{2}},$$

this inequality follows from the inequality in P 2.89 from Volume 2:

$$(xyz)^k(x^2+y^2+z^2) \le 3, \qquad k \ge \frac{1}{\sqrt{2}}$$

The proof is completed. The equality holds for $a=b=c=1$.

□

P 1.149. *Let $P(a,b,c)$ be a cyclic homogeneous polynomial of degree three. The inequality*

$$P(a,b,c) \ge 0$$

holds for all $a,b,c \ge 0$ if and only if the following two conditions are fulfilled:

(a) $P(1,1,1) \ge 0$;

(b) $P(0,b,c) \ge 0$ *for all* $b,c \ge 0$.

(Pham Kim Hung, 2007)

Solution. The conditions (a) and (b) are clearly necessary. Therefore, we will prove further that these conditions are also sufficient to have $P(a,b,c) \ge 0$. The polynomial $P(a,b,c)$ has the general form

$$P(a,b,c) = A(a^3+b^3+c^3) + B(a^2b+b^2c+c^2a) + C(ab^2+bc^2+ca^2) + 3Dabc.$$

Since

$$P(1,1,1) = 3(A+B+C+D), \quad P(0,1,1) = 2A+B+C, \quad P(0,0,1) = A,$$

the conditions (a) and (b) involves

$$A+B+C+D \ge 0, \quad 2A+B+C \ge 0, \quad A \ge 0.$$

Assume that $a = \min\{a, b, c\}$, and denote

$$b = a + p, \quad c = a + q, \quad p, q \ge 0.$$

For fixed p and q, define the function

$$f(a) = P(a, a + p, a + q), \quad a \ge 0.$$

Since

$$a' = b' = c' = 1,$$

we have the derivative

$$\begin{aligned} f'(a) &= 3A(a^2 + b^2 + c^2) + (B + C)(a + b + c)^2 + 3D(ab + bc + ca) \\ &= (3A + B + C)(a^2 + b^2 + c^2) + (2B + 2C + 3D)(ab + bc + ca) \\ &= (3A + B + C)(a^2 + b^2 + c^2 - ab - bc - ca) + 3(A + B + C + D)(ab + bc + ca). \end{aligned}$$

Because $f'(a) \ge 0$, f is increasing, hence $f(a) \ge f(0)$, which is equivalent to

$$P(a, b, c) \ge P(0, p, q) = P(0, b, c).$$

According to the condition (b), we have $P(0, b, c) \ge 0$, hence $P(a, b, c) \ge 0$.

Remark 1. From the proof of P 1.149, the following statement follows:

- *Let $P(a, b, c)$ be a cyclic homogeneous polynomial of degree three. The inequality*

$$P(a, b, c) \ge 0$$

holds for all nonnegative real numbers a, b, c satisfying

$$a \le b \le c$$

if and only if $P(1, 1, 1) \ge 0$ and $P(0, b, c) \ge 0$ for all $0 \le b \le c$.

Remark 2. From P 1.149, using the substitution

$$a = y + z, \quad b = z + x, \quad c = x + y, \qquad x, y, z \ge 0,$$

we get the following statement:

- *Let $P(a, b, c)$ be a cyclic homogeneous polynomial of degree three, where a, b, c are the lengths of the sides of a triangle. The inequality*

$$P(a, b, c) \ge 0$$

holds if and only if $P(1, 1, 1) \ge 0$ and $P(b + c, b, c) \ge 0$ for all $b, c \ge 0$.

□

P 1.150. *If a, b, c are nonnegative real numbers such that $a+b+c=3$, then*

$$8(a^2b+b^2c+c^2a)+9 \ge 11(ab+bc+ca).$$

Solution. Write the inequality in the homogeneous form $P(a,b,c) \ge 0$, where

$$P(a,b,c) = 24(a^2b+b^2c+c^2a)+(a+b+c)^3-11(a+b+c)(ab+bc+ca).$$

According to P 1.149, it suffices to show that $P(1,1,1) \ge 0$ and $P(0,b,c) \ge 0$ for all $b,c \ge 0$. We have

$$P(1,1,1)=0$$

and

$$\begin{aligned} P(0,b,c) &= 24b^2c+(b+c)^3-11bc(b+c) \\ &= b^3+16b^2c-8bc^2+c^3 \\ &\ge 16b^2c-8bc^2+c^3 = c(4b-c)^2 \ge 0. \end{aligned}$$

The equality holds for $a=b=c=1$.

□

P 1.151. *If a, b, c are nonnegative real numbers such that $a+b+c=6$, then*

$$a^3+b^3+c^3+8(a^2b+b^2c+c^2a) \ge 166.$$

(Vasile Cîrtoaje, 2010)

Solution. Write the inequality in the homogeneous form $P(a,b,c) \ge 0$, where

$$P(a,b,c) = a^3+b^3+c^3+8(a^2b+b^2c+c^2a)-166\left(\frac{a+b+c}{6}\right)^3.$$

According to P 1.149, it suffices to show that $P(1,1,1) \ge 0$ and $P(0,b,c) \ge 0$ for all $b,c \ge 0$. We have

$$P(1,1,1) = 27-\frac{83}{4} = \frac{25}{4} > 0$$

and

$$\begin{aligned} P(0,b,c) &= b^3+c^3+8b^2c-\frac{83}{108}(b+c)^3 \\ &= \frac{1}{108}(25b^3+615b^2c-249bc^2+25c^3) \\ &= \frac{1}{108}(5b-c)^2(b+25c) \ge 0. \end{aligned}$$

The equality holds for $a=0$, $b=1$, $c=5$ (or any cyclic permutation).

□

P 1.152. *If a, b, c are positive real numbers such that $abc = 1$, then*

$$a^{\frac{a}{b}} b^{\frac{b}{c}} c^{\frac{c}{a}} \geq 1.$$

(Vasile Cîrtoaje, 2004)

Solution. Write the inequality as

$$\frac{a}{b}\ln a + \frac{b}{c}\ln b + \frac{c}{a}\ln c \geq 0.$$

Since the function $f(x) = x\ln x$ is convex for $x > 0$, Jensen's inequality gives

$$\frac{1}{b}\cdot a\ln a + \frac{1}{c}\cdot b\ln b + \frac{1}{a}\cdot c\ln c \geq \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)\cdot \ln \frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{\frac{1}{b} + \frac{1}{c} + \frac{1}{a}}.$$

Since

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3,$$

it remains to show that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{1}{b} + \frac{1}{c} + \frac{1}{a},$$

which is the inequality from P 1.51, (a). The equality occurs for $a = b = c = 1$.

□

P 1.153. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + 7 \geq \frac{17}{3}\left(\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a}\right).$$

(Vasile Cîrtoaje, 2007)

Solution. Write the inequality as $P(a, b, c) \geq 0$, where

$$\begin{aligned} P(a,b,c) &= \sum (3a - 17b)(a+b)(a+c) + 21(a+b)(b+c)(c+a) \\ &= 3(a^3 + b^3 + c^3) - 10(a^2b + b^2c + c^2a) + 7(ab^2 + bc^2 + ca^2). \end{aligned}$$

According to P 1.149, it suffices to show that $P(1, 1, 1) \geq 0$ and $P(0, b, c) \geq 0$ for all $b, c \geq 0$. We have $P(1, 1, 1) = 0$ and

$$P(0, b, c) = 3(b^3 + c^3) - 10b^2c + 7bc^2.$$

Consider the nontrivial case $b, c > 0$. Setting $c = 1$, we need to show that $f(b) \geq 0$, where

$$f(b) = 3b^3 - 10b^2 + 7b + 3.$$

Case 1: $b \ge 3$. We have

$$f(b) > 3b^3 - 10b^2 + 7b = (b-1)(3b-7) > 0.$$

Case 2: $2 \le b \le 3$. We have

$$f(b) \ge 3b^3 - 10b^2 + 8b = b(b-2)(3b-4) \ge 0.$$

Case 3: $0 < b \le 2$. We have

$$f(b) \ge 3b^3 - 10b^2 + 7b + 1.5b = b(3b^2 - 10b + 8.5) > 3b(b - 5/3)^2 \ge 0.$$

The equality holds for $a = b = c$.

□

P 1.154. *Let a, b, c be nonnegative real numbers, no two of which are zero. If $0 \le k \le 5$, then*

$$\frac{ka+b}{a+c} + \frac{kb+c}{b+a} + \frac{kc+a}{c+b} \ge \frac{3}{2}(k+1).$$

(Vasile Cîrtoaje, 2007)

First Solution. Write the inequality as

$$\frac{b}{a+c} + \frac{c}{b+a} + \frac{a}{c+b} - \frac{3}{2} + k\left(\frac{a}{a+c} + \frac{b}{b+a} + \frac{c}{c+b} - \frac{3}{2}\right) \ge 0.$$

Since

$$\frac{b}{a+c} + \frac{c}{b+a} + \frac{a}{c+b} - \frac{3}{2} \ge 0,$$

it suffices to consider the case $k = 5$, when the inequality can be written as follows:

$$\sum (5a+b)(b+a)(c+b) \ge 9(a+c)(b+a)(c+b),$$

$$2\sum ab^2 + \sum a^3 \ge 3\sum a^2b,$$

$$2\sum ab^2 + \frac{4}{3}\sum a^3 - \frac{1}{3}\sum b^3 \ge 3\sum a^2b,$$

$$\sum (6ab^2 + 4a^3 - b^3 - 9a^2b) \ge 0,$$

$$(a-b)^2(4a-b) + (b-c)^2(4b-c) + (c-a)^2(4c-a) \ge 0.$$

Assume that $a = \min\{a, b, c\}$, and use the substitution

$$b = a + p, \quad c = a + q, \quad p, q \ge 0.$$

The inequality becomes

$$p^2(3a-p) + (p-q)^2(3a+4p-q) + q^2(3a+4q) \ge 0,$$

$$2Aa + B \geq 0,$$

where

$$A = p^2 - pq + q^2, \quad B = p^3 - 3p^2q + 2pq^2 + q^3.$$

Since $A \geq 0$, we only need to show that $B \geq 0$. For $q = 0$, we have $B = p^3 \geq 0$, while for $q > 0$, the inequality $B \geq 0$ is equivalent to

$$1 \geq x(x-1)(2-x),$$

where $x = p/q \geq 0$. For the non-trivial case $x \in [1, 2]$, we get this inequality by multiplying the obvious inequalities

$$1 \geq x - 1$$

and

$$1 \geq x(2-x).$$

The proof is completed. The equality holds for $a = b = c$.

Second Solution. We can write the inequality in the form $P(a, b, c) \geq 0$, where $P(a, b, c)$ is a cyclic homogeneous polynomial of degree three. According to P 1.149, it suffices to show that the desired inequality holds for $a = b = c$, and also for $a = 0$. If $a = 0$, then the inequality becomes

$$x + k + \frac{1}{x} + \frac{k}{1+x} \geq \frac{3}{2}(k+1),$$

$$2(x-1)^2 + x \geq \frac{kx(x-1)}{x+1},$$

where

$$x = \frac{b}{c} > 0.$$

For $0 < x \leq 1$, we have

$$2(x-1)^2 + x > 0 \geq \frac{kx(x-1)}{x+1}.$$

For $1 \leq x \leq 5$, it suffices to consider the case $k = 5$, when the inequality is equivalent to

$$2(x-1)^2 + x \geq \frac{5x(x-1)}{x+1},$$

$$x^3 - 3x^2 + 2x + 1 \geq 0,$$

$$x(x-2)^2 + (x-1)^2 \geq 0.$$

Remark. As in the second solution, we can prove that the inequality in P 1.154 holds for

$$0 \leq k \leq k_0, \qquad k_0 = \sqrt{13 + 16\sqrt{2}} \approx 5.969.$$

For $a = 0$ and $k = k_0$, the inequality becomes

$$2(x-1)^2 + x \geq \frac{kx(x-1)}{x+1}, \qquad x = \frac{b}{c} > 0,$$

$$2x^3-(k_0+1)x^2+(k_0-1)x+2\ge 0,$$

$$(x-x_0)^2\left(x+\frac{1}{x_0^2}\right)\ge 0,$$

where

$$x_0=\frac{1+\sqrt{2}+\sqrt{2\sqrt{2}-1}}{2}\approx 1.883.$$

If $k=k_0$, then the equality holds for $a=b=c$, and also for $a=0$ and $\frac{b}{c}+\frac{c}{b}=1+\sqrt{2}$ (or any cyclic permutation).

□

P 1.155. *Let a,b,c be nonnegative real numbers. Prove that*

(a) if $k\le 1-\frac{2}{5\sqrt{5}}$, then

$$\frac{ka+b}{2a+b+c}+\frac{kb+c}{a+2b+c}+\frac{kc+a}{a+b+2c}\ge\frac{3}{4}(k+1).$$

(b) if $k\ge 1+\frac{2}{5\sqrt{5}}$, then

$$\frac{ka+b}{2a+b+c}+\frac{kb+c}{a+2b+c}+\frac{kc+a}{a+b+2c}\le\frac{3}{4}(k+1).$$

(Vasile Cîrtoaje, 2007)

Solution. (a) Write the inequality in the form $P(a,b,c)\ge 0$, where $P(a,b,c)$ is a cyclic homogeneous polynomial of degree three. According to P 1.149, it suffices to show that the desired inequality holds for $a=b=c$, and also for $a=0$. For $a=0$, the inequality becomes

$$\frac{x}{x+1}+\frac{kx+1}{2x+1}+\frac{k}{x+2}\ge\frac{3}{4}(k+1),$$

$$(x+2)(2x^2-x+1)\ge k(x+1)(2x^2-x+2),$$

where

$$x=\frac{b}{c}\ge 0.$$

It suffices to consider the case $k=1-\frac{2}{5\sqrt{5}}$, when the inequality is equivalent to

$$(x-x_0)^2\left(x+\frac{2}{5\sqrt{5}\,x_0^2}\right)\ge 0,$$

where

$$x_0=\frac{3-\sqrt{5}}{2}.$$

The equality holds for $a=b=c$. If $k=1-\dfrac{2}{5\sqrt{5}}$, then the equality also holds for $a=0$ and $\dfrac{b}{c}+\dfrac{c}{b}=3$ (or any cyclic permutation).

(b) According to P 1.149, it suffices to show that the desired inequality holds for $a=b=c$, and also for $a=0$. If $a=0$, then the inequality becomes

$$\frac{x}{x+1}+\frac{kx+1}{2x+1}+\frac{k}{x+2}\le\frac{3}{4}(k+1),$$

$$(x+2)(2x^2-x+1)\le k(x+1)(2x^2-x+2),$$

where

$$x=\frac{b}{c}\ge 0.$$

It suffices to consider the case $k=1+\dfrac{2}{5\sqrt{5}}$, when the inequality is equivalent to

$$(x-x_1)^2\left(x+\frac{2}{5\sqrt{5}\,x_1^2}\right)\ge 0,$$

where

$$x_1=\frac{3+\sqrt{5}}{2}.$$

The equality holds for $a=b=c$. If $k=1+\dfrac{2}{5\sqrt{5}}$, then the equality also holds for $a=0$ and $\dfrac{b}{c}+\dfrac{c}{b}=3$ (or any cyclic permutation).

□

P 1.156. *Let a,b,c be nonnegative real numbers, no two of which are zero. If $k\le\dfrac{23}{8}$, then*

$$\frac{ka+b}{2a+c}+\frac{kb+c}{2b+a}+\frac{kc+a}{2c+b}\ge k+1.$$

(Vasile Cîrtoaje, 2007)

Solution. We can write the inequality in the form $P(a,b,c)\ge 0$, where $P(a,b,c)$ is a cyclic homogeneous polynomial of degree three. According to P 1.149, it suffices to show that the desired inequality holds for $a=b=c$, and also for $a=0$. For $a=0$, the inequality becomes

$$x+\frac{k}{2}+\frac{1}{2x}+\frac{k}{2+x}\ge k+1,$$

$$x^2+(x-1)^2\ge\frac{kx^2}{x+2},$$

where

$$x = \frac{b}{c} > 0.$$

It suffices to consider that $k = 23/8$, when the inequality is equivalent to

$$2x^2 - 2x + 1 \geq \frac{23x^2}{8(x+2)},$$

$$16x^3 - 7x^2 - 24x + 16 \geq 0,$$

$$16x(x-1)^2 + (5x-4)^2 \geq 0.$$

The equality holds for $a = b = c$.

Remark. For $k = 2$, we get the inequality in P 1.21.

□

P 1.157. *If a, b, c are positive real numbers such that $a \leq b \leq c$, then*

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 3 \geq 2\left(\frac{a+b}{b+c} + \frac{b+c}{c+a} + \frac{c+a}{a+b}\right).$$

Solution. Write the inequality as follows:

$$\sum \left(\frac{a}{b} - 1\right) \geq 2\sum \left(\frac{b+c}{c+a} - 1\right),$$

$$\sum (a-b)\left(\frac{1}{b} + \frac{2}{c+a}\right) \geq 0,$$

$$(a-b)\left(\frac{1}{b} + \frac{2}{c+a}\right) + (b-c)\left(\frac{1}{c} + \frac{2}{a+b}\right) + [(c-b)+(b-a)]\left(\frac{1}{a} + \frac{2}{b+c}\right) \geq 0,$$

$$(b-a)\left(\frac{1}{a} + \frac{2}{b+c} - \frac{1}{b} - \frac{2}{c+a}\right) + (c-b)\left(\frac{1}{a} + \frac{2}{b+c} - \frac{1}{c} - \frac{2}{a+b}\right) \geq 0,$$

$$(b-a)^2\left[\frac{1}{ab} - \frac{2}{(b+c)(c+a)}\right] + (c-b)(c-a)\left[\frac{1}{ac} - \frac{2}{(b+c)(a+b)}\right] \geq 0.$$

The inequality is true since

$$\frac{1}{ab} - \frac{2}{(b+c)(c+a)} = \frac{c(a+b+c)-ab}{(b+c)(c+a)} > \frac{a(c-b)}{(b+c)(c+a)} \geq 0$$

and

$$\frac{1}{ac} - \frac{2}{(b+c)(a+b)} = \frac{b(a+b+c)-ac}{(b+c)(a+b)} > \frac{c(b-a))}{(b+c)(a+b)} \geq 0.$$

The equality holds for $a = b = c$.

□

P 1.158. *If $a \ge b \ge c \ge 0$, then*

$$\frac{3a+b}{2a+c}+\frac{3b+c}{2b+a}+\frac{3c+a}{2c+b} \ge 4.$$

(Vasile Cîrtoaje, 2007)

First Solution. Write the inequality as follows:

$$\sum(3a+b)(2b+a)(2c+b) \ge 4(2a+c)(2b+a)(2c+b),$$

$$2\sum a^3+13\sum ab^2+7\sum a^2b+42abc \ge 4(4\sum ab^2+2\sum a^2b+9abc),$$

$$2\sum a^3+6abc \ge 3\sum ab^2+\sum a^2b,$$

$$2E(a,b,c) \ge F(a,b,c),$$

where

$$E(a,b,c)=\sum a^3+3abc-\sum ab^2-\sum a^2b,$$

$$F(a,b,c)=\sum ab^2-\sum a^2b.$$

The inequality is true since $E(a,b,c) \ge 0$ (by Schur's inequality of degree three) and

$$F(a,b,c)=(a-b)(b-c)(c-a) \le 0.$$

The equality holds for $a=b=c$, and also for $a=b$ and $c=0$.

Second Solution. Denote

$$x=a-b \ge 0, \qquad y=b-c \ge 0,$$

and write the inequality as follows

$$\sum\left(\frac{3a+b}{2a+c}-\frac{4}{3}\right) \ge 0,$$

$$\sum \frac{a+3b-4c}{2a+c} \ge 0,$$

$$\frac{a+3b-4c}{2a+c}+\frac{b+3c-4a}{2b+a}+\frac{c+3a-4b}{2c+b} \ge 0,$$

$$\frac{x+4y}{2a+c}-\frac{4x+3y}{2b+a}+\frac{3x-y}{2c+b} \ge 0,$$

$$xA+yB \ge 0,$$

where

$$A = \frac{1}{2a+c} - \frac{4}{2b+a} + \frac{3}{2c+b}$$
$$= \left(\frac{1}{2a+c} - \frac{1}{2b+a}\right) + 3\left(\frac{1}{2c+b} - \frac{1}{2b+a}\right)$$
$$= \frac{-x+y}{(2a+c)(2b+a)} + \frac{3(x+2y)}{(2b+a)(2c+b)}$$

and

$$B = \frac{4}{2a+c} - \frac{3}{2b+a} - \frac{1}{2c+b}$$
$$= 3\left(\frac{1}{2a+c} - \frac{1}{2b+a}\right) + \left(\frac{1}{2a+c} - \frac{1}{2c+b}\right)$$
$$= \frac{3(-x+y)}{(2a+c)(2b+a)} - \frac{2x+y}{(2a+c)(2c+b)}.$$

Thus, the inequality is equivalent to

$$x[(-x+y)(2c+b) + 3(x+2y)(2a+c) + y[3(-x+y)(2c+b) - (2x+y)(2b+a)] \geq 0,$$

$$x^2(6a-b+c) + xy(10a-6b+2c) - y^2(a-b-6c) \geq 0,$$

It suffices to show that

$$xy(10a-6b+2c) - y^2(a-b-6c) \geq 0,$$

which is true is

$$x(10a-6b+2c) - y(a-b-6c) \geq 0.$$

We have

$$x(10a-6b+2c) - y(a-b-6c) = x(10x+4y+6c) - y(x-6c)$$
$$= 10x^2 + 3xy + 6c(x+y) \geq 0.$$

Third Solution. According to Remark 1 from P 1.149, it suffices to prove that the inequality holds for $c = 0$ and $a \geq b$; that is, to show that

$$\frac{3}{2} + \frac{1}{2x} + \frac{3}{2+x} + x \geq 4,$$

where

$$x = \frac{a}{b} \geq 1.$$

The inequality is equivalent to

$$2x^3 - x^2 - 3x + 2 \geq 0,$$

$$(x-1)(2x^2 + x - 2) \geq 0.$$

□

P 1.159. *If* $a \ge b \ge c \ge 0$ *and* $ab + bc + ca = 2$, *then*

$$\sqrt{a+ab} + \sqrt{b+bc} + \sqrt{c+ca} \ge 3.$$

(KaiRain, 2020)

Solution. Consider the main case $a \ge b \ge c$ and show that

$$\sqrt{a+ab} + \sqrt{b+bc} + \sqrt{c+ca} \ge 3.$$

For $c = 0$, we need to show that $ab = 2$ involves

$$\sqrt{a+ab} + \sqrt{b} \ge 3,$$

that is

$$\sqrt{a+2} + \sqrt{\frac{2}{a}} \ge 3.$$

Denoting $x = \sqrt{\dfrac{a}{2}}$, we need to show that

$$\sqrt{2x^2+2} \ge 3 - \frac{1}{x}.$$

This is true if

$$2(x^2+1) \ge \left(3 - \frac{1}{x}\right)^2$$

for $x \ge 1/3$, which is equivalent to the obvious inequality

$$(x-1)^2(2x^2+4x-1) \ge 0.$$

Using this result, it suffices to show that

$$\sqrt{a+ab} + \sqrt{b+bc} + \sqrt{c+ca} \ge \sqrt{a+2} + \sqrt{\frac{2}{a}},$$

that is equivalent to

$$\sqrt{c+ca} \ge \sqrt{a+2} - \sqrt{a+ab} + \sqrt{\frac{2}{a}} - \sqrt{b+bc},$$

$$\sqrt{c+ca} \ge \frac{2-ab}{\sqrt{a+2}+\sqrt{a+ab}} + \frac{2-ab-abc}{\sqrt{2a}+a\sqrt{b+bc}},$$

$$\sqrt{c+ca} \ge \frac{c(a+b)}{\sqrt{a+2}+\sqrt{a+ab}} + \frac{c(a+b-ab)}{\sqrt{2a}+a\sqrt{b+bc}}.$$

So, we need to show that

$$\sqrt{1+a} \ge \frac{\sqrt{c}(a+b)}{\sqrt{a+2}+\sqrt{a+ab}} + \frac{\sqrt{c}(a+b-ab)}{\sqrt{2a}+a\sqrt{b+bc}}.$$

We get this inequality by summing the inequalities

$$\frac{\sqrt{1+a}}{2} \ge \frac{\sqrt{c}(a+b)}{\sqrt{a+2}+\sqrt{a+ab}}, \qquad \frac{\sqrt{1+a}}{2} \ge \frac{\sqrt{c}(a+b-ab)}{\sqrt{2a}+a\sqrt{b+bc}}.$$

From $ab+bc+ca=2$, it follows $\frac{2}{3} \le ab \le 2$ and $b \le \sqrt{2}$. Since

$$\sqrt{a+ab} \le \sqrt{a+2}$$

and

$$a\sqrt{b} \le \sqrt{2a}, \qquad a\sqrt{b} \le a\sqrt{b+bc},$$

it suffice to prove the inequalities

$$\sqrt{1+a} \ge \frac{\sqrt{c}(a+b)}{\sqrt{a+ab}}, \qquad \sqrt{1+a} \ge \frac{\sqrt{c}(a+b-ab)}{a\sqrt{b}}.$$

By squaring, the first inequality becomes

$$a(1+a)(1+b) \ge c(a+b)^2,$$

$$a(1+a)(1+b) \ge (a+b)(2-ab).$$

Since $2a \ge a+b$, it suffices to show that

$$(1+a)(1+b) \ge 2(2-ab),$$

that is

$$a+b+3ab \ge 3.$$

Indeed, we have

$$a+b+3ab \ge 2\sqrt{ab}+3ab \ge 2\sqrt{\frac{2}{3}}+2 > 3.$$

Since $\sqrt{b} \ge \sqrt{c}$, the second inequality is true if

$$a\sqrt{1+a} \ge a+b-ab,$$

that is

$$a(\sqrt{1+a}-1) \ge b(1-a).$$

For the nontrivial case $a \le 1$, it suffices to show that

$$a(\sqrt{1+a}-1) \ge a(1-a),$$

that is

$$\sqrt{1+a}+a \ge 2.$$

Since $3a^2 \ge ab+bc+ca = 2$, we have

$$\sqrt{1+a}+a \ge \sqrt{1+\sqrt{\frac{2}{3}}}+\sqrt{\frac{2}{3}} > 2.$$

The inequality is an equality for $a = 2$, $b = 1$, $c = 0$.

Remark. The following sharper inequality holds in the same conditions:

$$\sqrt{a+ab}+\sqrt{b}+\sqrt{c} \ge 3,$$

with equality for $a = 2$, $b = 1$, $c = 0$.

For fixed b, according to the relation $ab+bc+ca = 2$, we may consider that a is a function of c. Differentiating this equation, we get

$$a' = -\frac{a+b}{b+c},$$

$$a'' = \frac{(a+b+(b-c)a'}{(a+c)^2} = \frac{(a+b)(a-b+2c)}{(a+c)^3}.$$

Write the required inequality as $f(c) \ge 0$, where

$$f(c) = \sqrt{a+ab}+\sqrt{b}+\sqrt{c}-3, \qquad c \in [0,b].$$

We have

$$f'(c) = \frac{a'\sqrt{1+b}}{2\sqrt{a}}+\frac{1}{2\sqrt{c}},$$

$$f''(c) = \frac{(2aa''-(a')^2)\sqrt{1+b}}{4a^{3/2}}-\frac{1}{4c^{3/2}}$$

$$= \frac{(a+b)(a^2+3ac-3ab-bc)\sqrt{1+b}}{4a^{3/2}(a+c)^3}-\frac{1}{4c^{3/2}}.$$

Since

$$a^2+3ac-3ab-bc = a^2-3a(b-c)-bc < a^2,$$

we have

$$f''(c) < \frac{(a+b)\sqrt{a(1+b)}}{4(a+c)^3}-\frac{1}{4c^{3/2}}.$$

From $b^2 \le ab \le ab+bc+ca = 2$, we get $b \le \sqrt{2}$, $\sqrt{1+b} < 4$, hence

$$f''(c) < \frac{(a+b)\sqrt{a}}{(a+c)^3}-\frac{1}{4c^{3/2}} \le 2\left(\frac{\sqrt{a}}{a+c}\right)^3-\frac{1}{4(\sqrt{c})^3} \le 0.$$

Since f is concave and $0 \le c \le b$, it is enough to show that $f(0) \ge 0$ (for $c = 0$ and $ab = 2$) and $f(b) \ge 0$ (for $c = b$ and $2ab+b^2 = 2$). We have

$$f(0) = \sqrt{\frac{2+2b}{b}}+\sqrt{b}-3 = \frac{(1-\sqrt{b})^2(2+4\sqrt{b}-b)}{\sqrt{b(2+2b)}-b\sqrt{b}+3b} \ge 0.$$

For $c=b$, when $2=2ab+b^2\ge 3b^2$, hence $b\le\sqrt{\dfrac{2}{3}}$, we have

$$f(b)=\sqrt{\frac{(1+b)(2-b^2)}{2b}}+2\sqrt{b}-3=\frac{A}{\sqrt{2b(1+b)(2-b^2)}-4b\sqrt{b}+6b},$$

where, for $x=\sqrt{b}\le\sqrt[4]{\dfrac{2}{3}}<1$,

$$A=(1+x^2)(2-x^4)-2x^2(3-2x)^2=(1-x)(2+2x-14x^2+10x^3+x^4+x^5).$$

Since

$$2+2x-14x^2+10x^3+x^4+x^5=2-13x^2+13x^3+(1-x)^2x(2+3x+x^2)$$

$$>2+13x^3-13x^2=2+\frac{13x^3}{2}+\frac{13x^3}{2}-13x^2$$

$$\ge 3\sqrt[3]{2\cdot\frac{13x^3}{2}\cdot\frac{13x^3}{2}}-13x^2=\left(3\sqrt[3]{\frac{169}{2}}-13\right)x^2>0,$$

we have $A>0$, hence $f(b)>0$.

□

P 1.160. *If $a\ge b\ge c$ are nonnegative numbers such that $ab+bc+ca=3$, then*

$$\sqrt{a+2ab}+\sqrt{b+2bc}+\sqrt{c+2ca}\ \ge 4.$$

(Vasile Cîrtoaje, 2020)

Solution. We will prove the sharper inequality

$$\sqrt{a+2ab}+\sqrt{b+bc}+\sqrt{c+ca}\ \ge 4.$$

For $c=0$, we need to show that $ab=3$ involves

$$\sqrt{a+2ab}+\sqrt{b}\ge 4,$$

that is

$$\sqrt{a+6}+\sqrt{\frac{3}{a}}\ge 4.$$

It is easy to show that this inequality is true for all $a>0$. Using this result, it suffices to show that

$$\sqrt{a+2ab}+\sqrt{b+bc}+\sqrt{c+ca}\ge\sqrt{a+6}+\sqrt{\frac{3}{a}},$$

that is equivalent to

$$\sqrt{c+ca} \ge \sqrt{a+6} - \sqrt{a+2ab} + \sqrt{\frac{3}{a}} - \sqrt{b+bc},$$

$$\sqrt{c+ca} \ge \frac{2(3-ab}{\sqrt{a+6}+\sqrt{a+2ab}} + \frac{3-ab-abc}{\sqrt{3a}+a\sqrt{b+bc}},$$

$$\sqrt{c+ca} \ge \frac{2c(a+b)}{\sqrt{a+6}+\sqrt{a+2ab}} + \frac{c(a+b-ab)}{\sqrt{3a}+a\sqrt{b+bc}}.$$

So, we need to show that

$$\sqrt{1+a} \ge \frac{2\sqrt{c}(a+b)}{\sqrt{a+6}+\sqrt{a+2ab}} + \frac{\sqrt{c}(a+b-ab)}{\sqrt{3a}+a\sqrt{b+bc}}.$$

We get this inequality by summing the inequalities

$$k\sqrt{1+a} \ge \frac{2\sqrt{c}(a+b)}{\sqrt{a+6}+\sqrt{a+2ab}}, \qquad (1-k)\sqrt{1+a} \ge \frac{\sqrt{c}(a+b-ab)}{\sqrt{3a}+a\sqrt{b+bc}},$$

where

$$k = \sqrt{\frac{2}{3}}.$$

From $ab+bc+ca=3$, it follows $1 \le ab \le 3$ and $b \le \sqrt{3}$. Since

$$\sqrt{a+2ab} \le \sqrt{a+6},$$

the first inequality is true if

$$k\sqrt{1+a} \ge \frac{\sqrt{c}(a+b)}{\sqrt{a+2ab}},$$

that is

$$2a(1+a)(1+2b) \ge 3c(a+b)^2,$$

$$2a(1+a)(1+2b) \ge 3(3-ab)(a+b).$$

Since $2a \ge a+b$, it suffices to show that

$$(1+a)(1+2b) \ge 3(3-ab),$$

that is

$$(5b+1)a+2b \ge 8.$$

For $a \ge b \ge 1$, this inequality is obvious. For $0 \le b \le 1$, from

$$b \ge c = \frac{3-ab}{a+b}$$

we get

$$a \ge \frac{3-b^2}{2b}.$$

Therefore,

$$(5b+1)a+2b-8 \geq \frac{(5b+1)(3-b^2)}{2b}+2b$$

$$=\frac{3-b+3b^2-5b^3}{2b}=\frac{(1-b)(3+2b+5b^2)}{2b} \geq 0.$$

Since $1-k > \dfrac{1}{4}$, the second inequality is true if

$$\sqrt{1+a} \geq \frac{4\sqrt{c}(a+b-ab)}{\sqrt{3a}+a\sqrt{b+bc}},$$

Consider the nontrivial case $a+b-ab \geq 0$, and claim that $\sqrt{3a} \geq a\sqrt{b+bc}$, which is equivalent to $3 \geq ab+abc$. Indeed, we have

$$3-ab-abc=3-ab-\frac{ab(3-ab)}{a+b}=\frac{(3-ab)(a+b-ab)}{a+b} \geq 0.$$

Thus, it suffices to show that

$$\sqrt{1+a} \geq \frac{2\sqrt{c}(a+b-ab)}{a\sqrt{b+bc}}.$$

Since

$$\frac{a+b-ab}{a} \leq 1,$$

it suffices to show that

$$\sqrt{1+a} \geq 2\sqrt{\frac{c}{b(1+c)}},$$

that is

$$b(1+a)(1+c) \geq 4c.$$

Since $ab \geq 1$, we have

$$b(1+a) \geq b+1 \geq c+1,$$

therefore,

$$b(1+a)(1+c)-4c \geq (1+c)^2-4c=(1-c)^2 \geq 0.$$

The inequality is an equality for $a=3$, $b=1$, $c=0$.

□

P 1.161. *If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then*

$$\sqrt{a+3b}+\sqrt{b+3c}+\sqrt{c+3a} \geq 6.$$

Solution. Use the substitution

$$\sqrt{a+3b}=2x, \qquad \sqrt{b+3c}=2y, \qquad \sqrt{c+3a}=2z,$$

which yields

$$a=\frac{x^2-3y^2+9z^2}{7}, \qquad a=\frac{y^2-3z^2+9x^2}{7}, \qquad a=\frac{z^2-3x^2+9y^2}{7},$$

$$ab+bc+ca=\frac{-3(x^4+y^4+z^4)+10(x^2y^2+y^2z^2+z^2x^2)}{7}.$$

So, we need to show that

$$x+y+z\ge 3$$

for

$$3(x^4+y^4+z^4)+21=10(x^2y^2+y^2z^2+z^2x^2).$$

By the contradiction method, we need to prove that

$$x+y+z<3$$

involves

$$3(x^4+y^4+z^4)+21>10(x^2y^2+y^2z^2+z^2x^2).$$

It suffices to prove the homogeneous inequality $f(x,y,z)\ge 0$, where

$$f(x,y,z)=81(x^4+y^4+z^4)+7(x+y+z)^4-270(x^2y^2+y^2z^2+z^2x^2).$$

According to P 3.68 from Volume 1, it is enough to show that $f(0,y,z)\ge 0$ and $f(x,1,1)\ge 0$ for $x,y,z\ge 0$. We have

$$f(0,y,z)=81(y^4+z^4)+7(y+z)^4-270y^2z^2$$

$$\ge 162y^2z^2+112y^2z^2-270y^2z^2=4y^2z^2\ge 0$$

and

$$f(x,1,1)=81(x^4+2)+7(x+2)^4-540x^2=4(22x^4+14x^3-93x^2+56x+1)$$

$$=(x-1)^2(22x^2+58x+1)\ge 0.$$

The equality occurs for $a=b=c=1$.

□

P 1.162. *If a,b,c are the lengths of the sides of a triangle, then*

$$10\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)>9\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

Solution. According to Remark 2 from the proof of P 1.149, it suffices to show that $P(1,1,1) \ge 0$ and $P(b+c,b,c) \ge 0$ for $b,c \ge 0$, where

$$P(a,b,c) = 10(ab^2 + bc^2 + ca^2) - 9(a^2b + b^2c + c^2a).$$

We have $P(1,1,1) = 3 > 0$ and

$$P(b+c,b,c) = b^3 - 7b^2c + 12bc^2 + c^3.$$

We need to show that

$$x^3 - 7x^2 + 12x + 1 > 0,$$

where $x = b/c$, $x > 0$. For $x \in (0,3] \cup [4,\infty)$, we have

$$x^3 - 7x^2 + 12x + 1 > x^3 - 7x^2 + 12x = x(3-x)(4-x) \ge 0.$$

For $x \in (3,4)$, we have

$$x^3 - 7x^2 + 12x + 1 > x^3 - 7x^2 + 12x + \frac{x}{4} = \frac{x(2x-7)^2}{4} \ge 0.$$

□

P 1.163. *If a,b,c are the lengths of the sides of a triangle, then*

$$\frac{a}{3a+b-c} + \frac{b}{3b+c-a} + \frac{c}{3c+a-b} \ge 1.$$

Solution. Write the inequality as follows:

$$\sum \left(\frac{a}{3a+b-c} - \frac{1}{4} \right) \ge \frac{1}{4},$$

$$\sum \frac{a-b+c}{3a+b-c} \ge 1.$$

Applying the Cauchy-Schwarz inequality, we get

$$\sum \frac{a-b+c}{3a+b-c} \ge \frac{\left[\sum(a-b+c)\right]^2}{\sum(a-b+c)(3a+b-c)} = \frac{\left(\sum a\right)^2}{\sum a^2 + 2\sum ab} = 1.$$

The equality holds for $a = b = c$.

□

P 1.164. *If a, b, c are the lengths of the sides of a triangle, then*

$$\frac{a^2-b^2}{a^2+bc}+\frac{b^2-c^2}{b^2+ca}+\frac{c^2-a^2}{c^2+ab} \le 0.$$

(Vasile Cîrtoaje, 2007)

First Solution. Suppose that $a = \max\{a, b, c\}$. Since

$$c^2-a^2 = -(a^2-b^2)-(b^2-c^2),$$

the inequality can be written as follows:

$$(a^2-b^2)\left(\frac{1}{a^2+bc}-\frac{1}{c^2+ab}\right)+(b^2-c^2)\left(\frac{1}{b^2+ca}-\frac{1}{c^2+ab}\right) \le 0,$$

$$-\frac{(a^2-b^2)(a-c)(a-b+c)}{a^2+bc}-\frac{(b^2-c^2)(b-c)(b+c-a)}{a^2+bc} \le 0.$$

The equality holds for an equilateral triangle, and also for a degenerate triangle having a side equal to zero.

Second Solution. The sequences

$$\{a^2, \quad b^2, \quad c^2\}$$

and

$$\left\{\frac{1}{a^2+bc}, \quad \frac{1}{b^2+ca}, \quad \frac{1}{c^2+ab}\right\}$$

are reversely ordered. Indeed, if $a \ge b \ge c$, then

$$\frac{1}{a^2+bc} \le \frac{1}{b^2+ca} \le \frac{1}{c^2+ab},$$

because

$$\frac{1}{b^2+ca}-\frac{1}{a^2+bc} = \frac{(a-b)(a+b-c)}{(b^2+ca)(a^2+bc)} \ge 0,$$

$$\frac{1}{c^2+ab}-\frac{1}{b^2+ca} = \frac{(b-c)(b+c-a)}{(c^2+ab)(b^2+ca)} \ge 0.$$

Then, by the rearrangement inequality, we have

$$a^2\cdot\frac{1}{a^2+bc}+b^2\cdot\frac{1}{b^2+ca}+c^2\cdot\frac{1}{c^2+ab} \le$$

$$\le b^2\cdot\frac{1}{a^2+bc}+c^2\cdot\frac{1}{b^2+ca}+a^2\cdot\frac{1}{c^2+ab},$$

which is the desired inequality.

□

P 1.165. *If a, b, c are the lengths of the sides of a triangle, then*

$$a^2(a+b)(b-c)+b^2(b+c)(c-a)+c^2(c+a)(a-b) \ge 0.$$

(Vasile Cîrtoaje, 2006)

First Solution. Assume that

$$a = \max\{a, b, c\},$$

use the substitution

$$a = x+p+q, \quad b = x+p, \quad c = x+q, \qquad x, p, q \ge 0,$$

and write the inequality as

$$a^2b^2+b^2c^2+c^2a^2-abc(a+b+c) \ge ab^3+bc^3+ca^3-a^3b-b^3c-c^3a,$$

$$a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2 \ge 2(a+b+c)(a-b)(b-c)(c-a),$$

$$(x+p+q)^2(p-q)^2+(x+p)^2p^2+(x+q)^2q^2 \ge 2(3x+2p+2q)pq(q-p),$$

which is equivalent to

$$Ax^2+2Bx+C \ge 0,$$

where

$$A = p^2-pq+q^2 \ge 0,$$

$$B = p^3+q(p-q)^2 \ge 0,$$

$$C = (p^2+pq-q^2)^2 \ge 0.$$

The equality holds for an equilateral triangle, and also for a degenerate triangle with

$$\frac{a}{2} = \frac{b}{1+\sqrt{5}} = \frac{c}{3+\sqrt{5}}$$

(or any cyclic permutation).

Second Solution. Using the substitution

$$x = \sqrt{\frac{ca}{b}}, \quad y = \sqrt{\frac{ab}{c}}, \quad z = \sqrt{\frac{bc}{a}},$$

we can write the inequality as follows:

$$b^2c^2+c^2a^2+a^2b^2 \ge ab(b^2+c^2-a^2)+bc(c^2+a^2-b^2)+ca(a^2+b^2-c^2),$$

$$\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c} \ge 2b\cos A+2c\cos B+2a\cos C,$$

$$x^2+y^2+z^2 \ge 2yz\cos A+2zx\cos B+2xy\cos C,$$

$$(x-y\cos C-z\cos B)^2+(y\sin C-z\sin B)^2 \ge 0.$$

□

P 1.166. *If a, b, c are the lengths of the sides of a triangle, then*

$$a^2b + b^2c + c^2a \ge \sqrt{abc(a+b+c)(a^2+b^2+c^2)}.$$

(Vasile Cîrtoaje and Vo Quoc Ba Can, 2005)

Solution. Without loss of generality, assume that b is between a and c; that is

$$(b-a)(b-c) \le 0.$$

First Solution. By the AM-GM inequality, we have

$$4abc(a+b+c)(a^2+b^2+c^2) \le [ac(a+b+c) + b(a^2+b^2+c^2)]^2.$$

Thus, we only need to show that

$$2(a^2b+b^2c+c^2a) \ge ac(a+b+c) + b(a^2+b^2+c^2),$$

which is equivalent to

$$b[a^2-(b-c)^2] - ac(a+b-c) \ge 0,$$

$$(a+b-c)(a-b)(b-c) \ge 0.$$

The equality holds for an equilateral triangle, and also for a degenerate triangle with

$$c = a+b, \qquad b^3 = a^2(a+b)$$

(or any cyclic permutation).

Second Solution. The desired inequality is equivalent to $D \ge 0$, where D is the discriminant of the quadratic function

$$f(x) = (a^2+b^2+c^2)x^2 - 2(a^2b+b^2c+c^2a)x + abc(a+b+c).$$

For the sake of contradiction, assume that $D < 0$ for some a, b, c. Then, $f(x) > 0$ for all real x. This is not true because

$$f(b) = b(b-a)(b-c)(a+b-c) \le 0.$$

□

P 1.167. *If a, b, c are the lengths of the sides of a triangle, then*

$$a^2\left(\frac{b}{c}-1\right) + b^2\left(\frac{c}{a}-1\right) + c^2\left(\frac{a}{b}-1\right) \ge 0.$$

(Vasile Cîrtoaje, Moldova TST, 2006)

First Solution. Using the substitution

$$a = \frac{1}{x}, \quad b = \frac{1}{y}, \quad c = \frac{1}{z},$$

the inequality becomes

$$E(x, y, z) \ge 0,$$

where

$$E(x, y, z) = yz^2(z - y) + zx^2(x - z) + xy^2(y - x).$$

Without loss of generality, assume that

$$a = \min\{a, b, c\}, \qquad x = \max\{x, y, z\}.$$

We will show that

$$E(x, y, z) \ge E(y, y, z) \ge 0.$$

We have

$$\begin{aligned} E(x, y, z) - E(y, y, z) &= z(x^3 - y^3) - z^2(x^2 - y^2) + y^3(x - y) - y^2(x^2 - y^2) \\ &= (x - y)(x - z)(xz + yz - y^2) \ge 0, \end{aligned}$$

because

$$xz + yz - y^2 \ge y(2z - y) = \frac{2b - c}{b^2c} = \frac{(b - a) + (a + b - c)}{b^2c} \ge 0.$$

Also,

$$E(y, y, z) = yz(y - z)^2 \ge 0.$$

The equality holds for $a = b = c$.

Second Solution. Write the inequality as $F(a, b, c) \ge 0$, where

$$F(a, b, c) = a^3b^2 + b^3c^2 + c^3a^2 - abc(a^2 + b^2 + c^2).$$

Since

$$\begin{aligned} 2E(a, b, c) &= \left(\sum a^3b^2 + \sum a^2b^3 - 2abc\sum a^2\right) - \left(\sum a^2b^3 - \sum a^3b^2\right) \\ &= \left(\sum a^3b^2 + \sum a^3c^2 - 2abc\sum a^2\right) - \left(\sum a^2b^3 - \sum a^2c^3\right) \\ &= \sum a^3(b - c)^2 - \sum a^2(b^3 - c^3) \end{aligned}$$

and

$$\sum a^2(b^3 - c^3) = \sum a^2(b - c)^3,$$

we get

$$E(a, b, c) = \sum a^3(b - c)^2 - \sum a^2(b - c)^3 = \sum a^2(b - c)^2(a - b + c) \ge 0.$$

Third Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a^2b}{c} \geq \frac{\left(\sum a^2b\right)^2}{\sum a^2bc}.$$

Therefore, it suffices to show that

$$\left(\sum a^2b\right)^2 \geq abc(a+b+c)(a^2+b^2+c^2),$$

which is the inequality from the previous P 1.166.

□

P 1.168. *If a, b, c are the lengths of the sides of a triangle, then*

(a) $$a^3b + b^3c + c^3a \geq a^2b^2 + b^2c^2 + c^2a^2;$$

(b) $$3(a^3b + b^3c + c^3a) \geq (ab + bc + ca)(a^2 + b^2 + c^2);$$

(c) $$\frac{a^3b + b^3c + c^3}{3} \geq \left(\frac{a+b+c}{3}\right)^4.$$

Solution. (a) ***First Solution.*** Write the inequality as

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

Using the substitution

$$a = y+z, \quad b = z+x, \quad c = x+y, \qquad x, y, z \geq 0,$$

the inequality turns into

$$xy^3 + yz^3 + zx^3 \geq xyz(x+y+z),$$

which follows from the Cauchy-Schwarz inequality

$$(xy^3 + yz^3 + zx^3)(z+x+y) \geq xyz(y+z+x)^2.$$

The equality holds for an equilateral triangle, and also for a degenerate triangle with $a = 0$ and $b = c$ (or any cyclic permutation).

Second Solution. Multiplying by $a+b+c$, the inequality becomes as follows:

$$\sum a^4b + abc\sum a^2 \geq \sum a^2b^3 + abc\sum ab,$$

$$\sum b^4c + abc\sum a^2 \ge \sum b^2c^3 + abc\sum ab,$$
$$\sum \frac{b^3}{a} + \sum a^2 \ge \sum \frac{bc^2}{a} + \sum ab,$$
$$\sum a^2 \ge \sum \frac{b}{a}(c^2 + a^2 - b^2),$$
$$a^2 + b^2 + c^2 \ge 2bc\cos B + 2ca\cos C + 2ab\cos A,$$
$$(a - b\cos A - c\cos C)^2 + (b\sin A - c\sin C)^2 \ge 0.$$

(b) Write the inequality as

$$\sum a^2b(a-b) + \sum b^2(a-b)(a-c) \ge 0.$$

Since $\sum a^2b(a-b) \ge 0$ (according to the inequality in (a)), it suffices to show that

$$\sum b^2(a-b)(a-c) \ge 0.$$

This is a particular case ($x = c$, $y = a$, $z = b$) of the following inequality

$$(x-y)(x-z)a^2 + (y-z)(y-x)b^2 + (z-x)(z-y)c^2 \ge 0,$$

where x, y, z are real numbers. If two of x, y, z are equal, then the inequality is trivial. Otherwise, assume that $x > y > z$ and write the inequality as

$$\frac{a^2}{y-z} + \frac{c^2}{x-y} \ge \frac{b^2}{x-z}.$$

Applying the Cauchy-Schwarz inequality, we get

$$\frac{a^2}{y-z} + \frac{c^2}{x-y} \ge \frac{(a+c)^2}{(y-z)+(x-y)} = \frac{(a+c)^2}{x-z} \ge \frac{b^2}{x-z}.$$

The equality holds for $a = b = c$.

(c) According to the inequality (b), it suffices to show that

$$9(ab+bc+ca)(a^2+b^2+c^2) \ge (a+b+c)^4.$$

This is equivalent to

$$(A-B)(4B-A) \ge 0,$$

where

$$A = a^2+b^2+c^2, \qquad B = ab+bc+ca.$$

Since $A \ge B$ and

$$\begin{aligned} 4B - A &> 2(ab+bc+ca) - a^2 - b^2 - c^2 \\ &= a(2b+2c-a) - (b-c)^2 \\ &\ge a^2 - (b-c)^2 \\ &= (a-b+c)(a+b-c) \ge 0. \end{aligned}$$

the conclusion follows. The equality holds for $a = b = c$.

□

P 1.169. *If a, b, c are the lengths of the sides of a triangle, then*

$$2\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right) \ge \frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}+3.$$

Solution. Write the inequality as follows:

$$\sum \frac{a^2}{b^2} \ge 3+\sum \frac{b^2}{a^2}-\sum \frac{a^2}{b^2},$$

$$\sum \frac{b^2}{c^2} \ge 3+\sum \frac{c^2}{b^2}-\sum \frac{a^2}{b^2},$$

$$\sum \frac{b^2}{c^2} \ge \sum \left(1+\frac{c^2}{b^2}-\frac{a^2}{b^2}\right),$$

$$\sum \frac{b^2}{c^2} \ge 2\sum \frac{c}{b}\cos A.$$

Putting

$$x=\frac{b}{c}, \quad y=\frac{c}{a}, \quad z=\frac{a}{b},$$

we have $xyz=1$ and

$$\frac{c}{b}=\frac{1}{x}=yz, \quad \frac{a}{c}=\frac{1}{y}=zx, \quad \frac{b}{a}=\frac{1}{z}=xy.$$

Therefore, we can write the inequality as

$$x^2+y^2+z^2 \ge 2yz\cos A+2zx\cos B+2xy\cos C,$$

which is equivalent to the obvious inequality

$$(x-y\cos C-z\cos B)^2+(y\sin C-z\sin B)^2 \ge 0.$$

The equality occurs for $a=b=c$.

□

P 1.170. *If a, b, c are the lengths of the sides of a triangle such that $a<b<c$, then*

$$\frac{a^2}{a^2-b^2}+\frac{b^2}{b^2-c^2}+\frac{c^2}{c^2-a^2} \le 0.$$

(Vasile Cîrtoaje, 2003)

Solution. Write the inequality as

$$\frac{a^2}{b^2-a^2}+\frac{b^2}{c^2-b^2}\ge\frac{c^2}{c^2-a^2}.$$

Since $c\le a+b$, it suffices to show that

$$\frac{a^2}{b^2-a^2}+\frac{b^2}{c^2-b^2}\ge\frac{(a+b)^2}{c^2-a^2},$$

which is equivalent to

$$a^2\left(\frac{1}{b^2-a^2}-\frac{1}{c^2-a^2}\right)+b^2\left(\frac{1}{c^2-b^2}-\frac{1}{c^2-a^2}\right)\ge\frac{2ab}{c^2-a^2},$$

$$\frac{a^2(c^2-b^2)}{b^2-a^2}+\frac{b^2(b^2-a^2)}{c^2-b^2}\ge 2ab,$$

$$\left(a\sqrt{\frac{c^2-b^2}{b^2-a^2}}-b\sqrt{\frac{b^2-a^2}{c^2-b^2}}\right)^2\ge 0.$$

The equality occurs for a degenerate triangle with $c=a+b$ and $a=xb$, where $x\approx 0.53209$ is the positive root of the equation $x^3+3x^2-1=0$.

□

P 1.171. *If a,b,c are the lengths of the sides of a triangle, then*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3\ge 2\left(\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\right).$$

(Manlio Marangelli, 2008)

First Solution. Assume that $c=\max\{a,b,c\}$. If $a\le b\le c$, then the inequality follows from P 1.157. Consider further that

$$b\le a\le c.$$

Write the inequality as follows:

$$\sum\left(\frac{a}{b}-1\right)\ge 2\sum\left(\frac{b+c}{c+a}-1\right),$$

$$\sum(a-b)\left(\frac{1}{b}+\frac{2}{c+a}\right)\ge 0,$$

$$(a-b)\left(\frac{1}{b}+\frac{2}{c+a}\right)+[(b-a)+(a-c)]\left(\frac{1}{c}+\frac{2}{a+b}\right)+(c-a)\left(\frac{1}{a}+\frac{2}{b+c}\right)\ge 0,$$

$$(a-b)\left(\frac{1}{b}+\frac{2}{c+a}-\frac{1}{c}-\frac{2}{a+b}\right)+(c-a)\left(\frac{1}{a}+\frac{2}{b+c}-\frac{1}{c}-\frac{2}{a+b}\right)\ge 0,$$

$$(a-b)(c-b)\left[\frac{1}{bc}-\frac{2}{(a+b)(a+c)}\right]+(c-a)^2\left[\frac{1}{ac}-\frac{2}{(a+b)(b+c)}\right]\geq 0.$$

Since

$$\frac{1}{bc}-\frac{2}{(a+b)(a+c)}=\frac{c(a-b)+a(a+b)}{bc(a+b)(a+c)}\geq\frac{a(a+b)}{bc(a+b)(a+c)}=\frac{a}{bc(a+c)}$$

and

$$\frac{1}{ac}-\frac{2}{(a+b)(b+c)}=\frac{-c(a-b)+b(a+b)}{ac(a+b)(b+c)}>\frac{-c(a-b)}{ac(a+b)(b+c)}=\frac{-(a-b)}{a(a+b)(b+c)},$$

it suffices to show that

$$\frac{(a-b)(c-b)a}{bc(a+c)}-\frac{(c-a)^2(a-b)}{a(a+b)(b+c)}\geq 0,$$

which is true if

$$\frac{(c-b)a}{bc(a+c)}\geq\frac{(c-a)^2}{a(a+b)(b+c)}.$$

We can get this by multiplying the inequalities

$$c-b\geq c-a,$$

$$\frac{1}{b}\geq\frac{1}{a},$$

$$\frac{1}{c}\geq\frac{1}{a+b},$$

$$\frac{a}{a+c}\geq\frac{c-a}{b+c}.$$

The last inequality is true since

$$\frac{a}{a+c}-\frac{c-a}{b+c}\geq\frac{a}{a+c}-\frac{b}{b+c}=\frac{c(a-b)}{(a+c)(b+c)}\geq 0.$$

The equality holds for $a=b=c$.

Second Solution (*by Vo Quoc Ba Can*). Since

$$\sum\frac{a+b}{b+c}=\sum\left(1+\frac{a-c}{b+c}\right)=3+\sum\frac{a-c}{b+c},$$

we can write the desired inequality as

$$\sum\frac{a}{b}-3\geq 2\sum\frac{a-c}{b+c}.$$

Since

$$(ab+bc+ca)\left(\sum\frac{a}{b}-3\right)=\sum a^2-2\sum ab+\sum\frac{a^2c}{b}$$

and

$$(ab+bc+ca)\sum\frac{a-c}{b+c} = [a(b+c)+bc]\sum\frac{a-c}{b+c}$$
$$= \sum a^2 - \sum ab + \sum\frac{bc(a-c)}{b+c},$$

the inequality is equivalent to

$$\sum\frac{a^2c}{b} + 2\sum\frac{bc(c-a)}{b+c} \ge \sum a^2.$$

Since

$$\sum\frac{a^2c}{b} \ge \sum a^2$$

(see the inequality in P 1.167), we only need to show that

$$\sum\frac{bc(c-a)}{b+c} \ge 0.$$

Write this inequality as follows:

$$\sum bc(c^2-a^2)(a+b) \ge 0,$$

$$\sum(c^2-a^2)\left(1+\frac{b}{a}\right) \ge 0,$$

$$\sum(c^2-a^2)\frac{b}{a} \ge 0,$$

$$\sum\frac{bc^2}{a} \ge \sum ab.$$

According to P 1.167, we have

$$\sum\frac{bc^2}{a} \ge \sum a^2 \ge \sum ab.$$

□

P 1.172. *Let a,b,c be the lengths of the sides of a triangle. If $k \ge 2$, then*

$$a^kb(a-b)+b^kc(b-c)+c^ka(c-a) \ge 0.$$

(Vasile Cîrtoaje, 1986)

Solution *(by Darij Grinberg).* For $k = 2$, we get the known inequality (a) in P 1.168:

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

We will prove the following more general statement: if f is an increasing nonnegative function defined on $[0, \infty)$, then

$$E(a, b, c) \geq 0,$$

where

$$E(a, b, c) = a^2bf(a)(a-b) + b^2cf(b)(b-c) + c^2af(c)(c-a).$$

For $f(x) = x^{k-2}$, $k \geq 2$, we get the original inequality. In order to prove the claimed generalization, assume that $a = \max\{a, b, c\}$. There are two cases to consider.

Case 1: $a \geq b \geq c$. Since

$$f(a) \geq f(b) \geq f(c) \geq 0,$$

we have

$$\begin{aligned} E(a, b, c) &\geq a^2bf(c)(a-b) + b^2cf(c)(b-c) + c^2af(c)(c-a) \\ &= f(c)[a^2b(a-b) + b^2c(b-c) + c^2a(c-a)] \geq 0. \end{aligned}$$

Case 2: $a \geq c \geq b$. Since

$$f(a) \geq f(c) \geq f(b) \geq 0,$$

we have

$$\begin{aligned} E(a, b, c) &\geq a^2bf(a)(a-b) + b^2cf(a)(b-c) + c^2af(a)(c-a) \\ &= f(a)[a^2b(a-b) + b^2c(b-c) + c^2a(c-a)] \geq 0. \end{aligned}$$

The equality holds for $a = b = c$, and also for a degenerate triangle with $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.173. *Let a, b, c be the lengths of the sides of a triangle. If $k \geq 1$, then*

$$3(a^{k+1}b + b^{k+1}c + c^{k+1}a) \geq (a+b+c)(a^kb + b^kc + c^ka).$$

Solution. For $k = 1$, the inequality is equivalent to

$$2(a^2b + b^2c + c^2a) \geq ab^2 + bc^2 + ca^2 + 3abc,$$

$$(2c-a)b^2 + (2a^2 - 3ac - c^2)b - ac(a-2c) \geq 0.$$

Assuming that $a = \min\{a, b, c\}$ and making the substitution

$$b = x + \frac{a+c}{2},$$

this inequality becomes

$$(2c-a)x^2+\left(x+\frac{3a}{4}\right)(a-c)^2 \ge 0.$$

It is true since

$$4x+3a=a+4b-2c=2(a+b-c)+(2b-a)>0.$$

In order to prove the desired inequality for $k>1$, we rewrite it as

$$a^kb(2a-b-c)+b^kc(2b-c-a)+c^ka(2c-a-b)\ge 0.$$

We will prove that if f is an increasing nonnegative function defined on $[0,\infty)$, then $E(a,b,c)\ge 0$, where

$$E(a,b,c)=ab(2a-b-c)f(a)+bc(2b-c-a)f(b)+ca(2c-a-b)f(c).$$

For $f(x)=x^{k-1}$, $k\ge 1$, we get the original inequality. In order to prove this generalization, assume that $a=\max\{a,b,c\}$. There are two cases to consider.

Case 1: $a\ge b\ge c$. Since $f(a)\ge f(b)\ge f(c)\ge 0$, we have

$$\begin{aligned}E(a,b,c)&\ge ab(2a-b-c)f(b)+bc(2b-c-a)f(b)+ca(2c-a-b)f(c)\\&=b[2(a-b)(a-c)+ab-c^2]f(b)+ca(2c-a-b)f(c)\\&\ge b[2(a-b)(a-c)+ab-c^2]f(c)+ca(2c-a-b)f(c)\\&=[2(a^2b+b^2c+c^2a)-ab^2-bc^2-ca^2-3abc]f(c)\ge 0.\end{aligned}$$

Case 2: $a\ge c\ge b$. Since $f(a)\ge f(c)\ge f(b)\ge 0$, we have

$$\begin{aligned}E(a,b,c)&\ge ab(2a-b-c)f(c)+bc(2b-c-a)f(b)+ca(2c-a-b)f(c)\\&=a[(c-b)(2c-a)+b(a-b)]f(c)+bc(2b-c-a)f(b).\end{aligned}$$

Since

$$(c-b)(2c-a)+b(a-b)\ge(c-b)(b+c-a)+b(a-b)\ge 0,$$

we get

$$\begin{aligned}E(a,b,c)&\ge a[(c-b)(2c-a)+b(a-b)]f(b)+bc(2b-c-a)f(b)\\&=[2(a^2b+b^2c+c^2a)-ab^2-bc^2-ca^2-3abc]f(b)\ge 0.\end{aligned}$$

The equality holds for $a=b=c$.

Remark. For $k=1$, the inequality has the form

$$2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\ge\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3.$$

A sharper inequality is the following

$$3\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right) \ge 2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3.$$

Using the substitution

$$b = x + \frac{a+c}{2},$$

this inequality turns into

$$(3c-2a)x^2 + \left(x+a-\frac{c}{4}\right)(a-c)^2 \ge 0,$$

which is true since, on the assumption $a = \min\{a,b,c\}$, we have $3c - 2a > 0$ and

$$4x + 4a - c = 2a + 4b - 3c = 3(a+b-c) + (b-a) > 0.$$

□

P 1.174. *Let a, b, c, d be positive real numbers such that $a+b+c+d = 4$. Prove that*

$$\frac{a}{3+b}+\frac{b}{3+c}+\frac{c}{3+d}+\frac{d}{3+a} \ge 1.$$

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a}{3+b} \ge \frac{\left(\sum a\right)^2}{\sum a(3+b)} = \frac{16}{12+\sum ab}.$$

Therefore, it suffices to show that

$$ab+bc+cd+da \le 4.$$

Indeed,

$$ab+bc+cd+da = (a+c)(b+d) \le \left[\frac{(a+c)+(b+d)}{2}\right]^2 = 2.$$

The equality occurs for $a = b = c = d = 1$.

□

P 1.175. *Let a, b, c, d be positive real numbers such that $a+b+c+d = 4$. Prove that*

$$\frac{a}{1+b^2}+\frac{b}{1+c^2}+\frac{c}{1+d^2}+\frac{d}{1+a^2} \ge 2.$$

Solution. Since

$$\frac{a}{1+b^2} = a - \frac{ab^2}{1+b^2},$$

the inequality is equivalent to

$$\frac{ab^2}{1+b^2} + \frac{bc^2}{1+c^2} + \frac{cd^2}{1+d^2} + \frac{da^2}{1+a^2} \le 2.$$

Since

$$\frac{ab^2}{1+b^2} \le \frac{ab^2}{2b} = \frac{ab}{2},$$

it suffices to show that

$$ab + bc + cd + da \le 4.$$

Indeed, we have

$$ab + bc + cd + da = (a+c)(b+d) \le \left[\frac{(a+c)+(b+d)}{2}\right]^2 = 2.$$

The equality occurs for $a = b = c = d = 1$.

□

P 1.176. *If a, b, c, d are nonnegative real numbers such that $a + b + c + d = 4$, then*

$$a^2bc + b^2cd + c^2da + d^2ab \le 4.$$

(Song Yoon Kim, 2006)

Solution. Let (x, y, z, t) be a permutation of (a, b, c, d) such that

$$x \ge y \ge z \ge t,$$

hence

$$xyz \ge xyt \ge xzt \ge yzt.$$

By the rearrangement inequality, we have

$$\begin{aligned} a^2bc + b^2cd + c^2da + d^2ab &= a \cdot abc + b \cdot bcd + c \cdot cda + d \cdot dab \\ &\le x \cdot xyz + y \cdot xyt + z \cdot xzt + t \cdot yzt \\ &= (xy + zt)(xz + yt). \end{aligned}$$

Consequently, it suffices to show that $x + y + z + t = 4$ involves

$$(xy + zt)(xz + yt) \le 4.$$

Indeed, by the AM-GM inequality, we have

$$(xy + zt)(xz + yt) \le \frac{1}{4}(xy + zt + xz + yt)^2 = \frac{1}{4}(x+t)^2(y+z)^2 \le 4,$$

because

$$(x+t)(y+z) \le \frac{1}{4}(x+t+y+z)^2 = 4.$$

The equality holds for $a = b = c = d = 1$, and also for $a = 2$, $b = c = 1$ and $d = 0$ (or any cyclic permutation).

□

P 1.177. *If a, b, c, d are nonnegative real numbers such that $a + b + c + d = 4$, then*

$$a(b+c)^2 + b(c+d)^2 + c(d+a)^2 + d(a+b)^2 \le 16.$$

Solution *(by Vo Quoc Ba Can).* Write the inequality as

$$(a+b+c+d)^3 \ge 4[a(b+c)^2 + b(c+d)^2 + c(d+a)^2 + d(a+b)^2].$$

Since

$$(a+b+c+d)^2 \ge 4(a+b)(c+d),$$

we have

$$\begin{aligned}(a+b+c+d)^3 &\ge 4(a+b)(c+d)(a+b+c+d)\\ &= 4(c+d)(a+b)^2 + 4(a+b)(c+d)^2.\end{aligned}$$

Therefore, it suffices to show that

$$(c+d)(a+b)^2 + (a+b)(c+d)^2 \ge a(b+c)^2 + b(c+d)^2 + c(d+a)^2 + d(a+b)^2,$$

which is equivalent to

$$c(a+b)^2 + a(c+d)^2 \ge a(b+c)^2 + c(d+a)^2,$$

$$a[(c+d)^2 - (b+c)^2] + c[(a+b)^2 - (d+a)^2] \ge 0,$$

$$(b+d)(b-d)(c-a) \ge 0.$$

Similarly, due to cyclicity, the desired inequality is true if

$$(c+a)(c-a)(d-b) \ge 0.$$

Since one of the inequalities $(b-d)(c-a) \ge 0$ and $(c-a)(d-b) \ge 0$ is true, the conclusion follows. The equality holds for $a = c$ and $b = d$.

□

P 1.178. *If a, b, c, d are positive real numbers, then*

$$\frac{a-b}{b+c} + \frac{b-c}{c+d} + \frac{c-d}{d+a} + \frac{d-a}{a+b} \ge 0.$$

Solution. We have

$$\frac{a-b}{b+c}+\frac{c-d}{d+a}+2=\frac{a+c}{b+c}+\frac{a+c}{d+a}$$
$$=(a+c)\left(\frac{1}{b+c}+\frac{1}{d+a}\right)$$
$$\geq (a+c)\frac{4}{(b+c)+(d+a)}$$
$$=\frac{4(a+c)}{a+b+c+d}.$$

Similarly,

$$\frac{b-c}{c+d}+\frac{d-a}{a+b}+2\geq\frac{4(b+d)}{a+b+c+d}.$$

Adding these inequalities yields the desired inequality. The equality holds for $a=c$ and $b=d$.

Remark. It seems that the following inequality holds for a,b,c,d,e positive real numbers (*Vasile Cîrtoaje*, AMM, 5, 1998):

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+e}+\frac{d-e}{e+a}+\frac{e-a}{a+b}\geq 0.$$

The most difficult case (open) is $a\geq b\geq d\geq c\geq e$. □

P 1.179. *If a,b,c,d are positive real numbers, then*

(a) $$\frac{a-b}{a+2b+c}+\frac{b-c}{b+2c+d}+\frac{c-d}{c+2d+a}+\frac{d-a}{d+2a+b}\geq 0;$$

(b) $$\frac{a}{2a+b+c}+\frac{b}{2b+c+d}+\frac{c}{2c+d+a}+\frac{d}{2d+a+b}\leq 1.$$

Solution. (a) Write the inequality as

$$\sum\left(\frac{a-b}{a+2b+c}+\frac{1}{2}\right)\geq 2,$$
$$\sum\frac{3a+c}{a+2b+c}\geq 4.$$

By the Cauchy-Schwarz inequality, we get

$$\sum\frac{3a+c}{a+2b+c}\geq\frac{[\sum(3a+c)]^2}{\sum(3a+c)(a+2b+c)}$$
$$=\frac{16\left(\sum a\right)^2}{4\left(\sum a^2+2\sum ab+\sum ac\right)}$$
$$=\frac{4\left(\sum a\right)^2}{\left(\sum a\right)^2}=4.$$

The equality holds for $a = b = c = d$.

(b) Write the inequality as

$$\sum \left(\frac{1}{2} - \frac{a}{2a+b+c} \right) \ge 1,$$

$$\sum \frac{b+c}{2a+b+c} \ge 2.$$

By the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \sum \frac{b+c}{2a+b+c} &\ge \frac{[\sum (b+c)]^2}{\sum (b+c)(2a+b+c)} \\ &= \frac{4\left(\sum a\right)^2}{2\left(\sum a^2 + 2\sum ab + \sum ac\right)} \\ &= \frac{2\left(\sum a\right)^2}{\left(\sum a\right)^2} = 2. \end{aligned}$$

The equality holds for $a = b = c = d$.

Open problem 1. *If a, b, c, d, e are positive real numbers, then*

$$\frac{a-b}{a+2b+c} + \frac{b-c}{b+2c+d} + \frac{c-d}{c+2d+e} + \frac{d-e}{d+2e+a} + \frac{e-a}{e+2a+b} \ge 0.$$

Open problem 2 (by *Ando*). *If $a_1, a_2, \ldots, a_n$ $(n \ge 4)$ are positive real numbers, then*

$$\frac{a_1}{(n-2)a_1 + a_2 + a_3} + \frac{a_2}{(n-2)a_2 + a_3 + a_4} + \cdots + \frac{a_n}{(n-2)a_n + a_1 + a_2} \le 1.$$

□

P 1.180. *If a, b, c, d are positive real numbers such that $abcd = 1$, then*

$$\frac{1}{a(a+b)} + \frac{1}{b(b+c)} + \frac{1}{c(c+d)} + \frac{1}{d(d+a)} \ge 2.$$

(Vasile Cîrtoaje, G. Dospinescu, 2007)

Solution. Making the substitution

$$a = \sqrt{\frac{y}{x}}, \quad b = \sqrt{\frac{z}{y}}, \quad c = \sqrt{\frac{t}{z}}, \quad d = \sqrt{\frac{x}{t}},$$

where x, y, z, t are positive real numbers, the inequality can be rewritten as

$$\frac{x}{y+\sqrt{xz}} + \frac{y}{z+\sqrt{yt}} + \frac{z}{t+\sqrt{zx}} + \frac{t}{x+\sqrt{ty}} \ge 2.$$

Since

$$2\sqrt{xz} \le x+z, \qquad 2\sqrt{yt} \le y+t,$$

it suffices to show that

$$\frac{x}{x+2y+z} + \frac{y}{y+2z+t} + \frac{z}{z+2t+x} + \frac{t}{t+2x+y} \ge 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{x}{z+2y+z} \ge \frac{\left(\sum x\right)^2}{\sum x(x+2y+z)} = \frac{\left(\sum x\right)^2}{\sum x^2 + 2\sum xy + \sum xz} = 1.$$

The equality holds for $a = c = \dfrac{1}{b} = \dfrac{1}{d}$.

Open problem 1. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1a_2\cdots a_n = 1$, then*

$$\frac{1}{a_1^2 + a_1a_2} + \frac{1}{a_2^2 + a_2a_3} + \cdots + \frac{1}{a_n^2 + a_na_1} \ge \frac{n}{2}.$$

Open problem 2. *If $a_1, a_2, \ldots, a_n$ are positive real numbers, then*

$$\frac{1}{a_1^2 + a_1a_2} + \frac{1}{a_2^2 + a_2a_3} + \cdots + \frac{1}{a_n^2 + a_na_1} \ge \frac{n^2}{2(a_1a_2 + a_2a_3 + \cdots + a_na_1)}.$$

Remark 1. Using the substitution

$$a_1 = \frac{x_2}{x_1}, \quad a_2 = \frac{x_3}{x_2}, \ \ldots\ , \quad a_n = \frac{x_1}{x_n},$$

the inequality in Open problem 1 becomes

$$\frac{x_1^2}{x_2^2 + x_1x_3} + \frac{x_2^2}{x_3^2 + x_2x_4} + \cdots + \frac{x_n^2}{x_1^2 + x_nx_2} \ge \frac{n}{2},$$

where $x_1, x_2, \ldots, x_n > 0$. This cyclic inequality is like Shapiro's inequality

$$\frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_4} + \cdots + \frac{x_n}{x_1 + x_2} \ge \frac{n}{2},$$

which is true for even $n \le 12$ and for odd $n \le 23$.

Remark 2. By the AM-GM inequality, we have

$$a_1a_2 + a_2a_3 + \cdots + a_na_1 \ge n\sqrt[n]{a_1^2a_2^2\cdots a_n^2}.$$

Thus, the inequality in Open problem 2 is weaker than the inequality in Open problem 1. Therefore, if Open problem 1 is true, then Open problem 2 is also true.

□

P 1.181. *If a, b, c, d are positive real numbers, then*

$$\frac{1}{a(1+b)}+\frac{1}{b(1+c)}+\frac{1}{c(1+d)}+\frac{1}{d(1+a)}\ge\frac{16}{1+8\sqrt{abcd}}.$$

(Pham Kim Hung, 2007)

Solution. Let $p=\sqrt[4]{abcd}$. Putting

$$a=p\frac{x_2}{x_1},\qquad b=p\frac{x_3}{x_2},\qquad c=p\frac{x_4}{x_3},\qquad d=p\frac{x_1}{x_4},$$

where x_1, x_2, x_3, x_4 are positive real numbers, the inequality turns into

$$\sum\frac{x_1}{x_2+px_3}\ge\frac{16p}{1+8p^2}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{x_1}{x_2+px_3}\ge\frac{\left(\sum x_1\right)^2}{\sum x_1(x_2+px_3)}=\frac{\left(\sum x_1\right)^2}{(x_1+x_3)(x_2+x_4)+2p(x_1x_3+x_2x_4)}.$$

Since

$$x_1x_3+x_2x_4\le\left(\frac{x_1+x_3}{2}\right)^2+\left(\frac{x_2+x_4}{2}\right)^2,$$

it suffices to show that

$$\frac{(A+B)^2}{2AB+p(A^2+B^2)}\ge\frac{8p}{1+8p^2},$$

where

$$A=x_1+x_3,\qquad B=x_2+x_4.$$

This inequality is equivalent to

$$A^2+B^2+2(8p^2-8p+1)AB\ge 0,$$

which is true because

$$\begin{aligned}A^2+B^2+2(8p^2-8p+1)AB&\ge 2AB+2(8p^2-8p+1)AB\\&=4(2p-1)^2AB\ge 0.\end{aligned}$$

The equality holds for $a=b=c=d=\dfrac{1}{2}$.

□

P 1.182. *If a, b, c, d are nonnegative real numbers such that $a^2+b^2+c^2+d^2=4$, then*

(a) $$3(a+b+c+d)\ge 2(ab+bc+cd+da)+4;$$

(b) $$a+b+c+d-4\ge(2-\sqrt{2})(ab+bc+cd+da-4).$$

(Vasile Cîrtoaje, 2006)

Solution. Let $p = a + b + c + d$. By the Cauchy-Schwarz inequality

$$(1 + 1 + 1 + 1)(a^2 + b^2 + c^2 + d^2) \geq (a + b + c + d)^2,$$

we get $p \leq 4$, and by the inequality

$$(a + b + c + d)^2 \geq a^2 + b^2 + c^2 + d^2,$$

we get $p \geq 2$. In addition, we have

$$ab + bc + cd + da = (a + c)(b + d) \leq \frac{(a + c + b + d)^2}{4} = \frac{p^2}{4}.$$

(a) It suffices to show that

$$3p \geq \frac{p^2}{2} + 4.$$

Indeed,

$$3p - \frac{p^2}{2} - 4 = \frac{(4 - p)(p - 2)}{2} \geq 0.$$

The equality holds for $a = b = c = d = 1$.

(b) It suffices to show that

$$p - 4 \geq (2 - \sqrt{2}) \left(\frac{p^2}{4} - 4 \right).$$

This inequality is equivalent to

$$(4 - p)(p - 2\sqrt{2}) \geq 0,$$

which is true for $p \geq 2\sqrt{2}$. So, it remains to consider the case $2 \leq p < 2\sqrt{2}$. Since

$$2(ab + bc + cd + da) \leq (a + b + c + d)^2 - (a^2 + b^2 + c^2 + d^2) = p^2 - 4,$$

it is enough to prove that

$$p - 4 \geq (2 - \sqrt{2}\,) \left(\frac{p^2 - 4}{2} - 4 \right).$$

Write this inequality as

$$(2 + \sqrt{2}\,)(p - 4) \geq p^2 - 12,$$

$$(2\sqrt{2} - p)(p - 2 + \sqrt{2}\,) \geq 0.$$

The equality holds for $a = b = c = d = 1$, and also for $a = b = 0$ and $c = d = \sqrt{2}$ (or any cyclic permutation).

□

P 1.183. *Let a, b, c, d be positive real numbers.*

(a) If $a, b, c, d \geq 1$, then

$$\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{d}\right)\left(d+\frac{1}{a}\right) \geq (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right);$$

(b) If $abcd = 1$, then

$$\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{d}\right)\left(d+\frac{1}{a}\right) \leq (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

(Vasile Cîrtoaje and Ji Chen, 2011)

Solution. Let

$$\begin{aligned} A =&(1+ab)(1+bc)(1+cd)(1+da) \\ =&1+\sum ab+\sum a^2bd+2abcd+abcd\sum ab+a^2b^2c^2d^2 \\ =&(1-abcd)^2+4abcd+(1+abcd)\sum ab+\sum a^2bd \\ =&(1-abcd)^2+4abcd+(1+abcd)(a+c)(b+d)+\sum a^2bd \end{aligned}$$

and

$$\begin{aligned} B =&(a+b+c+d)(abc+bcd+cda+dab) \\ =&4abcd+\sum a^2(bc+cd+db) \\ =&4abcd+\sum a^2c(b+d)+\sum a^2bd \\ =&4abcd+(ac+bd)(a+c)(b+d)+\sum a^2bd. \end{aligned}$$

Thus,

$$\begin{aligned} A-B &= (1-abcd)^2+(1+abcd)(a+c)(b+d)-(ac+bd)(a+c)(b+d) \\ &= (1-abcd)^2+(1-ac)(1-bd)(a+c)(b+d). \end{aligned}$$

(a) The inequality $A \geq B$ is clearly true for $a, b, c, d \geq 1$. The equality holds for $a = b = c = d = 1$.

(b) For $abcd = 1$, we have

$$B-A = \frac{1}{ac}(1-ac)^2(a+c)(b+d) \geq 0.$$

The equality holds for $ac = bd = 1$.

□

P 1.184. *If a, b, c, d are positive real numbers, then*

$$\left(1+\frac{a}{a+b}\right)^2+\left(1+\frac{b}{b+c}\right)^2+\left(1+\frac{c}{c+d}\right)^2+\left(1+\frac{d}{d+a}\right)^2>7.$$

(Vasile Cîrtoaje, 2012)

First Solution. Assume that $d=\max\{a,b,c,d\}$. We get the desired inequality by summing the inequalities

$$\left(1+\frac{a}{a+b}\right)^2+\left(1+\frac{b}{b+c}\right)^2+\left(1+\frac{c}{c+a}\right)^2>6$$

and

$$\left(1+\frac{c}{c+d}\right)^2+\left(1+\frac{d}{d+a}\right)^2>1+\left(1+\frac{c}{c+a}\right)^2.$$

Let

$$x=\frac{a-b}{a+b},\quad y=\frac{b-c}{b+c},\quad z=\frac{c-a}{c+a}.$$

We have $-1<x,y,z<1$ and

$$x+y+z+xyz=0.$$

Since

$$\frac{a}{a+b}=\frac{x+1}{2},\quad \frac{b}{b+c}=\frac{y+1}{2},\quad \frac{c}{c+a}=\frac{z+1}{2},$$

we can write the first inequality as follows:

$$(x+3)^2+(y+3)^2+(z+3)^2>24,$$

$$x^2+y^2+z^2+6(x+y+z)+3>0,$$

$$x^2+y^2+z^2+3>6xyz.$$

By the AM-GM inequality, we have

$$x^2+y^2+z^2+3\ge 6\sqrt[6]{x^2y^2z^2}>6xyz.$$

Write now the second inequality as

$$\left(1+\frac{c}{c+d}\right)^2-1>\left(\frac{c}{c+a}-\frac{d}{d+a}\right)\left(2+\frac{c}{c+a}+\frac{d}{d+a}\right).$$

Since

$$\frac{c}{c+a}-\frac{d}{d+a}=\frac{a(c-d)}{(c+a)(d+a)}\le 0,$$

we have

$$\left(1+\frac{c}{c+d}\right)^2-1>0\ge\left(\frac{c}{c+a}-\frac{d}{d+a}\right)\left(2+\frac{c}{c+a}+\frac{d}{d+a}\right).$$

Second Solution. Using the inequality

$$(1+x)^2 > 1+3x^2, \quad 0<x<1,$$

we have

$$\left(1+\frac{a}{a+b}\right)^2+\left(1+\frac{b}{b+c}\right)^2+\left(1+\frac{c}{c+d}\right)^2+\left(1+\frac{d}{d+a}\right)^2 >$$

$$> 4+3\left[\left(\frac{a}{a+b}\right)^2+\left(\frac{b}{b+c}\right)^2+\left(\frac{c}{c+d}\right)^2+\left(\frac{d}{d+a}\right)^2\right].$$

Therefore, it suffices to prove that

$$\left(\frac{a}{a+b}\right)^2+\left(\frac{b}{b+c}\right)^2+\left(\frac{c}{c+d}\right)^2+\left(\frac{d}{d+a}\right)^2 \ge 1,$$

which is equivalent to the known inequality in P 1.191 from Volume 2:

$$\frac{1}{(1+x)^2}+\frac{1}{(1+y)^2}+\frac{1}{(1+z)^2}+\frac{1}{(1+t)^2} \ge 1,$$

where

$$x=\frac{a}{b}, \quad y=\frac{b}{c}, \quad z=\frac{c}{d}, \quad t=\frac{d}{a}, \qquad xyzt=1.$$

□

P 1.185. *If a,b,c,d are positive real numbers, then*

$$\frac{a^2-bd}{b+2c+d}+\frac{b^2-ca}{c+2d+a}+\frac{c^2-db}{d+2a+b}+\frac{d^2-ac}{a+2b+c} \ge 0.$$

(Vo Quoc Ba Can, 2009)

Solution. Write the inequality as follows:

$$\sum\left(\frac{4a^2-4bd}{b+2c+d}+b+d-2a\right) \ge 0,$$

$$\sum\frac{(b-d)^2+2(a-c)(2a-b-d)}{b+2c+d} \ge 0.$$

It suffices to show that

$$\sum\frac{(a-c)(2a-b-d)}{b+2c+d} \ge 0.$$

This inequality is equivalent to

$$(a-c)\left(\frac{2a-b-d}{b+2c+d}-\frac{2c-d-b}{d+2a+b}\right)+(b-d)\left(\frac{2b-c-a}{c+2d+a}-\frac{2d-a-c}{a+2b+c}\right) \ge 0,$$

which can be written as

$$\frac{(a-c)(a^2-c^2)}{(b+2c+d)(d+2a+b)}+\frac{(b-d)(b^2-d^2)}{(c+2d+a)(a+2b+c)}\ge 0.$$

The equality occurs for $a=c$ and $b=d$.

□

P 1.186. *If a,b,c,d are positive real numbers such that $a\le b\le c\le d$, then*

$$\sqrt{\frac{2a}{a+b}}+\sqrt{\frac{2b}{b+c}}+\sqrt{\frac{2c}{c+d}}+\sqrt{\frac{2d}{d+a}}\le 4.$$

(Vasile Cîrtoaje, 2009)

Solution. According to the inequality in P 1.74, we have

$$\sqrt{\frac{2a}{a+b}}+\sqrt{\frac{2b}{b+c}}+\sqrt{\frac{2c}{c+a}}\le 3.$$

Therefore, it suffices to show that

$$\sqrt{\frac{2c}{c+d}}+\sqrt{\frac{2d}{d+a}}\le 1+\sqrt{\frac{2c}{c+a}}.$$

By squaring, this inequality becomes

$$\frac{2c}{c+d}+\frac{2d}{d+a}+2\sqrt{\frac{4cd}{(c+d)(d+a)}}\le 1+\frac{2c}{c+a}+2\sqrt{\frac{2c}{c+a}}.$$

We can get it by summing the inequalities

$$\frac{2c}{c+d}+\frac{2d}{d+a}\le 1+\frac{2c}{c+a},$$

$$2\sqrt{\frac{4cd}{(c+d)(d+a)}}\le 2\sqrt{\frac{2c}{c+a}}.$$

The former inequality is true since

$$\frac{2c}{c+d}+\frac{2d}{d+a}-1-\frac{2c}{c+a}=\frac{(a-d)(d-c)(c-a)}{(c+d)(d+a)(a+c)}\le 0,$$

while the second inequality reduces to

$$c(a-d)(d-c)\le 0.$$

The equality holds for $a=b=c=d$.

□

P 1.187. *Let a,b,c,d be nonnegative real numbers, and let*

$$x=\frac{a}{b+c},\quad y=\frac{b}{c+d},\quad z=\frac{c}{d+a},\quad t=\frac{d}{a+b}.$$

Prove that

(a) $$\sqrt{xz}+\sqrt{yt}\le 1;$$

(b) $$x+y+z+t+4(xz+yt)\ge 4.$$

(Vasile Cîrtoaje, 2004)

Solution. (a) Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sqrt{xz}+\sqrt{yt}=&\frac{\sqrt{ac}}{\sqrt{(b+c)(d+a)}}+\frac{\sqrt{bd}}{\sqrt{(c+d)(a+b)}}\\ \le&\frac{\sqrt{ac}}{\sqrt{ac}+\sqrt{bd}}+\frac{\sqrt{bd}}{\sqrt{ac}+\sqrt{bd}}=1.\end{aligned}$$

The equality holds for $a=b=c=d$, for $a=c=0$, and for $b=d=0$

(b) Write the inequality as

$$A+B\ge 6,$$

where

$$\begin{aligned}A=x+z+4xz+1=&\frac{(a+b)(c+d)+(a+c)^2+ab+2ac+cd}{(b+c)(d+a)}\\ =&\frac{(a+b)(c+d)}{(b+c)(d+a)}+\frac{(a+c)^2}{(b+c)(d+a)}+\frac{a}{d+a}+\frac{c}{b+c},\end{aligned}$$

$$B=y+t+4yt+1=\frac{(b+c)(d+a)}{(c+d)(a+b)}+\frac{(b+d)^2}{(c+d)(a+b)}+\frac{b}{a+b}+\frac{d}{c+d}.$$

Since

$$\frac{(a+b)(c+d)}{(b+c)(d+a)}+\frac{(b+c)(d+a)}{(c+d)(a+b)}\ge 2,$$

it suffices to show that

$$\frac{(a+c)^2}{(b+c)(d+a)}+\frac{(b+d)^2}{(c+d)(a+b)}+\sum\frac{a}{d+a}\ge 4.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{(a+c)^2}{(b+c)(d+a)}+\frac{(b+d)^2}{(c+d)(a+b)}\ge\frac{(a+b+c+d)^2}{C},$$

$$\sum\frac{a}{d+a}\ge\frac{(a+b+c+d)^2}{D},$$

where

$$C = (b+c)(d+a) + (c+d)(a+b),$$

$$D = \sum a(d+a) = a^2 + b^2 + c^2 + d^2 + ab + bc + cd + da,$$

$$C + D = (a+b+c+d)^2.$$

Thus, it is enough to show that

$$(C+D)\left(\frac{1}{C} + \frac{1}{D}\right) \ge 4,$$

which is clearly true. The equality holds for $a = b = c = d$.

□

P 1.188. *If a, b, c, d are nonnegative real numbers, then*

$$\left(1 + \frac{2a}{b+c}\right)\left(1 + \frac{2b}{c+d}\right)\left(1 + \frac{2c}{d+a}\right)\left(1 + \frac{2d}{a+b}\right) \ge 9.$$

(Vasile Cîrtoaje, 2004)

Solution. We can rewrite the inequality as

$$\left(1 + \frac{a+c}{a+b}\right)\left(1 + \frac{a+c}{c+d}\right)\left(1 + \frac{b+d}{b+c}\right)\left(1 + \frac{b+d}{d+a}\right) \ge 9.$$

Using the Cauchy-Schwarz inequality and the AM-GM inequality yields

$$\left(1 + \frac{a+c}{a+b}\right)\left(1 + \frac{a+c}{c+d}\right) \ge \left[1 + \frac{a+c}{\sqrt{(a+b)(c+d)}}\right]^2 \ge \left(1 + \frac{2a+2c}{a+b+c+d}\right)^2,$$

$$\left(1 + \frac{b+d}{b+c}\right)\left(1 + \frac{b+d}{d+a}\right) \ge \left[1 + \frac{b+d}{\sqrt{(b+c)(d+a)}}\right]^2 \ge \left(1 + \frac{2b+2d}{a+b+c+d}\right)^2.$$

Thus, it suffices to show that

$$\left(1 + \frac{2a+2c}{a+b+c+d}\right)\left(1 + \frac{2b+2d}{a+b+c+d}\right) \ge 3.$$

This is equivalent to the obvious inequality

$$\frac{4(a+c)(b+d)}{(a+b+c+d)^2} \ge 0.$$

The equality holds for $a = c = 0$ and $b = d$, as well as for $b = d = 0$ and $a = c$.

□

P 1.189. *Let a, b, c, d be nonnegative real numbers. If $k > 0$, then*

$$\left(1+\frac{ka}{b+c}\right)\left(1+\frac{kb}{c+d}\right)\left(1+\frac{kc}{d+a}\right)\left(1+\frac{kd}{a+b}\right) \ge (1+k)^2.$$

(Vasile Cîrtoaje, 2004)

Solution. Let us denote

$$x = \frac{a}{b+c}, \quad y = \frac{b}{c+d}, \quad z = \frac{c}{d+a}, \quad t = \frac{d}{a+b}.$$

Since

$$\prod(1+kx) \ge 1 + k(x+y+z+t) + k^2(xy+yz+zt+tx+xz+yt),$$

it suffices to show that

$$x+y+z+t \ge 2$$

and

$$xy+yz+zt+tx+xz+yt \ge 1.$$

The inequality $x+y+z+t \ge 2$ is the well-known Shapiro's inequality for 4 positive real numbers. This can be proved by the Cauchy-Schwarz inequality, as follows:

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b} \ge \frac{(a+b+c+d)^2}{a(b+c)+b(c+d)+c(d+a)+d(a+b)} \ge 2.$$

The right inequality reduces to the obvious inequality

$$(a-c)^2 + (b-d)^2 \ge 0.$$

To prove the inequality $xy+yz+zt+tx+xz+yt \ge 1$, we will use the inequalities

$$\frac{x+z}{2} \ge xz,$$

$$\frac{y+t}{2} \ge yt,$$

and the identity

$$xz(1+y+t) + yt(1+x+z) = 1.$$

If these are true, then

$$\begin{aligned} xy+yz+zt+tx+xz+yt &= \frac{x+z}{2}(y+t) + \frac{y+t}{2}(x+z) + xz + yt \\ &\ge xz(y+t) + yt(x+z) + xz + yt \\ &= xz(1+y+t) + yt(1+x+z) = 1. \end{aligned}$$

We have

$$\frac{x+z}{2} - xz = \frac{bc+da+(a-c)^2}{2(b+c)(d+a)} \ge 0$$

and

$$\frac{y+t}{2} - yt = \frac{ab+cd+(b-d)^2}{2(a+b)(c+d)} \geq 0.$$

To prove the identity above, we rewrite it as

$$\sum xyz + xz + yt = 1,$$

and see that

$$\sum xyz = \frac{\sum abc(a+b)}{A} = \frac{\sum a^2bc + \sum a^2bd}{A}$$

and

$$xz + yt = \frac{ac(a+b)(c+d) + bd(b+c)(d+a)}{A} = \frac{\sum a^2cd + (ac+bd)^2}{A},$$

where

$$A = \prod(a+b) = \sum a^2bc + \sum a^2bd + \sum a^2cd + (ac+bd)^2.$$

Thus, the proof is completed. The equality holds for $a = c = 0$ and $b = d$, as well as for $b = d = 0$ and $a = c$.

Remark. For $k = 2$, we get the inequality in P 1.188. For $k = 1$, we get the following known inequality

$$(a+b+c)(b+c+d)(c+d+a)(d+a+b) \geq 4(a+b)(b+c)(c+d)(d+a).$$

A proof of this inequality starts from the inequalities

$$(a+b+c)^2 \geq (2a+b)(2c+b)$$

and

$$(2a+b)(2b+a) \geq 2(a+b)^2.$$

We have

$$\begin{aligned}\prod(a+b+c)^2 &\geq \prod(2a+b) \cdot \prod(2c+b) \\ &= \prod(2a+b)(2b+a) \\ &\geq 2^4 \prod(a+b)^2,\end{aligned}$$

hence

$$\prod(a+b+c) \geq 4\prod(a+b).$$

□

P 1.190. *If a, b, c, d are positive real numbers such that $a+b+c+d = 4$, then*

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{cd} + \frac{1}{da} \geq a^2 + b^2 + c^2 + d^2.$$

(Vasile Cîrtoaje, 2007)

Solution. Write the inequality as

$$(a+c)(b+d) \geq abcd(a^2+b^2+c^2+d^2).$$

From $(a-c)^4 \geq 0$ and $(b-d)^4 \geq 0$, we get

$$(a+c)^4 \geq 8ac(a^2+c^2), \quad (b+d)^4 \geq 8bd(b^2+d^2),$$

hence

$$bd(a+c)^4 + ac(b+d)^4 \geq 8abcd(a^2+b^2+c^2+d^2).$$

Therefore, it suffices to show that

$$8(a+c)(b+d) \geq bd(a+c)^4 + ac(b+d)^4.$$

Since $4bd \leq (b+d)^2$ and $4ac \leq (a+c)^2$, we only need to show that

$$32(a+c)(b+d) \geq (b+d)^2(a+c)^4 + (a+c)^2(b+d)^4.$$

Denoting $a+c = 2x$ and $b+d = 2y$, this inequality is equivalent to

$$2 \geq xy(x^2+y^2),$$

$$(x+y)^4 \geq 8xy(x^2+y^2),$$

$$(x-y)^4 \geq 0.$$

The equality occurs for $a = b = c = d = 1$.

□

P 1.191. *If a, b, c, d are positive real numbers, then*

$$\frac{a^2}{(a+b+c)^2} + \frac{b^2}{(b+c+d)^2} + \frac{c^2}{(c+d+a)^2} + \frac{d^2}{(d+a+b)^2} \geq \frac{4}{9}.$$

(Pham Kim Hung, 2006)

First Solution. By Hölder's inequality, we have

$$\sum \frac{a^2}{(a+b+c)^2} \geq \frac{\left(\sum a^{4/3}\right)^3}{\left[\sum a(a+b+c)\right]^2}.$$

Since

$$\sum a(a+b+c) = (a+c)^2 + (b+d)^2 + (a+c)(b+d)$$

and

$$\sum a^{4/3} = \left(a^{4/3} + c^{4/3}\right) + \left(b^{4/3} + d^{4/3}\right) \geq 2\left(\frac{a+c}{2}\right)^{4/3} + 2\left(\frac{b+d}{2}\right)^{4/3},$$

it suffices to show that

$$9\left[(a+c)^{4/3}+(b+d)^{4/3}\right]^3 \ge 8[(a+c)^2+(b+d)^2+(a+c)(b+d)]^2.$$

Due to homogeneity, we may assume that $b+d=1$. Putting $a+c=t^3$, $t>0$, the inequality becomes

$$9(t^4+1)^3 \ge 8(t^6+1+t^3)^2,$$

$$9\left(t^2+\frac{1}{t^2}\right)^3 \ge 8\left(t^3+\frac{1}{t^3}+1\right)^2.$$

Setting

$$x = t+\frac{1}{t}, \qquad x \ge 2,$$

the inequality turns into

$$9(x^2-2)^3 \ge 8(x^3-3x+1)^2,$$

which is equivalent to

$$(x-2)^2(x^4+4x^3+6x^2-8x-20) \ge 0.$$

This is true since

$$x^4+4x^3+6x^2-8x-20 = x^4+4x^2(x-2)+4x(x-2)+10(x^2-2) > 0.$$

Thus, the proof is completed. The equality holds for $a=b=c=d$.

Second Solution. Due to homogeneity, we may assume that

$$a+b+c+d=1.$$

In this case, we write the inequality as

$$\left(\frac{a}{1-d}\right)^2+\left(\frac{b}{1-a}\right)^2+\left(\frac{c}{1-b}\right)^2+\left(\frac{d}{1-c}\right)^2 \ge \frac{4}{9}.$$

Let (x,y,z,t) be a permutation of (a,b,c,d) such that

$$x \ge y \ge z \ge t.$$

Since

$$\frac{1}{(1-t)^2} \le \frac{1}{(1-z)^2} \le \frac{1}{(1-y)^2} \le \frac{1}{(1-x)^2},$$

by the rearrangement inequality, we have

$$\left(\frac{x}{1-t}\right)^2+\left(\frac{y}{1-z}\right)^2+\left(\frac{z}{1-y}\right)^2+\left(\frac{t}{1-x}\right)^2 \le$$

$$\le \left(\frac{a}{1-d}\right)^2+\left(\frac{b}{1-a}\right)^2+\left(\frac{c}{1-b}\right)^2+\left(\frac{d}{1-c}\right)^2.$$

Therefore, it suffices to show that $x+y+z+t=1$ involves

$$U+V \ge \frac{4}{9},$$

where

$$U = \left(\frac{x}{1-t}\right)^2 + \left(\frac{t}{1-x}\right)^2,$$

$$V = \left(\frac{y}{1-z}\right)^2 + \left(\frac{z}{1-y}\right)^2.$$

Let

$$s = x+t, \quad p = xt, \quad s \in (0,1),$$

Since

$$x^2+t^2 = s^2-2p, \quad x^3+t^3 = s^3-3ps, \quad x^4+t^4 = s^4-4ps^2+2p^2,$$

we get

$$U = \frac{x^2+t^2-2(x^3+t^3)+x^4+t^4}{(1-s+p)^2}$$

$$= \frac{2p^2-2(1-s)(1-2s)p+s^2(1-s)^2}{p^2+2(1-s)p+(1-s)^2},$$

$$(2-U)p^2-2(1-s)(1-2s+U)p+(1-s)^2(s^2-U)=0.$$

The quadratic trinomial in p has the discriminant

$$D = (1-s)^2[(1-2s+U)^2-(2-U)(s^2-U)].$$

From the necessary condition $D \ge 0$, we get

$$U \ge \frac{4s-1-2s^2}{(2-s)^2}.$$

Analogously,

$$V \ge \frac{4r-1-2r^2}{(2-r)^2},$$

where $r = y+z$. Taking into account that

$$s+r=1,$$

we get

$$U+V \ge \frac{4s-1-2s^2}{(2-s)^2} + \frac{4r-1-2r^2}{(2-r)^2}$$

$$= \frac{4s-1-2s^2}{(1+r)^2} + \frac{4r-1-2r^2}{(1+s)^2}$$

$$= \frac{5(s^2+r^2)-2(s^4+r^4)}{(2+sr)^2}$$

$$= \frac{5(s^2+r^2)-2(s^2+r^2)^2+4s^2r^2}{(2+sr)^2},$$

hence

$$U+V-\frac{4}{9} \ge \frac{5(s^2+r^2)-2(s^2+r^2)^2+4s^2r^2}{(2+sr)^2}-\frac{4}{9}$$
$$=\frac{5(s^2+r^2)-2(s^2+r^2)^2}{(2+sr)^2}+\frac{2(1-4sr)^2-18}{9(2+sr)^2}$$
$$\ge \frac{5(s^2+r^2)-2(s^2+r^2)^2-2}{(2+sr)^2}$$
$$=\frac{(2-s^2-r^2)(2s^2+2r^2-1)}{(2+sr)^2}.$$

Thus, we need to show that $(2-s^2-r^2)(2s^2+2r^2-1)\ge 0$. This is true since since

$$2-s^2-r^2>2-(s+r)^2=1,$$
$$2s^2+2r^2-1\ge (s+r)^2-1=0.$$

□

P 1.192. *If a,b,c,d are positive real numbers such that $a+b+c+d=3$, then*

$$ab(b+c)+bc(c+d)+cd(d+a)+da(a+b)\le 4.$$

(Pham Kim Hung, 2007)

Solution. Write the inequality as

$$\sum ab^2+\sum abc\le 4,$$
$$(ab^2+cd^2+bcd+dab)+(bc^2+da^2+abc+cda)\le 4,$$
$$(b+d)(ab+cd)+(a+c)(bc+da)\le 4.$$

Without loss of generality, assume that $a+c\le b+d$. Since

$$(ab+cd)+(bc+da)=(a+c)(b+d),$$

we can rewrite the inequality as

$$(a+c)(b+d)^2+(a+c-b-d)(bc+da)\le 4.$$

Since $a+c-b-d\le 0$, it suffices to show that

$$(a+c)(b+d)^2\le 4.$$

Indeed, by the AM-GM inequality, we have

$$(a+c)\left(\frac{b+d}{2}\right)\left(\frac{b+d}{2}\right)\le \frac{1}{27}\left(a+c+\frac{b+d}{2}+\frac{b+d}{2}\right)^3=1.$$

The equality holds for $a=b=0$, $c=1$ and $d=2$ (or any cyclic permutation).

□

P 1.193. *If $a \ge b \ge c \ge d \ge 0$ and $a+b+c+d=2$, then*

$$ab(b+c)+bc(c+d)+cd(d+a)+da(a+b) \le 1.$$

(Vasile Cîrtoaje, 2007)

Solution. Write the inequality as

$$\sum ab^2 + \sum abc \le 1.$$

Since

$$\sum ab^2 - \sum a^2b = (ab^2+bc^2+ca^2-a^2b-b^2c-c^2a)+(cd^2+da^2+ac^2-c^2d-d^2a-a^2c)$$

$$= (a-b)(b-c)(c-a)+(c-d)(d-a)(a-c) \le 0,$$

it suffices to show that

$$\sum ab^2 + \sum a^2b + 2\sum abc \le 2.$$

Indeed,

$$\begin{aligned}\sum ab^2 + \sum a^2b + 2\sum abc &= \sum (ab^2+a^2b+abc+abd)\\ &= (a+b+c+d)\sum ab\\ &= 2(a+c)(b+d)\\ &\le 2\left[\frac{(a+c)+(b+d)}{2}\right]^2 = 2.\end{aligned}$$

The equality holds for $a=b=t$ and $c=d=1-t$, where $t \in \left[\dfrac{1}{2}, 1\right]$.

□

P 1.194. *Let a,b,c,d be nonnegative real numbers such that $a+b+c+d=4$. If $k \ge \dfrac{37}{27}$, then*

$$ab(b+kc)+bc(c+kd)+cd(d+ka)+da(a+kb) \le 4(1+k).$$

(Vasile Cîrtoaje, 2007)

Solution. Write the inequality in the homogeneous form

$$ab(b+kc)+bc(c+kd)+cd(d+ka)+da(a+kb) \le \frac{(1+k)(a+b+c+d)^3}{16}.$$

Assume that $d = \min\{a,b,c,d\}$ and use the substitution

$$a = d+x, \quad b = d+y, \quad c = d+z,$$

where $x, y, z \geq 0$. The inequality can be restated as

$$4Ad + B \geq 0,$$

where

$$A = (3k-1)(x^2+y^2+z^2) - 2(k+1)y(x+z) + (6-2k)xz,$$

$$B = (1+k)(x+y+z)^3 - 16(xy^2+yz^2+kxyz).$$

It suffices to show that $A \geq 0$ and $B \geq 0$. We have

$$\begin{aligned}
A &= (3k-1)y^2 + (3k-1)(x+z)^2 - 2(k+1)y(x+z) - 8(k-1)xz \\
&\geq (3k-1)y^2 + (3k-1)(x+z)^2 - 2(k+1)y(x+z) - 2(k-1)(x+z)^2 \\
&= (3k-1)y^2 + (k+1)(x+z)^2 - 2(k+1)y(x+z) \\
&\geq 2\sqrt{(3k-1)(k+1)}y(x+z) - 2(k+1)y(x+z) \\
&= 2\sqrt{k+1}\left(\sqrt{3k-1} - \sqrt{k+1}\right)y(x+z) \geq 0.
\end{aligned}$$

Since

$$(x+y+z)^3 - 16xyz \geq 0,$$

the inequality $B \geq 0$ holds for all $k \geq \frac{37}{27}$ if it holds for $k = \frac{37}{27}$. In this particular case, the inequality $B \geq 0$ can be written as

$$4\left(\frac{x+y+z}{3}\right)^3 \geq xy^2 + yz^2 + \frac{37}{27}xyz.$$

Actually, the following sharper inequality holds (see P 2.24)

$$4\left(\frac{x+y+z}{3}\right)^3 \geq xy^2 + yz^2 + \frac{3}{2}xyz.$$

Thus, the proof is completed. The equality holds for $a = b = c = d = 1$. If $k = \frac{37}{27}$, then the equality also holds for $a = \frac{4}{3}$, $b = \frac{8}{3}$ and $c = d = 0$ (or any cyclic permutation).

□

P 1.195. *If a, b, c, d are nonnegative real numbers such that $a+b+c+d = 4$, then*

$$\sqrt{\frac{3a}{b+2}} + \sqrt{\frac{3b}{c+2}} + \sqrt{\frac{3c}{d+2}} + \sqrt{\frac{3d}{a+2}} \leq 4.$$

(Vasile Cîrtoaje, 2020)

Solution. (after an idea of *Michael Rozenberg*) Let $(a1, a_2, a_3, a_4)$ be an increasing permutation of (a, b, c, d). Since the sequences

$$(a1, a_2, a_3, a_4) \qquad \textit{and} \qquad \left(\frac{1}{a_4+2}\ , \frac{1}{a_3+2}\ , \frac{1}{a_2+2}\ , \frac{1}{a_1+2}\right)$$

are increasing, according to the rearrangement inequality, we have

$$\sqrt{\frac{3a}{b+2}} + \sqrt{\frac{3b}{c+2}} + \sqrt{\frac{3c}{d+2}} + \sqrt{\frac{3d}{a+2}} \le$$

$$\le \sqrt{\frac{3a_1}{a_4+2}} + \sqrt{\frac{3a_2}{a_3+2}} + \sqrt{\frac{3a_3}{a_2+2}} + \sqrt{\frac{3a_4}{a_1+2}} = A + B,$$

where

$$A = \sqrt{\frac{3a_1}{a_4+2}} + \sqrt{\frac{3a_4}{a_1+2}}\ , \qquad B = \sqrt{\frac{3a_2}{a_3+2}} + +\sqrt{\frac{3a_3}{a_2+2}}\ .$$

We need to show that $A + B \le 2$. According to Lemma below, we have

$$A + B \le \frac{a_1 + a_4 + 4}{3} + \frac{a_2 + a_3 + 4}{3} = 4.$$

The equality holds for $a = b = c = d = 1$.

Lemma. *If a, b are nonnegative real numbers, then*

$$\sqrt{\frac{3a}{b+2}} + \sqrt{\frac{3b}{a+2}} \le \frac{a+b+4}{3}.$$

Proof. Use the substitution

$$x = \sqrt{\frac{3a}{b+2}}, \qquad y = \sqrt{\frac{3b}{a+2}},$$

which yields $xy < 3$ and

$$a = \frac{2x^2(y^2+3)}{9-x^2y^2}, \qquad b = \frac{2y^2(x^2+3)}{9-x^2y^2}, \qquad a + b = \frac{4x^2y^2 + 6(x^2+y^2)}{9-x^2y^2}.$$

Thus, we need to show that

$$3(x+y) \le \frac{4x^2y^2 + 6(x^2+y^2)}{9-x^2y^2} + 4,$$

which is equivalent to

$$2(x+y)^2 - (9 - x^2y^2)(x+y) + 12 - 4xy \ge 0,$$

$$\left(4x + 4y - 9 + x^2y^2\right)^2 + 15 - 32xy + 18x^2y^2 - x^4y^4 \ge 0,$$

$$\left(4x + 4y - 9 + x^2y^2\right)^2 + (1 - xy)^2(3 - xy)(5 + xy) \ge 0.$$

The equality holds for $a = b = 1$.

□

P 1.196. *Let a,b,c,d be positive real numbers such that $a\le b\le c\le d$. Prove that*

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)\ge 4+\frac{a}{c}+\frac{c}{a}+\frac{b}{d}+\frac{d}{b}.$$

(Vasile Cîrtoaje, 2012)

First Solution. Let

$$E(a,b,c,d)=2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)-4-\frac{a}{c}-\frac{c}{a}-\frac{b}{d}-\frac{d}{b}.$$

We show that

$$E(a,b,c,d)\ge E(b,b,c,d)\ge E(b,b,c,c).$$

We have

$$E(a,b,c,d)-E(b,b,c,d)=(b-a)\left(\frac{1}{c}+\frac{2d}{ab}-\frac{2}{b}-\frac{c}{ab}\right)\ge 0,$$

since

$$\frac{1}{c}+\frac{2d}{ab}-\frac{2}{b}-\frac{c}{ab}\ge\frac{1}{c}+\frac{2c}{ab}-\frac{2}{b}-\frac{c}{ab}$$
$$=\frac{1}{c}+\frac{c}{ab}-\frac{2}{b}\ge\frac{1}{c}+\frac{c}{b^2}-\frac{2}{b}=\frac{(b-c)^2}{b^2c}\ge 0.$$

Also,

$$E(b,b,c,d)-E(b,b,c,c)=(d-c)\left(\frac{1}{b}-\frac{2c-b}{cd}\right)\ge 0,$$

since

$$\frac{1}{b}-\frac{2c-b}{cd}\ge\frac{1}{b}-\frac{2c-b}{c^2}=\frac{(b-c)^2}{bc^2}\ge 0.$$

Because $E(b,b,c,c)=0$, the proof is completed. The equality holds for $a=b$ and $c=d$.

Second Solution. Using the substitution

$$x=\frac{a}{b},\quad y=\frac{b}{c},\quad z=\frac{c}{d},\quad 0<x,y,z\le 1,$$

the inequality becomes as follows:

$$2\left(x+y+z+\frac{1}{xyz}\right)\ge 4+xy+\frac{1}{xy}+yz+\frac{1}{yz},$$

$$y(2-x-z)+\frac{1}{y}\left(\frac{2}{xz}-\frac{1}{x}-\frac{1}{z}\right)-2(2-x-z)\ge 0,$$

$$(2-x-z)\left(y+\frac{1}{xyz}-2\right)\ge 0.$$

The last inequality is true since $2-x-y\ge 0$ and

$$y+\frac{1}{xyz}-2\ge y+\frac{1}{y}-2\ge 0.$$

□

P 1.197. *Let a, b, c, d be positive real numbers such that*

$$a \le b \le c \le d, \qquad abcd = 1.$$

Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \ge ab + bc + cd + da.$$

(Vasile Cîrtoaje, 2012)

Solution. Write the inequality as follows:

$$a^2cd + b^2da + c^2ab + d^2bc \ge ab + bc + cd + da,$$

$$ac(ad + bc) + bd(ab + cd) \ge (ad + bc) + (ab + cd),$$

$$(ac - 1)(ad + bc) + (bd - 1)(ab + cd) \ge 0.$$

Since

$$ac - 1 = \frac{1}{bd} - 1 \ge 1 - bd$$

and

$$bd \ge \sqrt{abcd} = 1,$$

we have

$$(ac - 1)(ad + bc) + (bd - 1)(ab + cd) \ge (1 - bd)(ad + bc) + (bd - 1)(ab + cd)$$
$$= (bd - 1)(a - c)(b - d) \ge 0.$$

The equality holds for $a = b = \dfrac{1}{c} = \dfrac{1}{d} \le 1$.

□

P 1.198. *Let a, b, c, d be positive real numbers such that*

$$a \le b \le c \le d, \qquad abcd = 1.$$

Prove that

$$4 + \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \ge 2(a + b + c + d).$$

(Vasile Cîrtoaje, 2012)

Solution. Making the substitution

$$x = \sqrt[4]{\frac{a}{b}}, \quad y = \sqrt{\frac{b}{c}}, \quad z = \sqrt[4]{\frac{c}{d}}, \qquad 0 < x, y, z \le 1,$$

we need to show that $E(x, y, z) \ge 0$, where

$$E(x, y, z) = 4 + x^4 + z^4 + y^2 + \frac{1}{x^4y^2z^4} - 2\left(x^3yz + \frac{yz}{x} + \frac{z}{xy} + \frac{1}{xyz^3}\right).$$

We will show that

$$E(x,y,z) \ge E(x,1,z) \ge E(x,1,1) \ge 0. \tag{*}$$

The left inequality is equivalent to

$$(1-y)E_1(x,y,z) \ge 0,$$

where

$$E_1(x,y,z) = -1 - y + \frac{1+y}{x^4y^2z^4} + 2\left(x^3z + \frac{z}{x}\right) - \frac{2}{y}\left(\frac{z}{x} + \frac{1}{xz^3}\right).$$

To prove it, we show that

$$E_1(x,y,z) \ge E_1(x,1,z) \ge 0.$$

We have

$$E_1(x,1,z) = 2(1-x^3z)\left(\frac{1}{x^4z^4} - 1\right) \ge 0.$$

Since

$$E_1(x,y,z - E_1(x,1,z) = (1-y)E_2(x,y,z),$$

where

$$E_2(x,y,z) = 1 + \frac{1+2y}{x^4y^2z^4} - \frac{2}{y}\left(\frac{z}{x} + \frac{1}{xz^3}\right),$$

we need to show $E_2(x,y,z) \ge 0$. Indeed,

$$\begin{aligned}
E_2(x,y,z) &= 1 + \frac{1}{x^4y^2z^4} - \frac{2}{y}\left(\frac{z}{x} + \frac{1}{xz^3} - \frac{1}{x^4z^4}\right)\\
&\ge \frac{2}{x^2yz^2} - \frac{2}{y}\left(\frac{z}{x} + \frac{1}{xz^3} - \frac{1}{x^4z^4}\right)\\
&= \frac{2}{xyz}\left(\frac{1}{xz} - z^2 - \frac{1}{z^2} + \frac{1}{x^3z^3}\right)\\
&\ge \frac{2}{xyz}\left(\frac{1}{z} - z^2 - \frac{1}{z^2} + \frac{1}{z^3}\right)\\
&= \frac{2}{xyz}\left(\frac{1-z^3}{z} + \frac{1-z}{z^3}\right) \ge 0.
\end{aligned}$$

The middle inequality in (*) is equivalent to

$$(1-z)F(x,z) \ge 0,$$

where

$$F(x,z) = (1+z+z^2+z^3)\left(\frac{1}{x^4z^4} - 1\right) + 2\left(x^3 + \frac{2}{x}\right) - \frac{1+z+z^2}{xz}.$$

It is true since

$$F(x,z) > \frac{1}{x^4z^4} - 1 + \frac{3}{x} - \frac{1+z+z^2}{xz}$$
$$\geq \frac{1}{xz} - 1 + \frac{3}{x} - \frac{1+z+z^2}{xz}$$
$$= \frac{2-x-z}{x} \geq 0.$$

The right inequality in (*) is also true since

$$x^4E(x,1,1) = x^8 - 2x^7 + 6x^4 - 6x^3 + 1$$
$$= (x-1)^2(x^6 - x^4 - 2x^3 + 3x^2 + 2x + 1)$$
$$\geq (x-1)^2(x^6 - x^4 - 2x^3 + 2x^2)$$
$$= x^2(x-1)^4(x^2 + 2x + 2) \geq 0.$$

The proof is completed. The equality holds for $a = b = c = d = 1$.

□

P 1.199. *Let $A = \{a_1, a_2, a_3, a_4\}$ be a set of real numbers such that*

$$a_1 + a_2 + a_3 + a_4 = 0.$$

Prove that there exists a permutation $B = \{a, b, c, d\}$ of A such that

$$a^2 + b^2 + c^2 + d^2 + 3(ab + bc + cd + da) \geq 0.$$

Solution. Write the desired inequality as

$$a^2 + b^2 + c^2 + d^2 + 3(ab + bc + cd + da) \geq (a + b + c + d)^2,$$

$$ab + bc + cd + da \geq 2(ac + bd),$$

$$(ab + cd - ac - bd) + (bc + da - ac - bd) \geq 0.$$

$$(a-d)(b-c) + (a-b)(d-c) \geq 0.$$

Clearly, this inequality is true for $a \leq b \leq d \leq c$. The equality occurs when A has three equal elements.

□

P 1.200. *If a, b, c, d, e are positive real numbers, then*

$$\frac{a}{a+2b+2c} + \frac{b}{b+2c+2d} + \frac{c}{c+2d+2e} + \frac{d}{d+2e+2a} + \frac{e}{e+2a+2b} \geq 1.$$

Solution. The inequality follows by applying the Cauchy-Schwarz inequality:

$$\sum \frac{a}{a+2b+2c} \ge \frac{\left(\sum a\right)^2}{\sum a(a+2b+2c)} = \frac{\left(\sum a\right)^2}{\sum a^2+2\sum ab+2\sum ac} = 1.$$

The equality holds for $a=b=c=d=e$.

□

P 1.201. *Let a,b,c,d,e be positive real numbers such that $a+b+c+d+e=5$. Prove that*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}+\frac{e}{a} \le 1+\frac{4}{abcde}.$$

Solution. Let (x,y,z,t,u) be a permutation of (a,b,c,d,e) such that $x \ge y \ge z \ge t \ge u$. By the rearrangement inequality, we have

$$\begin{aligned}\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}+\frac{e}{a} &\le \frac{x}{u}+\frac{y}{t}+\frac{z}{z}+\frac{t}{y}+\frac{u}{x}\\ &= \left(\frac{x}{u}+\frac{u}{x}+2\right)+\left(\frac{y}{t}+\frac{t}{y}+2\right)-3\\ &= 4(p+q)-3,\end{aligned}$$

where

$$p=\frac{1}{4}\left(\frac{x}{u}+\frac{u}{x}+2\right) \ge 1, \qquad q=\frac{1}{4}\left(\frac{y}{t}+\frac{t}{y}+2\right) \ge 1.$$

From $(p-1)(q-1) \ge 0$, we get

$$p+q \le 1+pq,$$

$$4(p+q)-3 \le 1+4pq,$$

hence

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}+\frac{e}{a} \le 1+4pq.$$

Thus, it suffices to show that

$$pq \le \frac{1}{xyztu},$$

which is is equivalent to

$$z\left(\frac{x+u}{2}\right)^2\left(\frac{y+t}{2}\right)^2 \le 1.$$

Indeed, by the AM-GM inequality, we get

$$z\left(\frac{x+u}{2}\right)^2\left(\frac{y+t}{2}\right)^2 \le \left(\frac{z+\dfrac{x+u}{2}+\dfrac{x+u}{2}+\dfrac{y+t}{2}+\dfrac{y+t}{2}}{5}\right)^5 = 1.$$

The equality holds for $a = b = c = d = e = 1$.

Remark. Similarly, we can prove the following generalization (*Michael Rozenberg*):

- *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then*

$$n - 4 + \frac{4}{a_1 a_2 \cdots a_n} \ge \frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_n}{a_1}.$$

□

P 1.202. *If a, b, c, d, e are real numbers such that $a + b + c + d + e = 0$, then*

$$\frac{-\sqrt{5}-1}{4} \le \frac{ab + bc + cd + de + ea}{a^2 + b^2 + c^2 + d^2 + e^2} \le \frac{\sqrt{5}-1}{4}.$$

Solution. From

$$(a + b + c + d + e)^2 = 0,$$

we get

$$\sum a^2 + 2\sum ab + 2\sum ac = 0.$$

Therefore, for any real k, we have

$$\sum a^2 + (2k + 2)\sum ab = \sum 2a(kb - c).$$

By the AM-GM inequality, we get

$$2a(kb - c) \le a^2 + (kb - c)^2,$$

hence

$$\sum a^2 + (2k + 2)\sum ab \le \sum [a^2 + (kb - c)^2] = (k^2 + 2)\sum a^2 - 2k\sum ab,$$

which is equivalent to

$$\sum a^2 \ge \frac{2(2k + 1)}{k^2 + 1}\sum ab.$$

Choosing $k = \dfrac{-1-\sqrt{5}}{2}$ and $k = \dfrac{-1+\sqrt{5}}{2}$, we get the desired inequalities. The equality in both inequalities occurs when

$$a = kb - c, \quad b = kc - d, \quad c = kd - e, \quad d = ke - a, \quad e = ka - b;$$

that is, when

$$a = x, \quad b = y, \quad c = -x + ky, \quad d = -k(x + y), \quad e = kx - y,$$

where x and y are real numbers.

□

P 1.203. *Let a, b, c, d, e be positive real numbers such that*

$$a^2+b^2+c^2+d^2+e^2=5.$$

Prove that

$$\frac{a^2}{b+c+d}+\frac{b^2}{c+d+e}+\frac{c^2}{d+e+a}+\frac{d^2}{e+a+b}+\frac{e^2}{a+b+c}\ge\frac{5}{3}.$$

(Pham Van Thuan, 2005)

Solution. By the AM-GM Inequality, we get

$$2(b+c+d)\le(b^2+1)+(c^2+1)+(d^2+1)=8-a^2-e^2.$$

Therefore, it suffices to show that

$$\sum\frac{a^2}{8-a^2-e^2}\ge\frac{5}{6}.$$

By the Cauchy-Schwarz Inequality, we have

$$\sum\frac{a^2}{8-a^2-e^2}\ge\frac{\left(\sum a^2\right)^2}{\sum a^2(8-a^2-e^2)}=\frac{25}{40-\sum a^4-\sum a^2e^2}$$
$$=\frac{50}{80-\sum(a^2+e^2)^2}\ge\frac{50}{80-\frac{1}{5}\left[\sum(a^2+e^2)\right]^2}=\frac{5}{6}.$$

The equality holds for $a=b=c=d=e=1$.

□

P 1.204. *Let a,b,c,d,e be nonnegative real numbers such that $a+b+c+d+e=5$. Prove that*

$$(a^2+b^2)(b^2+c^2)(c^2+d^2)(d^2+e^2)(e^2+a^2)\le\frac{729}{2}.$$

(Vasile Cîrtoaje, 2007)

Solution. Write the inequality as

$$E(a,b,c,d,e)\le 0,$$

and, without loss of generality, assume that

$$e=\min\{a,b,c,d,e\}.$$

We claim that it suffices to prove the desired inequality for the case $e = 0$. To prove this, it suffices to show that

$$E(a,b,c,d,e) \le E\left(a + \frac{e}{2}, b, c, d + \frac{e}{2}, 0\right), \tag{*}$$

which is equivalent to

$$(a^2+b^2)(c^2+d^2)(d^2+e^2)(e^2+a^2) \le$$
$$\le \left[\left(a+\frac{e}{2}\right)^2 + b^2\right]\left[c^2 + \left(d+\frac{e}{2}\right)^2\right]\left(d+\frac{e}{2}\right)^2\left(a+\frac{e}{2}\right)^2.$$

Since

$$a^2+b^2 \le \left(a+\frac{e}{2}\right)^2 + b^2,$$

$$c^2+d^2 \le c^2 + \left(d+\frac{e}{2}\right)^2,$$

$$d^2+e^2 \le d^2+de \le \left(d+\frac{e}{2}\right)^2,$$

$$e^2+a^2 \le ae+a^2 \le \left(a+\frac{e}{2}\right)^2,$$

the conclusion follows. Thus, we only need to show that

$$a+b+c+d = 5$$

involves

$$E(a,b,c,d,0) \le 0,$$

where

$$E(a,b,c,d,0) = a^2d^2(a^2+b^2)(b^2+c^2)(c^2+d^2) - \frac{729}{2}.$$

Without loss of generality, assume that

$$c = \min\{b,c\}.$$

We claim that it suffices to prove the inequality $E(a,b,c,d,0) \le 0$ for the case $c = 0$. To prove this, it suffices to show that

$$E(a,b,c,d,0) \le E\left(a, b+\frac{c}{2}, 0, d+\frac{c}{2}, 0\right), \tag{**}$$

which is equivalent to

$$d^2(a^2+b^2)(b^2+c^2)(c^2+d^2) \le \left(d+\frac{c}{2}\right)^2\left[a^2+\left(b+\frac{c}{2}\right)^2\right]\left(b+\frac{c}{2}\right)^2\left(d+\frac{c}{2}\right)^2.$$

This is true since

$$d^2(c^2+d^2) \le \left(d+\frac{c}{2}\right)^4,$$

$$a^2+b^2 \le a^2+\left(b+\frac{c}{2}\right)^2,$$

$$b^2+c^2 \le b^2+bc \le \left(b+\frac{c}{2}\right)^2.$$

Thus, we only need to show that

$$a+b+d=5$$

involves

$$E(a,b,0,d,0) \le 0,$$

where

$$E(a,b,0,d,0)=a^2b^2d^4(a^2+b^2)-\frac{729}{2}.$$

We will show that

$$E(a,b,0,d,0) \le E\left(\frac{a+b}{2},\frac{a+b}{2},0,d,0\right) \le 0. \tag{***}$$

The left inequality is true if

$$32a^2b^2(a^2+b^2) \le (a+b)^6.$$

Indeed, we have

$$(a+b)^6-32a^2b^2(a^2+b^2) \ge 4ab(a+b)^4-32a^2b^2(a^2+b^2)=4ab(a-b)^4 \ge 0.$$

To prove the right inequality, denote

$$u=\frac{a+b}{2}.$$

We need to show that

$$2u+d=5$$

implies

$$E(u,u,0,d,0) \le 0;$$

that is,

$$u^6d^4 \le \frac{729}{4},$$

$$u^3d^2 \le \frac{27}{2}.$$

By the AM-GM inequality, we have

$$5=\frac{2u}{3}+\frac{2u}{3}+\frac{2u}{3}+\frac{d}{2}+\frac{d}{2} \ge 5\sqrt[5]{\left(\frac{2u}{3}\right)^3\left(\frac{t}{2}\right)^2},$$

from which the conclusion follows. The equality holds for $a=b=\frac{3}{2}$, $c=0$, $d=2$ and $e=0$ (or any cyclic permutation).

□

P 1.205. *If $a, b, c, d, e \in [1, 5]$, then*

$$\frac{a-b}{b+c} + \frac{b-c}{c+d} + \frac{c-d}{d+e} + \frac{d-e}{e+a} + \frac{e-a}{a+b} \geq 0.$$

(Vasile Cîrtoaje, 2002)

Solution. Write the inequality as

$$\sum \left(\frac{a-b}{b+c} + \frac{2}{3} \right) \geq \frac{10}{3},$$

$$\sum \frac{3a-b+2c}{b+c} \geq 10.$$

Since

$$3a - b + 2c \geq 3 - 5 + 2 = 0,$$

we may apply the Cauchy-Schwarz inequality to get

$$\sum \frac{3a-b+2c}{b+c} \geq \frac{[\sum(3a-b+2c)]^2}{\sum(b+c)(3a-b+2c)} = \frac{16\left(\sum a\right)^2}{\sum a^2 + 4\sum ab + 3\sum ac}.$$

Therefore, it suffices to show that

$$8\left(\sum a\right)^2 \geq 5\sum a^2 + 20\sum ab + 15\sum ac.$$

Since

$$\left(\sum a\right)^2 = \sum a^2 + 2\sum ab + 2\sum ac,$$

this inequality is equivalent to

$$3\sum a^2 + \sum ac \geq 4\sum ab.$$

Indeed,

$$3\sum a^2 + \sum ac - 4\sum ab = \frac{1}{2}\sum(a-2b+c)^2 \geq 0.$$

The equality holds for $a = b = c = d = e$.

□

P 1.206. *If $a, b, c, d, e, f \in [1, 3]$, then*

$$\frac{a-b}{b+c} + \frac{b-c}{c+d} + \frac{c-d}{d+e} + \frac{d-e}{e+f} + \frac{e-f}{f+a} + \frac{f-a}{a+b} \geq 0.$$

(Vasile Cîrtoaje, 2002)

Solution. Write the inequality as

$$\sum \left(\frac{a-b}{b+c} + \frac{1}{2} \right) \geq 3,$$

$$\sum \frac{2a-b+c}{b+c} \geq 6.$$

Since

$$2a-b+c \geq 2-3+1 = 0,$$

we may apply the Cauchy-Schwarz inequality to get

$$\sum \frac{2a-b+c}{b+c} \geq \frac{\left[\sum(2a-b+c)\right]^2}{\sum(b+c)(2a-b+c)} = \frac{2\left(\sum a\right)^2}{\sum ab + \sum ac}.$$

Thus, we still have to show that

$$\left(\sum a\right)^2 \geq 3\left(\sum ab + \sum ac\right).$$

Let

$$x = a+d, \quad y = b+e, \quad z = c+f.$$

Since

$$\sum ab + \sum ac = xy + yz + zx,$$

we have

$$\left(\sum a\right)^2 - 3\left(\sum ab + \sum ac\right) = (x+y+z)^2 - 3(xy+yz+zx) \geq 0.$$

The equality holds for $a = c = e$ and $b = d = f$.

□

P 1.207. *If* $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ *are positive real numbers, then*

$$\sum_{i=1}^{n} \frac{a_i}{a_{i-1} + 2a_i + a_{i+1}} \leq \frac{n}{4},$$

where $a_0 = a_n$ *and* $a_{n+1} = a_1$.

(Vasile Cîrtoaje, 2008)

Solution. Applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sum_{i=1}^{n}\frac{a_i}{a_{i-1}+2a_i+a_{i+1}}&=\sum_{i=1}^{n}\frac{a_i}{(a_{i-1}+a_i)+(a_i+a_{i+1})}\\&\le\frac{1}{4}\sum_{i=1}^{n}a_i\left(\frac{1}{a_{i-1}+a_i}+\frac{1}{a_i+a_{i+1}}\right)\\&=\frac{1}{4}\left(\sum_{i=1}^{n}\frac{a_i}{a_{i-1}+a_i}+\sum_{i=1}^{n}\frac{a_i}{a_i+a_{i+1}}\right)\\&=\frac{1}{4}\left(\sum_{i=1}^{n}\frac{a_{i+1}}{a_i+a_{i+1}}+\sum_{i=1}^{n}\frac{a_i}{a_i+a_{i+1}}\right)=\frac{n}{4}.\end{aligned}$$

The equality holds for $a_1=a_2=\cdots=a_n$.

□

P 1.208. *Let* $a_1, a_2, \ldots, a_n$ $(n\ge 3)$ *be positive real numbers such that* $a_1a_2\cdots a_n=1$. *Prove that*

$$\frac{1}{n-2+a_1+a_2}+\frac{1}{n-2+a_2+a_3}+\cdots+\frac{1}{n-2+a_n+a_1}\le 1.$$

(Vasile Cîrtoaje, 2008)

First Solution. Let $r=\dfrac{n-2}{n}$. We can get the desired inequality by summing the following inequalities

$$\frac{n-2}{n-2+a_1+a_2}\le\frac{a_3^r+a_4^r+\cdots+a_n^r}{a_1^r+a_2^r+\cdots+a_n^r},$$

$$\frac{n-2}{n-2+a_2+a_3}\le\frac{a_1^r+a_4^r+\cdots+a_n^r}{a_1^r+a_2^r+\cdots+a_n^r},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{n-2}{n-2+a_n+a_1}\le\frac{a_2^r+a_3^r+\cdots+a_{n-1}^r}{a_1^r+a_2^r+\cdots+a_n^r}.$$

The first inequality is equivalent to

$$(a_1+a_2)(a_3^r+a_4^r+\cdots+a_n^r)\ge(n-2)(a_1^r+a_2^r).$$

By the AM-GM inequality, we have

$$a_3^r+a_4^r+\cdots+a_n^r\ge(n-2)(a_3a_4\cdots a_n)^{\frac{r}{n-2}}=\frac{n-2}{(a_1a_2)^{\frac{r}{n-2}}}.$$

Therefore, it suffices to show that

$$a_1+a_2\ge(a_1a_2)^{\frac{r}{n-2}}(a_1^r+a_2^r),$$

or, equivalently,

$$a_1 + a_2 \geq (a_1a_2)^{\frac{1}{n}} \left(a_1^{\frac{n-2}{n}} + a_2^{\frac{n-2}{n}} \right).$$

This is equivalent to the obvious inequality

$$\left(a_1^{\frac{n-1}{n}} - a_2^{\frac{n-1}{n}} \right) \left(a_1^{\frac{1}{n}} - a_2^{\frac{1}{n}} \right) \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n$.

Second Solution. Since

$$\frac{n-2}{n-2+a_1+a_2} = 1 - \frac{a_1+a_2}{n-2+a_1+a_2},$$

we can write the desired inequality as

$$\sum_{i=1}^{n} \frac{a_i + a_{i+1}}{a_i + a_{i+1} + n - 2} \geq 2,$$

where $a_{n+1} = a_1$. Using the Cauchy-Schwarz inequality, we get

$$\sum_{i=1}^{n} \frac{a_i + a_{i+1}}{a_i + a_{i+1} + n - 2} \geq \frac{\left(\sum_{i=1}^{n} \sqrt{a_i + a_{i+1}} \right)^2}{\sum_{i=1}^{n} (a_i + a_{i+1} + n - 2)}$$

$$= \frac{2\sum_{i=1}^{n} a_i + 2 \sum_{1 \leq i < j \leq n} \sqrt{(a_i + a_{i+1})(a_j + a_{j+1})}}{2\sum_{i=1}^{n} a_i + n(n-2)}.$$

Therefore, it suffices to prove that

$$\sum_{1 \leq i < j \leq n} \sqrt{(a_i + a_{i+1})(a_j + a_{j+1})} \geq \sum_{i=1}^{n} a_i + n(n-2).$$

Setting $a_{n+2} = a_2$, by the Cauchy-Schwarz inequality and the AM-GM inequality, we have

$$\sum_{1\le i<j\le n} \sqrt{(a_i+a_{i+1})(a_j+a_{j+1})} =$$

$$= \sum_{i=1}^{n} \sqrt{(a_i+a_{i+1})(a_{i+1}+a_{i+2})} + \sum_{\substack{1\le i<j\le n\\ j\ne i+1}} \sqrt{(a_i+a_{i+1})(a_j+a_{j+1})}$$

$$\ge \sum_{i=1}^{n} (a_{i+1}+\sqrt{a_i a_{i+2}}) + n(n-3)\sqrt[n]{a_1a_2\cdots a_n}$$

$$= \sum_{i=1}^{n} a_i + n(n-3) + \sum_{i=1}^{n} \sqrt{a_i a_{i+2}}$$

$$\ge \sum_{i=1}^{n} a_i + n(n-3) + n\sqrt[n]{a_1a_2\cdots a_n} = \sum_{i=1}^{n} a_i + n(n-2).$$

□

P 1.209. *If $a_1, a_2, ..., a_n \ge 1$, then*

$$\prod \left(a_1 + \frac{1}{a_2} + n - 2\right) \ge n^{n-2}(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right).$$

(Vasile Cîrtoaje, 2011)

Solution. Write the inequality as $E(a_1, a_2, ..., a_n) \ge 0$, and denote

$$A = \left(a_2+\frac{1}{a_3}+n-2\right)\left(a_3+\frac{1}{a_4}+n-2\right)\cdots\left(a_{n-1}+\frac{1}{a_n}+n-2\right).$$

We will prove that

$$E(a_1, a_2, ..., a_n) \ge E(1, a_2, ..., a_n).$$

If this is true, then

$$E(a_1, a_2, ..., a_n) \ge E(1, a_2, ..., a_n) \ge E(1, 1, a_3, ..., a_n) \ge \cdots \ge E(1, 1, ..., 1, a_n) = 0.$$

We have

$$E(a_1, a_2, ..., a_n) - E(1, a_2, ..., a_n) = (a_1 - 1)\left(B - \frac{C}{a_1}\right),$$

where

$$B = A(a_n+n-2) - n^{n-2}\left(\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n}\right),$$

$$C = A\left(\frac{1}{a_2}+n-2\right) - n^{n-2}(a_2+a_3+\cdots+a_n).$$

Since $a_1 - 1 \geq 0$, we need to show that

$$a_1 B - C \geq 0.$$

According to the AM-GM inequality, we have

$$A \geq \left(n\sqrt[n]{\frac{a_2}{a_3}}\right)\left(n\sqrt[n]{\frac{a_3}{a_4}}\right)\cdots\left(n\sqrt[n]{\frac{a_{n-1}}{a_n}}\right) = n^{n-2}\sqrt[n]{\frac{a_2}{a_n}},$$

$$a_n + n - 2 \geq (n-1)\sqrt[n-1]{a_n},$$

$$A(a_n + n - 2) \geq (n-1)n^{n-2}\sqrt[n]{a_2 a_n^{\frac{1}{n-1}}} \geq (n-1)n^{n-2},$$

therefore

$$B \geq n^{n-2}\left(n - 1 - \frac{1}{a_2} - \frac{1}{a_3} - \cdots - \frac{1}{a_n}\right) \geq 0$$

and

$$a_1 B - C \geq B - C = A\left(a_n - \frac{1}{a_2}\right) + n^{n-2}\left(a_2 - \frac{1}{a_2}\right) + \cdots + n^{n-2}\left(a_n - \frac{1}{a_n}\right) \geq 0.$$

The equality holds when $n-1$ of the numbers $a_1, a_2, ..., a_n$ are equal to 1.

□

P 1.210. *If* $a_1, a_2, ..., a_n \geq 1$, *then*

$$\left(a_1 + \frac{1}{a_1}\right)\left(a_2 + \frac{1}{a_2}\right)\cdots\left(a_n + \frac{1}{a_n}\right) + 2^n \geq 2\left(1 + \frac{a_1}{a_2}\right)\left(1 + \frac{a_2}{a_3}\right)\cdots\left(1 + \frac{a_n}{a_1}\right).$$

(Vasile Cîrtoaje, 2011)

Solution. Write the inequality as $E(a_1, a_2, ..., a_n) \geq 0$, and denote

$$A = \left(a_2 + \frac{1}{a_2}\right)\cdots\left(a_n + \frac{1}{a_n}\right),$$

$$B = \left(1 + \frac{a_2}{a_3}\right)\cdots\left(1 + \frac{a_{n-1}}{a_n}\right).$$

We will prove that

$$E(a_1, a_2, ..., a_n) \geq E(1, a_2, ..., a_n).$$

If this is true, then

$$E(a_1, a_2, ..., a_n) \geq E(1, a_2, ..., a_n) \geq E(1, 1, a_3, ..., a_n) \geq \cdots \geq E(1, 1, ..., 1, a_n) = 0.$$

We have

$$E(a_1, a_2, ..., a_n) - E(1, a_2, ..., a_n) = (a_1 - 1)\left(C - \frac{D}{a_1}\right),$$

where

$$C = A - \frac{2B}{a_2},$$

$$D = A - 2Ba_n.$$

Since $a_1 - 1 \geq 0$, we need to show that

$$a_1C - D \geq 0.$$

First, we prove that $C \geq 0$; that is,

$$(a_2^2 + 1) \cdots (a_n^2 + 1) \geq 2(a_2 + a_3) \cdots (a_{n-1} + a_n).$$

By squaring, this inequality becomes

$$(a_2^2 + 1)[(a_2^2 + 1)(a_3^2 + 1)] \cdots [(a_{n-1}^2 + 1)(a_n^2 + 1)](a_n^2 + 1) \geq$$

$$\geq 4(a_2 + a_3)^2 \cdots (a_{n-1} + a_n)^2.$$

By the Cauchy-Schwarz inequality, we have

$$(a_2^2 + 1)(a_3^2 + 1) \geq (a_2 + a_3)^2, \quad ... \ , \ (a_{n-1}^2 + 1)(a_n^2 + 1) \geq (a_{n-1} + a_n)^2.$$

Therefore, we still have to show that

$$(a_2^2 + 1)(a_n^2 + 1) \geq 4,$$

which is clearly true for $a_2 \geq 1$ and $a_n \geq 1$. Finally, we have

$$a_1C - D \geq C - D = 2B\left(a_n - \frac{1}{a_2}\right) \geq 0.$$

The equality holds when $n - 1$ of $a_1, a_2, ..., a_n$ are equal to 1.

□

P 1.211. *Let k and n be positive integers with $k < n$, and let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 \leq a_2 \leq \cdots \leq a_n$. Then*

$$(a_1 + a_2 + \cdots + a_n)^2 \geq n(a_1a_{k+1} + a_2a_{k+2} + \cdots + a_na_{n+k})$$

(where $a_{n+i} = a_i$ for any positive integer i) in the following cases:

(a) $n = 2k$;

(b) $n = 4k$.

(Vasile Cîrtoaje, Crux Mathematicorum, 5, 2005)

Solution. (a) We have to prove that

$$(a_1 + a_2 + \cdots + a_{2k})^2 \geq 4k(a_1a_{k+1} + a_2a_{k+2} + \cdots + a_ka_{2k}).$$

Let x be a real number such that $a_k \leq x \leq a_{k+1}$. Then, obviously,

$$(x - a_1)(a_{k+1} - x) + (x - a_2)(a_{k+2} - x) + \cdots + (x - a_k)(a_{2k} - x) \geq 0.$$

Expanding, rearranging and multiplying by $4k$, we obtain

$$4kx(a_1 + a_2 + \cdots + a_{2k}) \geq 4k^2x^2 + 4k(a_1a_{k+1} + a_2a_{k+2} + \cdots + a_ka_{2k}).$$

On the other hand, by the AM-GM inequality, we have

$$(a_1 + a_2 + \cdots + a_{2k})^2 + 4k^2x^2 \geq 4kx(a_1 + a_2 + \cdots + a_{2k}).$$

Adding these inequalities, we obtain the desired inequality. The equality holds for

$$a_{j+1} = a_{j+2} = \cdots = a_{j+k} = \frac{a_1 + a_2 + \cdots + a_{2k}}{2k},$$

where $j \in \{1, 2, \cdots, k-1\}$.

(b) Let

$$b_i = a_i + a_{2k+i}, \quad i = 1, 2, ..., 2k.$$

Clearly, $b_1 \leq b_2 \leq \cdots \leq b_{2k}$. Applying the inequality from part (a), we obtain

$$(a_1 + a_2 + \cdots + a_{4k})^2 \geq 4k(a_1a_{k+1} + a_2a_{k+2} + \cdots + a_{4k}a_k).$$

$$(b_1 + b_2 + \cdots + b_{2k})^2 \geq 4k(b_1b_{k+1} + b_2b_{k+2} + \cdots + b_kb_{2k}),$$

which is the desired inequality. The equality occurs for

$$\begin{cases} a_{j+1} = a_{j+2} = \cdots = a_{j+k} = a \\ a_{j+2k+1} = a_{j+2k+2} = \cdots = a_{j+3k} = b \\ a_1 + a_2 + \cdots + a_{4k} = 2k(a + b) \end{cases},$$

where $a \leq b$ are real numbers, and $j \in \{1, 2, \cdots, k-1\}$.

Remark. Actually, the inequality holds for any integer k satisfying $\dfrac{n}{4} \leq k \leq \dfrac{n}{2}$ (see Crux Mathematicorum, 2008, volume 34, issue 4).

□

P 1.212. *If* $a_1,\ a_2,\ \dots,\ a_n \in [1,\ 2]$, *then*

$$\sum_{i=1}^{n} \frac{3}{a_i + 2a_{i+1}} \ge \sum_{i=1}^{n} \frac{2}{a_i + a_{i+1}},$$

where $a_{n+1} = a_1$.

(Vasile Cîrtoaje, 2005)

Solution. Rewrite the inequality as follows

$$\sum_{i=1}^{n} \frac{a_i - a_{i+1}}{(a_i + a_{i+1})(a_i + 2a_{i+1})} \ge 0,$$

$$\sum_{i=1}^{n} \left[\frac{k(a_i - a_{i+1})}{(a_i + a_{i+1})(a_i + 2a_{i+1})} + \frac{1}{a_i} - \frac{1}{a_{i+1}}\right] \ge 0, \qquad k > 0,$$

$$\sum_{i=1}^{n} \frac{(a_i - a_{i+1})[(k-3)a_i a_{i+1} - a_i^2 - 2a_{i+1}^2]}{a_i a_{i+1}(a_i + a_{i+1})(a_i + 2a_{i+1})} \ge 0,$$

Setting $k = 6$, the inequality becomes

$$\sum_{i=1}^{n} \frac{(a_i - a_{i+1})^2(2a_{i+1} - a_i)}{a_i a_{i+1}(a_i + a_{i+1})(a_i + 2a_{i+1})} \ge 0.$$

Since $1 \le a_i \le 2$, we have $2a_{i+1} - a_i \ge 0$ for all $i = 1, 2, \dots, n$. Thus, the inequality is proved. The equality holds for $a_1 = a_2 = \cdots = a_n$.

□

P 1.213. *If* $a_1, a_2, \dots, a_n \quad (n \ge 3)$ *are real numbers such that* $a_1 \ge a_2 \ge \cdots \ge a_n$ *and*

$$a_1 a_2 + a_2 a_3 + \cdots + a_n a_1 = n,$$

then

$$(3 - a_1)^2 + (3 - a_2)^2 + \cdots + (3 - a_n)^2 \ge 4n.$$

(Vasile Cîrtoaje, GMA, no. 3-4, 2022)

Solution. Let

$$a = \frac{a_2 + a_3 + \cdots + a_{n-1}}{n - 2}.$$

By Jensen's inequality applied to the convex function $f(x) = (3 - x)^2$, we have

$$(3 - a_2)^2 + (3 - a_3)^2 + \cdots + (3 - a_{n-1})^2 \ge (n - 2)(3 - a)^2.$$

Thus, it suffices to show that

$$(3 - a_1)^2 + (3 - a_n)^2 + (n - 2)(3 - a)^2 \ge 4n.$$

Using the substitutions

$$A = \frac{a+a_1}{2}, \qquad B = \frac{a+a_n}{2},$$

the inequality becomes as follows:

$$(3+a-2A)^2 + (3+a-2B)^2 + (n-2)(3-a)^2 \geq 4n,$$

$$4(A^2+B^2) - 4(3+a)(A+B) + 2(3+a)^2 + (n-2)(3-a)^2 - 4n \geq 0,$$

$$4(A+B)^2 - 4(3+a)(A+B) + 2(3+a)^2 + (n-2)(3-a)^2 - 4n \geq 8AB,$$

$$(2A+2B-3-a)^2 + (3+a)^2 + (n-2)(3-a)^2 - 4n \geq 8AB.$$

It is true if

$$(3+a)^2 + (n-2)(3-a)^2 - 4n \geq 8AB.$$

By Lemma below, we have:

$$4AB + (n-4)a^2 \leq n.$$

So, it suffices to show that

$$(3+a)^2 + (n-2)(3-a)^2 - 4n \geq 2n - 2(n-4)a^2,$$

which is equivalent to

$$3(n-3)(a-1)^2 \geq 0.$$

The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.

Lemma. *If $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ are real numbers such that $a_1 \geq a_2 \geq \cdots \geq a_n$, then*

$$(a+a_1)(a+a_n) + (n-4)a^2 \leq a_1a_2 + a_2a_3 + \cdots + a_na_1,$$

where

$$a = \frac{a_2 + a_3 + \cdots + a_{n-1}}{n-2}.$$

Proof. For $n = 3$, the inequality is an identity. For $n > 3$, since the sequences $(a_2, \ldots, a_{n-2})$ and $(a_3, \ldots, a_{n-1})$ are decreasing, by Chebyshev's inequality we have

$$(n-3)(a_2a_3 + \cdots + a_{n-2}a_{n-1}) \geq (a_2 + \cdots + a_{n-2})(a_3 + \cdots + a_{n-1}).$$

Thus, the desired inequality is true if

$$(a_1+a_n)a + (n-3)a^2 \leq a_1a_2 + a_{n-1}a_n + \frac{1}{n-3}(a_2 + \cdots + a_{n-2})(a_3 + \cdots + a_{n-1}),$$

which is equivalent to

$$a_1(a-a_2) + a_n(a-a_{n-1}) + (n-3)a^2 \leq \frac{1}{n-3}[(n-2)a - a_{n-1}][(n-2)a - a_2].$$

Since $a - a_2 \le 0$ and $a - a_{n-1} \ge 0$, it suffices to show that

$$a_2(a - a_2) + a_{n-1}(a - a_{n-1}) + (n-3)a^2 \le \frac{1}{n-3}[(n-2)a - a_{n-1}][(n-2)a - a_2],$$

that is

$$(2n-5)a^2 - (2n-5)(a_2 + a_{n-1})a + (n-3)(a_2^2 + a_{n-1}^2) + a_2 a_{n-1} \ge 0,$$

$$(2n-5)\left(a - \frac{a_2 + a_{n-1}}{2}\right)^2 + \frac{2n-7}{4}(a_2 - a_{n-1})^2 \ge 0.$$

Remark 1. Actually, 3 is the largest real value of k such that

$$(k - a_1)^2 + (k - a_2)^2 + \cdots + (k - a_n)^2 \ge n(k-1)^2$$

for any real numbers a_i with $a_1 \ge a_2 \ge \cdots \ge a_n$ and $a_1a_2 + a_2a_3 + \cdots + a_na_1 = n$. Choosing $a_1 > a_2 = \cdots = a_{n-1} = 1 > a_n > 0$ and denoting $s = (a_1 + a_n)/2$, the equality constraint becomes $a_1a_n + 2s = 3$. From $3 = a_1a_n + 2s > 2s$ and $3 = a_1a_n + 2s < s^2 + 2s$, we get $s \in (1, 3/2)$. The inequality can be written as follows:

$$(k - a_1)^2 + (k - a_n)^2 \ge 2(k-1)^2, \qquad (s-1)(s+2-k) \ge 0, \qquad s + 2 - k \ge 0.$$

Taking $s \to 1$, we get the necessary condition $k \le 3$.

Remark 2. We can write the inequalities from P 1.213 as

$$a_1^2 + a_2^2 + \cdots + a_n^2 + 5n \ge 6(a_1 + a_2 + \cdots + a_n).$$

Since

$$\frac{a_1 + a_2 + \cdots + a_n}{n} + \frac{n}{a_1 + a_2 + \cdots + a_n} \ge 2$$

for $a_1 + a_2 + \cdots + a_n > 0$, the following inequality follows:

- *If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are nonnegative real numbers such that*

$$a_1a_2 + a_2a_3 + \cdots + a_na_1 = n, \qquad a_1 \ge a_2 \ge \cdots \ge a_n,$$

then

$$a_1^2 + a_2^2 + \cdots + a_n^2 + \frac{6n^2}{a_1 + a_2 + \cdots + a_n} \ge 7n.$$

□

P 1.214. *Let a, b, c, d be positive real numbers such that $ab + bc + cd + da = 4$.*

(a) *If $a \ge b \ge 1 \ge c \ge d$, then*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + 8 \ge 3(a + b + c + d).$$

(b) *If $a \ge b \ge c \ge 1 \ge d$, then the inequality above holds true.*

(Vasile Cîrtoaje, SSMJ, 4 and 6, 2024)

Solution. Write the constraint as

$$(a+c)(b+d)=4.$$

(a) Let

$$x=\frac{a+c}{2}, \qquad y=\frac{b+d}{2}, \qquad xy=1.$$

From $(a-1)(c-1)\le 0$ and $(b-1)(d-1)\le 0$, we get

$$ac\le 2x-1, \qquad bd\le 2y-1, \qquad \frac{1}{2}<y\le x<2.$$

Since

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=\frac{a+c}{ac}+\frac{b+d}{bd}\ge\frac{2x}{2x-1}+\frac{2y}{2y-1},$$

it suffices to show that

$$\frac{x}{2x-1}+\frac{y}{2y-1}+4\ge 3(x+y),$$

which is equivalent to

$$\frac{x}{2x-1}+\frac{1}{2-x}+4\ge 3\left(x+\frac{1}{x}\right),$$

$$(x-1)^4\ge 0.$$

The equality occurs for $a=b=c=d=1$.

(b) Let

$$x=\frac{a+d}{2}, \qquad y=\frac{b+c}{2}, \qquad y\ge 1.$$

Since

$$\frac{1}{b}+\frac{1}{c}\ge\frac{4}{b+c}=\frac{2}{y},$$

it suffices to show that

$$\frac{1}{a}+\frac{1}{d}+\frac{2}{y}+8\ge 3(a+d+2y),$$

that is

$$\frac{x}{ad}+\frac{1}{y}+4\ge 3(x+y).$$

From

$$4xy-4=(a+d)(b+c)-(a+c)(b+d)=(a-b)(c-d)\ge 0,$$

we get

$$xy\ge 1,$$

and from

$$4-(a+y)(y+d)=(a+c)(b+d)-(a+y)(y+d)=a(b-y)+(bc-y^2)+d(c-y)$$

$$= \frac{a(b-c)}{2} - \frac{(b-c)^2}{4} - \frac{d(b-c)}{2} = \frac{(b-c)(2a+c-b-2d)}{4} \geq 0,$$

we get $(a+y)(y+d) \leq 4$, hence

$$ad \leq 4 - y^2 - 2xy.$$

So, it suffices to show that

$$\frac{x}{4-y^2-2xy} + \frac{1}{y} + 4 \geq 3(x+y),$$

that is

$$x + (4-y^2-2xy)\left(\frac{1}{y} + 4 - 3x - 3y\right) \geq 0.$$

For fixed y, the inequality is true if $f(x) \geq 0$, where

$$f(x) = 6y^2x^2 + y(9y^2 - 8y - 13)x + (y^2-4)(3y^2-4y-1).$$

Since

$$f'(x) = 12y^2x + y(9y^2-8y-13) \geq 12y + y(9y^2-8y-13)$$
$$= y(9y^2-8y-1) = y(y-1)(9y+1) \geq 0,$$

f is an increasing function, therefore

$$f(x) \geq f\left(\frac{1}{y}\right) = 3y^4 - 4y^3 - 4y^2 + 8y - 3 = (y-1)^2(3y^2+2y-3) \geq 0.$$

The equality occurs for $a=b=c=d=1$.

Remark 1. Points (a) and (b) can be combined as follows:

- *Let* $a \geq b \geq c \geq d > 0$ *such that* $ab+bc+cd+da = 4$. *If* $b \geq 1$, *then*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + 8 \geq 3(a+b+c+d).$$

Note that the inequality

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} - 4 \geq k(a+b+c+d-4)$$

does not hold true for $k > 3$. To prove this assertion, we set $b=c=1$, when the inequality becomes

$$\frac{1}{a} + \frac{1}{d} - 2 \geq k(a+d-2)$$

under the constraint

$$ad + a + d = 3.$$

Denoting $x = (a+d)/2$, we have $ad = 3-2x$ and $x \in [1, 3/2)$. The condition $x \geq 1$ follows from

$$3 = ad + S \leq x^2 + x.$$

Consider now that $x \in (1, 3/2)$, and write the inequality as follows:

$$\frac{x}{ad} - 1 \geq k(x-1),$$

$$\frac{x}{3-2x} - 1 \geq k(x-1),$$

$$k \leq \frac{3}{3-2x}.$$

From this, we get the necessary condition

$$k \leq \lim_{x \to 1} \frac{3}{3-2x} = 3.$$

Remark 2. Since

$$\frac{a+b+c+d}{4} + \frac{4}{a+b+c+d} \geq 2,$$

the following statement follows:

- *Let* $a \geq b \geq c \geq d > 0$ *such that* $ab+bc+cd+da = 4$. *If* $b \geq 1$, *then*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{48}{a+b+c+d} \geq 16.$$

□

P 1.215. *If* a, b, c, d *are nonnegative real numbers such that*

$$ab + bc + cd + da = 4, \qquad a \geq b \geq c \geq d,$$

then

$$\frac{4}{3} \leq \frac{1}{a+2} + \frac{1}{b+2} + \frac{1}{c+2} + \frac{1}{d+2} \leq \frac{3}{2}.$$

(Vasile Cîrtoaje, Math. Reflections, 4, 2024)

Solution. The hypothesis is equivalent to

$$(a+c)(b+d) = 4.$$

Therefore,

$$a + c \geq 2 \geq b + d.$$

(a) To prove the right inequality, we see that

$$\frac{1}{a+2}+\frac{1}{c+2}=\frac{a+c+4}{ac+2(a+c)+4}\le\frac{a+c+4}{2(a+c+2)}$$

and

$$\frac{1}{b+2}+\frac{1}{d+2}=\frac{b+d+4}{bd+2(b+d)+4}\le\frac{b+d+4}{2(b+d+2)}=\frac{a+c+1}{a+c+2},$$

hence

$$\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}+\frac{1}{d+2}\le\frac{a+c+4}{2(a+c+2)}+\frac{a+c+1}{a+c+2}=\frac{3}{2}.$$

The equality occurs for $ab=4$ and $c=d=0$.

(b) Let

$$x=\frac{a+d}{2},\qquad y=\frac{b+c}{2}.$$

From

$$4xy-4=(a+d)(b+c)-(a+c)(b+d)=(a-b)(c-d)\ge 0,$$

we get

$$xy\ge 1,$$

and from

$$4-(a+y)(y+d)=(a+c)(b+d)-(a+y)(y+d)=a(b-y)+(bc-y^2)+d(c-y)$$

$$=\frac{a(b-c)}{2}-\frac{(b-c)^2}{4}-\frac{d(b-c)}{2}=\frac{(b-c)(2a+c-b-2d)}{4}\ge 0,$$

we get $(a+y)(y+d)\le 4$, hence

$$ad\le 4-2xy-y^2.$$

Since

$$\frac{1}{b+2}+\frac{1}{c+2}\ge\frac{2}{y+2},$$

we only need to show that

$$\frac{1}{a+2}+\frac{1}{d+2}+\frac{2}{y+2}\ge\frac{4}{3},$$

that is

$$\frac{x+2}{ad+4x+4}+\frac{1}{y+2}\ge\frac{2}{3}.$$

So, it suffices to show that

$$\frac{x+2}{8+4x-2xy-y^2}+\frac{1}{y+2}\ge\frac{2}{3},$$

that is

$$x+2+(8+4x-2xy-y^2)\left(\frac{1}{y+2}-\frac{2}{3}\right)\ge 0,$$

$$x(4y^2-3y+2)+2y^3+y^2-10y+4\ge 0.$$

Since $xy\ge 1$, it is true if

$$\frac{4y^2-3y+2}{y}+2y^3+y^2-10y+4\ge 0,$$

which is equivalent to

$$2y^4+y^3-6y^2+y+2\ge 0,$$

$$(y-1)^2(2y^2+5y+2)\ge 0.$$

The proof is completed. The equality occurs for $a=b=c=d=1$.

□

P 1.216. *If $n\ge 6$ and $a_1\ge 1\ge a_2\ge\cdots\ge a_n$ such that $a_1a_2+a_2a_3+\cdots+a_na_1=n$, then*

$$\frac{1}{a_1+3}+\frac{1}{a_2+3}+\cdots+\frac{1}{a_n+3}\ge\frac{n}{4}.$$

(Vasile Cîrtoaje, Crux Mathematicorum, 6, 2024)

Solution. Let

$$S=\frac{a_1+a_n}{2},\qquad x=\frac{a_2+a_3+\cdots+a_{n-1}}{n-2},$$

where

$$a_1\ge 1\ge x\ge a_n.$$

By the AM-HM inequality, we have

$$\frac{1}{a_2+3}+\cdots+\frac{1}{a_{n-1}+3}\ge\frac{(n-2)^2}{(a_2+3)+\cdots+(a_{n-1}+3)}=\frac{n-2}{x+3},$$

and it suffices to show that

$$\frac{1}{a_1+3}+\frac{1}{a_n+3}+\frac{n-2}{x+3}\ge\frac{n}{4}.$$

By Lemma below, we have $(n-3)x^2+x(a_1+a_n)+a_1a_n\le n$. Since the left hand side of the desired inequality decreases when a_1 increases, we may replace this inequality constraint with the equality constraint $(n-3)x^2+x(a_1+a_n)+a_1a_n=n$, i.e.

$$a_1a_n=n-2xS-(n-3)x^2.$$

From $(a_1-x)(a_n-x)\le 0$, we get $a_1a_n\le 2xS-x^2$, and from $n-2xS-(n-3)x^2=a_1a_n\le 2xS-x^2$, we get

$$S\ge S_1=\frac{n-(n-4)x^2}{4x}.$$

Since

$$\frac{1}{a_1+3}+\frac{1}{a_n+3}=\frac{2S+6}{a_1a_n+9+6S}=\frac{2S+6}{n+9+2(3-x)S-(n-3)x^2},$$

we need to show that

$$\frac{2S+6}{n+9+2(3-x)S-(n-3)x^2}+\frac{n-8-nx}{4(x+3)}\geq 0,$$

which can be written as $2A(x)S+B(x)\geq 0$, where

$$A(x)=nx^2-4(n-3)x+3(n-4)=(x-1)(nx-3n+12),$$

$$B(x)=n(n-3)x^3-(n-3)(n-8)x^2-(n^2+9n-24)x+n(n+1)$$
$$=(x-1)[n(n-3)x^2+8(n-3)x-n(n+1)].$$

Since $x\leq 1$ and $3n-12-nx\geq 3n-12-n=2(n-6)\geq 0$, we have $A(x)\geq 0$. So, it suffices to show that $2A(x)S_1+B(x)\geq 0$, which is equivalent to the obvious inequality $(x-1)^2h(x)\geq 0$, where

$$h(x)=(n-2)x^2+2(2n-5)x+3(n-4)>0.$$

Thus, the proof is completed. For $k_n=3$, the equality occurs when $a_2=\cdots=a_{n-1}=1$ and $a_1+a_n+a_1a_n=3$ such that $a_1\geq 1\geq a_n$.

Lemma. *Let $a_1\geq a_2\geq\ldots\geq a_n\geq 0$ such that $a_1a_2+a_2a_3+\cdots+a_na_1=n$. If $n\geq 4$ and $x=\dfrac{a_2+\cdots+a_{n-1}}{n-2}$, then*

$$(n-3)x^2+x(a_1+a_n)+a_1a_n\leq n,$$

with equality for $a_2=\cdots=a_{n-1}$.

Proof. Write the desired inequality as follows:

$$(n-3)x^2+x(a_1+a_n)+a_1a_n\leq a_1a_2+a_2a_3+\cdots+a_na_1,$$

$$(n-3)x^2+a_1(x-a_2)+a_n(x-a_{n-1})\leq a_2a_3+\cdots+a_{n-2}a_{n-1}.$$

Since $x-a_2\leq 0$ and $x-a_{n-1}\geq 0$, it suffices to show that

$$(n-3)x^2+a_2(x-a_2)+a_{n-1}(x-a_{n-1})\leq a_2a_3+\cdots+a_{n-2}a_{n-1},$$

which can be rewritten as

$$a_2a_3+\cdots a_{n-2}a_{n-1}\geq (n-3)x^2+(a_2+a_{n-1})x-a_2^2-a_{n-1}^2.$$

Since the sequences $a_2,a_3,\ldots,a_{n-2}$ and $a_3,a_4,\ldots,a_{n-1}$ are decreasing, by Chebyshev's inequality we have

$$(n-3)(a_2a_3+\cdots+a_{n-2}a_{n-1})\geq(a_2+\cdots+a_{n-2})(a_3+\cdots+a_{n-1})=((n-2)x-a_{n-1})((n-2)x-a_2).$$

Thus, it suffices to show that

$$\frac{((n-2)x-a_{n-1})\,((n-2)x-a_2)}{n-3} \ge (n-3)x^2+(a_2+a_{n-1})x-a_2^2-a_{n-1}^2,$$

which is equivalent to

$$(2n-5)x^2-(2n-5)(a_2+a_{n-1})x+(n-3)(a_2^2+a_{n-1}^2)+a_2a_{n-1}\ge 0,$$

$$(2n-5)(2x-a_2-a_{n-1})^2+(2n-7)(a_2-a_{n-1})^2\ge 0.$$

Clearly, the last inequality is true.

Remark. Note that 3 is the largest positive value of k such that the inequality

$$\frac{1}{a_1+k}+\frac{1}{a_2+k}+\cdots+\frac{1}{a_n+k}\ge\frac{n}{1+k}$$

holds for $n\ge 6$ and all nonnegative numbers $a_1,a_2,\ldots,a_n$ satisfying

$$a_1a_2+a_2a_3+\cdots+a_na_1=n, \qquad a_1\ge 1\ge a_2\ge\cdots\ge a_n.$$

Indeed, by setting $a_1=3$, $a_2=\cdots=a_{n-1}=1$ and $a_n=0$, the desired inequality leads to the necessary condition $k\le 3$.

□

P 1.217. *If x_1,x_2,x_3,x_4,x_5 are nonnegative real numbers such that*

$$x_1x_2+x_2x_3+x_3x_4+x_4x_5+x_5x_1=5,$$

then

$$\frac{1}{5x_1+4}+\frac{1}{5x_2+4}+\frac{1}{5x_3+4}+\frac{1}{5x_4+4}+\frac{1}{5x_5+4}\ge\frac{5}{9}.$$

(Vasile Cîrtoaje, AMM, 6, 2023)

Solution. According to Lemma below, it is sufficient to show that

$$\frac{1}{5a+4}+\frac{1}{5b+4}+\frac{1}{5c+4}+\frac{1}{5d+4}+\frac{1}{5e+4}\ge\frac{5}{9}$$

for $a\ge b\ge c\ge d\ge e\ge 0$ such that $ae+ad+be+bc+cd=5$. We will prove the extended inequality

$$\frac{1}{a+k}+\frac{1}{b+k}+\frac{1}{c+k}+\frac{1}{d+k}+\frac{1}{e+k}\ge\frac{5}{1+k}$$

for

$$0<k\le k_0, \qquad k_0=\frac{\sqrt{54-2\sqrt{5}}-3\sqrt{5}+3}{4}\approx 0.83234991.$$

Denote
$$x=\frac{a+b}{2}, \qquad y=\frac{d+e}{2}, \qquad a\ge x\ge b\ge c\ge d\ge y\ge e.$$
Replacing a and e with $2x-b$ and $2y-d$, respectively, we have
$$5=a(d+e)+be+bc+cd=2(2x-b)y+b(2y-d)+bc+cd=4xy+bc-(b-c)d.$$
From this, we get
$$5\ge 4xy+bc-(b-c)c=4xy+c^2,$$
hence
$$4xy\le 5-c^2, \qquad c<\sqrt{5},$$
and
$$5=4xy+bc-(b-c)d\le 4xy+bc-(b-c)y=4xy+b(c-y)+cy$$
$$\le 4xy+x(c-y)+cy=3xy+c(x+y)\le \frac{3}{4}(5-c^2)+c(x+y),$$
hence
$$4c(x+y)\ge 3c^2+5.$$
By the AM-HM inequality, we have
$$\frac{1}{a+k}+\frac{1}{b+k}\ge\frac{4}{a+b+2k}, \qquad \frac{1}{d+k}+\frac{1}{e+k}\ge\frac{4}{x_4+x_5+2k}.$$
So, it suffices to show that the conditions
$$4xy\le 5-c^2, \qquad 4c(x+y)\ge 3c^2+5, \qquad x\ge c\ge y\ge 0, \qquad c\le\sqrt{2}$$
involve
$$\frac{2}{x+k}+\frac{2}{y+k}+\frac{1}{c+k}\ge\frac{5}{1+k},$$
that is
$$\frac{2(x+y)+4k}{xy+k(x+y)+k^2}+\frac{1}{c+k}\ge\frac{5}{1+k}.$$
Since $4xy\le 5-c^2$, it suffices to show that
$$A+\frac{1}{c+k}\ge\frac{5}{1+k},$$
where
$$A=\frac{8(x+y)+16k}{5-c^2+4k(x+y)+4k^2}=\frac{2}{k}\cdot\frac{4k(x+y)+8k^2}{5-c^2+4k(x+y)+4k^2}$$
$$=\frac{2}{k}\left[1+\frac{4k^2-5+c^2}{5-c^2+4k(x+y)+4k^2}\right].$$

Case 1: $4k^2-5+c^2\ge 0$. Since
$$A\ge\frac{2}{k},$$

we need to show that

$$\frac{2}{k}+\frac{1}{c+k}\ge\frac{5}{1+k},$$

which is true when

$$\frac{2}{k}+\frac{1}{\sqrt{5}+k}\ge\frac{5}{1+k},$$

hence when $0<k\le k_0$.

Case 2: $4k^2-5+c^2\le 0$. Since $4c(x+y)\ge 3c^2+5$, it suffices to consider the case $4c(x+y)=3c^2+5$, when

$$A\ge\frac{2}{k}\left[1+\frac{4k^2+5-c^2}{5-c^2+k(3c^2+5)/c+4k^2}\right]=\frac{6c^2+16kc+10}{5k+(4k^2+5)c+3kc^2-c^3}.$$

Thus, we need to prove that

$$\frac{6c^2+16kc+10}{5k+(4k^2+5)c+3kc^2-c^3}+\frac{1}{c+k}\ge\frac{5}{1+k},$$

which is equivalent to

$$c^4+(1-k)c^3-(2k^2-5k+5)c^2+(4k^2-7k+3)c-2k^2+3k\ge 0,$$

$$(c-1)^2[c^2+(3-k)c+k(3-2k)]\ge 0.$$

The proof is completed. The equality $E(a,b,c,d,e)=\dfrac{4}{19}$ occurs for $a=b=c=d=e=1$, while the original inequality is an equality for $x_1=x_2=x_3=x_4=x_5=1$.

Lemma. *Let x_1,x_2,x_3,x_4,x_5 be nonnegative real numbers such that $x_1x_2+x_2x_3+x_3x_4+x_4x_5+x_5x_1=5$, and let $E(x_1,x_2,x_3,x_4,x_5)$ be a symmetric and decreasing function with respect to each variable. If $E(a,b,c,d,e)\ge 0$ for any $a\ge b\ge c\ge d\ge e\ge 0$ such that $ae+ad+be+bc+cd=5$, then $E(x_1,x_2,x_3,x_4,x_5)\ge 0$.*

Proof. Let $T=(T_1,T_2,T_3,T_4,T_5)$ and $t=(t_1,t_2,t_3,t_4,t_5)$ be two decreasing sequences of nonnegative real numbers. By Karamata majorization inequality applied to the convex function $f(x)=e^x$, if $T_1\cdots T_j\ge t_1\cdots t_j$ for $j=1,2,3,4,5$, then

$$T_1+T_2+T_3+T_4+T_5\ge t_1+t_2+t_3+t_4+t_5.$$

If (a,b,c,d,e) is a permutation of (x_1,x_2,x_3,x_4,x_5) such that $a\ge b\ge c\ge d\ge e\ge 0$, then

$$E(a,b,c,d,e)=E(x_1,x_2,x_3,x_4,x_5)$$

and

$$\sum_{sym}ab=\sum_{sym}x_1x_2,$$

where

$$\sum_{sym}x_1x_2=\sum_{1\le i<j\le 5}x_ix_j.$$

Let $T = (ab, ac, bd, ce, de)$ be a decreasing sequence, and t a decreasing permutation of the sequence $(x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1)$. Since $T_1 \cdots T_j \ge t_1 \cdots t_j$ for $j = 1, 2, 3, 4, 5$, by Karamata's inequality we have

$$ab + ac + bd + ce + de \ge x_1x_3 + x_3x_5 + x_5x_2 + x_2x_4 + x_4x_1,$$

which is equivalent to

$$\sum_{sym} ab - (ab + ac + bd + ce + de) \le \sum_{sym} x_1x_2 - (x_1x_3 + x_3x_5 + x_5x_2 + x_2x_4 + x_4x_1),$$

i.e.

$$ae + ad + be + bc + cd \le x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 = 5.$$

In the case $ae+ad+be+bc+cd < 5$, by increasing the numbers a, b, c, d, e to have $ae+ad+be+bc+cd = 5$ and to keep the constraint $a \ge b \ge c \ge d \ge e \ge 0$, the function $E(a, b, c, d, e)$ decreases, therefore

$$E(a, b, c, d, e) \le E(x_1, x_2, x_3, x_4, x_5).$$

On the other hand, by hypothesis, $E(a, b, c, d, e) \ge 0$. So, we have

$$E(x_1, x_2, x_3, x_4, x_5) \ge E(a, b, c, d, e) \ge 0.$$

□

P 1.218. *If a, b, c, d, e are nonnegative real numbers such that*

$$ab + bc + cd + de + ea = 5,$$

then

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} + \frac{1}{e+1} \ge \frac{5}{2}.$$

(Vasile Cîrtoaje, AMM, 6, 2023)

Solution. Assume that $a = \max\{a, b, c, d, e\}$, $a \ge 1$. Since

$$(a + c)b + (a + d)e = 5 - cd, \qquad cd \le 5,$$

$$(a + c)(b + 1) + (a + d)(e + 1) = 2a + c + d + 5 - cd,$$

by the Cauchy-Schwarz inequality we have

$$\frac{1}{b+1} + \frac{1}{e+1} \ge \frac{(\sqrt{a+c} + \sqrt{a+d})^2}{(a+c)(b+1) + (a+d)(e+1)}$$

$$= \frac{2a + c + d + 2\sqrt{(a+c)(a+d)}}{2a + c + d + 5 - cd} \ge \frac{4a + c + d + 2\sqrt{cd}}{2a + c + d + 5 - cd}.$$

Thus, it suffices to prove that

$$\frac{1}{a+1}+\frac{1}{c+1}+\frac{1}{d+1}+\frac{4a+c+d+2\sqrt{cd}}{2a+c+d+5-cd}\ge\frac{5}{2} \qquad (*)$$

for $a=\max\{a,c,d\}$, $a\ge 1$ and $cd\le 5$. Due to symmetry, we may assume that $c\ge d$, hence

$$a\ge c\ge d, \qquad a\ge 1, \qquad cd\le 5.$$

Denote

$$x=\frac{c+d}{2}, \qquad p=\sqrt{cd}.$$

From $c+d\ge 2\sqrt{cd}$ and $(a-c)(a-d)\ge 0$, we get $x\ge p$ and $a^2-2ax+p^2\ge 0$, therefore

$$p\le x\le\frac{a^2+p^2}{2a}.$$

For fixed a and p, the inequality (*) can be written as

$$\frac{1}{a+1}+\frac{2(x+1)}{1+2x+p^2}+\frac{4a+2x+2p}{2a+2x+5-p^2}\ge\frac{5}{2},$$

which is equivalent to $f(x)\ge 0$, where f is a polynomial of second order with the expression

$$f(x)=4(1-a)x^2+B(a,p)x+C(a,p).$$

Since f is concave, it suffices to prove the inequality $f(x)\ge 0$ for $x=p$ and $x=\dfrac{a^2+p^2}{2a}$, therefore for $c=d$ and for $c=a$.

Case 1: $c=d\le\sqrt{5}$. We need to prove the inequality

$$\frac{1}{a+1}+\frac{2}{c+1}+\frac{4a+4c}{2a+2c+5-c^2}\ge\frac{5}{2},$$

which is equivalent to

$$2(3-c)a^2+(5c^3-c^2-17c+5)a+3c^3+c^2-5c+5\ge 0,$$

$$2(3-c)(a-c)^2+5(c^3-c^2-c+1)a+5(c^3-c^2-c+1)\ge 0,$$

$$2(3-c)(a-c)^2+5(c-1)^2(c+1)(a+1)\ge 0.$$

Case 2: $c=a$. We need to show that

$$\frac{2}{a+1}+\frac{1}{d+1}+\frac{5a+d+2\sqrt{ad}}{3a+d+5-ad}\ge\frac{5}{2},$$

which is equivalent to

$$5a^2d^2-2ad(a+d)+a^2+d^2-20ad-2(a+d)+5+4(a+1)(d+1)\sqrt{ad}\ge 0,$$

$$(a-1)^2(d-1)^2+4\left[a^2d^2-6ad+1+(a+1)(d+1)\sqrt{ad}\right]\ge 0,$$

$$\frac{1}{4}(a-1)^2(d-1)^2+(ad-1)^2+\sqrt{ad}\left[(\sqrt{a}-\sqrt{d})^2+(\sqrt{ad}-1)^2\right]\ge 0.$$

The proof is completed. The equality occurs for $a=b=c=d=e=1$.

Remark. In our opinion, the inequality

$$\frac{1}{a+k}+\frac{1}{b+k}+\frac{1}{c+k}+\frac{1}{d+k}+\frac{1}{e+k}\ge\frac{5}{1+k}$$

holds for $k\in[0,k_1]$, where

$$k_1=\frac{3\sqrt{5}-4+4\sqrt{4-\sqrt{5}}}{2}\approx 1.1570$$

is a root of the equation

$$\frac{2}{\sqrt{5}+k}+\frac{2}{k}=\frac{5}{1+k}.$$

In addition, the condition $k\le k_1$ is necessary. Indeed, by choosing $a=b=\sqrt{5}$ and $c=e=0$, the equality constraint is satisfied and the inequality becomes

$$\frac{2}{\sqrt{5}+k}+\frac{1}{d+k}\ge\frac{3k-2}{k(1+k)}.$$

Clearly, the inequality is true for all $d\ge 0$ if and only if

$$\frac{2}{\sqrt{5}+k}\ge\frac{3k-2}{k(1+k)},$$

that is $k\le k_1$.

□

P 1.219. *If* $a_1,a_2,\dots,a_8$ *are nonnegative real numbers such that* $a_1a_2+a_2a_3+\cdots+a_8a_1=8$, *then*

$$\frac{1}{5a_1+3}+\frac{1}{5a_2+3}+\cdots+\frac{1}{5a_8+3}\ge 1.$$

(Vasile Cîrtoaje, GMA, no. 3-4, 2024)

Solution. We first show that

$$64\left(\frac{1}{5a_1+3}+\frac{1}{5a_2+3}\right)+5a_1a_2\ge 21.$$

Denoting $s=\dfrac{a_1+a_2}{2}$ and $p=\sqrt{a_1a_2}$, the inequality becomes as follows:

$$\frac{64(10s+6)}{25p^2+30s+9}+5p^2\ge 21,$$

$$2(15p^2+1)s+25p^4-96p^2+39\ge 0.$$

Since $s\ge p$, we have

$$2(15p^2+1)s+25p^4-96p^2+39\ge 2(15p^2+1)p+25p^4-96p^2+39$$

$$=25P^4+30p^3-96p^2+2p+39=(p-1)^2(25p^2+80p+39)\ge 0.$$

Thus, from

$$\sum_{cyc}\left[64\left(\frac{1}{5a_1+3}+\frac{1}{5a_2+3}\right)+5a_1a_2-21\right]\ge 0,$$

we get the desired inequality. The equality occurs for $a_1=a_2=\cdots=a_8=1$.

Remark 1. Similarly, we can prove the following statement:

- *If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers such that*

$$a_1a_2+a_2a_3+\cdots+a_na_1=n,$$

then

$$\frac{1}{5a_1+3}+\frac{1}{5a_2+3}+\cdots+\frac{1}{5a_n+3}\ge\frac{n}{8}.$$

Remark 2. The following stronger inequality is true.

- *If $k\ge\dfrac{\sqrt{5}+1}{2}\approx 1.618$ and $a_1,a_2,\ldots,a_n$ are nonnegative real numbers such that*

$$a_1a_2+a_2a_3+\cdots+a_na_1=n,$$

then

$$\frac{1}{ka_1+1}+\frac{1}{ka_2+1}+\cdots+\frac{1}{ka_n+1}\ge\frac{n}{k+1}.$$

The proof is based on the inequality

$$(k+1)^2\left(\frac{1}{ka_1+1}+\frac{1}{ka_2+1}\right)+ka_1a_2\ge 3k+2,$$

which is equivalent to

$$2(k^2-k-1+kp^2)s+k^2p^4-(3k^2+2k-1)p^2+2k+1\ge 0,$$

where $s=\dfrac{a_1+a_2}{2}$ and $p=\sqrt{a_1a_2}$. For $p=0$, the inequality becomes

$$2(k^2-k-1)s+2k+1\ge 0,$$

and it is true for $k\ge\dfrac{\sqrt{5}+1}{2}$. For $p>0$, it suffices to show that

$$2(k^2-k-1+kp^2)p+k^2p^4-(3k^2+2k-1)p^2+2k+1\ge 0,$$

which is equivalent to

$$k^2p^4 + 2kp^3 - (3k^2 + 2k - 1)p^2 + 2(k^2 - k - 1)p + 2k + 1 \ge 0,$$

$$(p-1)^2[k^2p^2 + 2(k+1)p + 2k + 1] \ge 0.$$

Remark 3. The following nice *open inequality* is true.

- *If $a_1, a_2, \dots, a_7$ are nonnegative real numbers such that $a_1a_2 + a_2a_3 + \cdots + a_7a_1 = 7$, then*

$$\frac{1}{a_1+1} + \frac{1}{a_2+1} + \cdots + \frac{1}{a_7+1} \ge \frac{7}{2}.$$

Note that the inequality

$$\frac{1}{a_1+k} + \frac{1}{a_2+k} + \cdots + \frac{1}{a_7+k} \ge \frac{7}{1+k}$$

doesn't hold for $k > 1$. Choosing $a_1 = a_4 = 3$, $a_2 = a_3 = 1$ and $a_5 = a_7 = 0$, the equality constraint is satisfied and the inequality becomes:

$$\frac{1}{a_6+k} + \frac{2}{3+k} + \frac{2}{k} \ge \frac{5}{1+k}.$$

Moreover, for $a_6 \to \infty$, we get the necessary condition

$$\frac{2}{3+k} + \frac{2}{k} \ge \frac{5}{1+k},$$

which involves $k \le 1$.

Choosing $a_5 = a_7 = 0$ and then $a_6 \to \infty$, we get the following very nice inequality (see P 2.119):

- *If a_1, a_2, a_3, a_4 are nonnegative real numbers such that $a_1a_2 + a_2a_3 + a_3a_4 = 7$, then*

$$\frac{1}{a_1+1} + \frac{1}{a_2+1} + \frac{1}{a_3+1} + \frac{1}{a_4+1} \ge \frac{3}{2},$$

with equality for $a_1 = a_4 = 3$ and $a_2 = a_3 = 1$.

As a final remark, the inequality

$$\frac{1}{a_1+1} + \frac{1}{a_2+1} + \cdots + \frac{1}{a_n+1} \ge \frac{n}{2}$$

with $a_1a_2 + a_2a_3 + \cdots + a_na_1 = n$, does not hold for $n = 6$ and for $n = 8$.

□

P 1.220. *If a, b, c, d, e are nonnegative real numbers such that*

$$ab + bc + cd + de + ea = 5, \qquad a \ge b \ge c \ge 1 \ge d \ge e,$$

then

$$\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3} + \frac{1}{d+3} + \frac{1}{e+3} \ge \frac{5}{4}.$$

(Vasile Cîrtoaje, Arhimede Math. J., No. 2, 2024)

Solution. For fixed a, d and e, from the equality constraint we may assume that b is a function of c. By differentiating the constraint, we get

$$(a+c)b' + b + d = 0, \qquad -b' = \frac{b+d}{a+c} \le 1.$$

Denoting the left side of the desired inequality by $f(c)$, we have

$$f'(c) = \frac{-b'}{(b+3)^2} - \frac{1}{(c+3)^2} \le \frac{1}{(b+3)^2} - \frac{1}{(c+3)^2} \le 0.$$

Thus, $f(c)$ is decreasing and has the minimum value when c is maximum, that is when $c = b$. So, we only need to show that

$$\frac{1}{a+3} + \frac{2}{b+3} + \frac{1}{d+3} + \frac{1}{e+3} \ge \frac{5}{4}$$

for

$$ab + b^2 + bd + de + ea = 5, \qquad a \ge b \ge 1 \ge d \ge e.$$

For fixed a and e, from the equality constraint, we may assume that b is a decreasing function of d. By differentiating the constraint, we get

$$(a+2b+d)b' + b + e = 0, \qquad -b' = \frac{b+e}{a+2b+d} \le \frac{1}{2}.$$

Denoting the left side of the desired inequality by $g(d)$, we have

$$g'(d) = \frac{-2b'}{(b+3)^2} - \frac{1}{(d+3)^2} \le \frac{1}{(b+3)^2} - \frac{1}{(d+3)^2} \le 0.$$

Thus, $g(d)$ is decreasing and has the minimum value when d is maximum (b is minimum), that is when $d = 1$ (because $d \le 1$) or $b = 1$ (because $b \ge 1$). So, it suffices to consider these cases.

Case 1: $d = 1$. We need to show that

$$\frac{1}{a+3} + \frac{2}{b+3} + \frac{1}{e+3} \ge 1$$

for

$$ab + b^2 + b + e + ea = 5, \qquad a \ge b \ge 1 \ge e.$$

Since

$$e = \frac{5 - b - b^2 - ab}{1 + a}, \qquad e + 3 = \frac{8 - b - b^2 + (3 - b)a}{1 + a},$$

we need to show that

$$\frac{1}{a+3} + \frac{2}{b+3} + \frac{1+a}{8 - b - b^2 + (3-b)a} \geq 1,$$

which is equivalent to

$$\frac{1}{a+3} + \frac{1+a}{8 - b - b^2 + (3-b)a} \geq \frac{b+1}{b+3},$$

$$(b-1)[ba^2 + (b^2 + 5b - 4)a + 2b^2 + 4b - 9] \geq 0.$$

Since $a \geq b \geq 1$, we have

$$ba^2 + (b^2 + 5b - 4)a + 2b^2 + 4b - 9 \geq a^2 + 2a - 3 = (a-1)(a+3) \geq 0.$$

Case 2: $b = 1$. We need to show that

$$\frac{1}{a+3} + \frac{1}{d+3} + \frac{1}{e+3} \geq \frac{3}{4}$$

for

$$(a+d)(e+1) = 4, \qquad a \geq 1 \geq d \geq e.$$

Write the desired inequality as

$$\frac{a+d+6}{(a+3)(d+3)} + \frac{1}{e+3} \geq \frac{3}{4}.$$

From $(a-1)(d-1) \leq 0$, we get $ad \leq a + d - 1$, hence

$$(a+3)(d+3) = (a-1)(d-1) + 4(a+d) + 8 \leq 4(a+d) + 8$$

and

$$\frac{a+d+6}{(a+3)(d+3)} \geq \frac{a+d+6}{4(a+d)+8} = \frac{4/(e+1)+6}{16/(e+1)+8} = \frac{3e+5}{4(e+3)}.$$

So, it suffices to show that

$$\frac{3e+5}{4(e+3)} + \frac{1}{e+3} \geq \frac{3}{4},$$

which is an identity.

The equality occurs for $b = c = d = 1$ and $a + e + ae = 3$, $a \geq 1 \geq e$.

□

P 1.221. *Prove that 3 is the largest positive value of the constant k such that*

$$\frac{1}{a+k}+\frac{1}{b+k}+\frac{1}{c+k}+\frac{1}{d+k}+\frac{1}{e+k}\ge\frac{5}{1+k}$$

for any $a\ge b\ge c\ge d\ge 1\ge e\ge 0$ satisfying $ab+bc+cd+de+ea=5$.

(Vasile Cîrtoaje, RMM, 38, 2025)

Solution. Choosing $a=3$, $b=c=d=1$ and $e=0$, the constraints are satisfied, while the inequality becomes

$$\frac{1}{3+k}+\frac{1}{k}\ge\frac{2}{1+k},$$

i.e. $k\le 3$. To prove that 3 is the largest positive value of k, we need to show that

$$\frac{1}{a+3}+\frac{1}{b+3}+\frac{1}{c+3}+\frac{1}{d+3}+\frac{1}{e+3}\ge\frac{5}{4}.$$

Denote $x=\dfrac{b+c+d}{3}$ and $y=\dfrac{a+e}{2}$. We have

$$a\ge x\ge 1\ge e.$$

By the AM-HM inequality, we have

$$\frac{1}{b+3}+\frac{1}{c+3}+\frac{1}{d+3}\ge\frac{9}{(b+3)+(c+3)+(d+3)}=\frac{3}{x+3}.$$

Thus, it suffices to show that

$$\frac{1}{a+3}+\frac{1}{e+3}+\frac{3}{x+3}\ge\frac{5}{4},$$

i.e.

$$\frac{2y+6}{ae+6y+9}+\frac{3}{x+3}\ge\frac{5}{4}.$$

First, we show that $2x^2+2xy+ae\le 5$. Indeed, we have

$$9(5-2x^2-2xy-ae)=9(ab+bc+cd+de+ea)-2(b+c+d)^2-3(b+c+d)(a+e)-9ae$$

$$=-2(b+c+d)^2+3a(2b-c-d)-3e(b+c-2d)+9c(b+d)$$

$$\ge -2(b+c+d)^2+3b(2b-c-d)-3d(b+c-2d)+9c(b+d)$$

$$=4(b-d)^2+2(b-c)(c-d)\ge 0.$$

Since the left side of the desired inequality decreases when x increases, we may replace the inequality constraint $2x^2+2xy+ae\le 5$ with the equality constraint

$$2x^2+2xy+ae=5.$$

So, since $ae = 5 - 2xy - 2x^2$, it suffices to show that

$$\frac{2y+6}{14+6y-2xy-2x^2} + \frac{3}{x+3} \geq \frac{5}{4},$$

which is equivalent to $(x-1)E \geq 0$, where

$$E = (5x-3)y + 5x^2 + 8x - 15.$$

Since $x \geq 1$, we need to show that $E \geq 0$. From $(x-a)(x-e) \leq 0$, we get $ae \leq 2xy - x^2$, and from

$$5 = 2x^2 + 2xy + ae \leq 2x^2 + 2xy + 2xy - x^2,$$

we get $y \geq \dfrac{5-x^2}{4x}$. Thus,

$$E \geq \frac{(5x-3)(5-x^2)}{4x} + 5x^2 + 8x - 15 = \frac{5(3x^3+7x^2-7x-3)}{4x} = \frac{5(x-1)(3x^2+10x+3)}{4x} \geq 0.$$

For $k = 3$, the equality occurs when $b = c = d = 1$ and $a + e + ae = 3$ with $a \geq 1 \geq e$.

□

P 1.222. *If a, b, c, d are nonnegative real numbers such that*

$$ab + ac + ad + bc + bd + cd = 6,$$

then

$$\frac{1}{ab+3} + \frac{1}{bc+3} + \frac{1}{cd+3} + \frac{1}{da+3} \geq 1.$$

(Vasile Cîrtoaje, Math. Reflections, 3, 2023)

Solution. By the AM-GM inequality, we have

$$6 = ab + ac + ad + bc + bd + cd \geq 6\sqrt[6]{a^3b^3c^3d^3},$$

hence

$$abcd \leq 1.$$

Write the required inequality as follows:

$$\frac{1}{ab+3} + \frac{1}{cd+3} + \frac{1}{bc+3} + \frac{1}{da+3} \geq 1,$$

$$\frac{ab+cd+6}{(ab+3)(cd+3)} + \frac{bc+da+6}{(bc+3)(da+3)} \geq 1,$$

$$\frac{3(ab+cd)+18}{abcd+3(ab+cd)+9} + \frac{3(bc+da)+18}{abcd+3(bc+da)+9} \geq 3,$$

$$1+\frac{9-abcd}{abcd+3(ab+cd)+9}+1+\frac{9-abcd}{abcd+3(bc+da)+9}\ge 3,$$

$$\frac{1}{abcd+3(ab+cd)+9}+\frac{1}{abcd+3(bc+da)+9}\ge \frac{1}{9-abcd}.$$

According to the AM-HM inequality, it is sufficient to show that

$$\frac{4}{[abcd+3(ab+cd)+9]+[abcd+3(bc+da)+9]}\ge \frac{1}{9-abcd},$$

which is equivalent to

$$6\ge ab+bc+cd+da+2abcd,$$

$$ac+bd\ge 2abcd.$$

Indeed, we have

$$ac+bd\ge 2\sqrt{abcd}\ge 2abcd.$$

The proof is completed. The equality occurs for $a=b=c=d=1$.

Remark. The inequality

$$\frac{1}{ab+k}+\frac{1}{bc+k}+\frac{1}{cd+k}+\frac{1}{da+k}\ge \frac{4}{1+k}$$

does not hold for $k>3$. By choosing $b=d=\sqrt{ac}$, the constraint $ab+ac+ad+bc+bd+cd=6$ becomes $(a+c)\sqrt{ac}+ac=3$, while the inequality can be written as follows:

$$\frac{1}{a\sqrt{ac}+k}+\frac{1}{c\sqrt{ac}+k}\ge \frac{2}{1+k},$$

$$\frac{(a+c)\sqrt{ac}+2k}{a^2c^2+k(a+c)\sqrt{ac}+k^2}\ge \frac{2}{1+k}.$$

Denoting $x=ac$, we have $3=(a+c)\sqrt{ac}+ac\ge 3ac=3x$, hence $x\in(0,1]$. Since $(a+c)\sqrt{ac}=3-x$, the inequality becomes

$$\frac{3+2k-x}{x^2-kx+k^2+3k}\ge \frac{2}{1+k},$$

which is equivalent to

$$(x-1)(2x+3-k)\le 0.$$

It is true if and only if $2x+3-k\ge 0$ for $x\in(0,1)$. For $x\to 1$, we get the necessary condition $k\le 3$.

□

P 1.223. *If a, b, c, d are nonnegative real numbers such that*

$$ab + ac + ad + bc + bd + cd = 6, \qquad a \ge b \ge c \ge d,$$

then

$$\frac{1}{ab+5} + \frac{1}{bc+5} + \frac{1}{cd+5} + \frac{1}{da+5} \ge \frac{2}{3}.$$

(Vasile Cîrtoaje, AMM, 1, 2023)

Solution. Write the inequality as follows:

$$\frac{ab+cd+10}{(ab+5)(cd+5)} + \frac{bc+ad+10}{(bc+5)(ad+5)} \ge \frac{2}{3},$$

$$\frac{5(ab+cd)+50}{abcd+5(ab+cd)+25} + \frac{5(bc+ad)+50}{abcd+5(bc+ad)+25} \ge \frac{10}{3},$$

$$1 + \frac{25-abcd}{abcd+5(ab+cd)+25} + 1 + \frac{25-abcd}{abcd+5(bc+ad)+25} \ge \frac{10}{3},$$

$$\frac{1}{abcd+5(ab+cd)+25} + \frac{1}{abcd+5(bc+ad)+25} \ge \frac{4}{3(25-abcd)}.$$

Since

$$ab+cd = 6-(bc+ad)-(ac+bd) = 6-2(bc+ad)-(a-b)(c-d) \le 6-2(bc+ad),$$

it suffices to show that

$$\frac{1}{abcd-10(bc+ad)+55} + \frac{1}{abcd+5(bc+ad)+25} \ge \frac{4}{3(25-abcd)}.$$

Using the substitution

$$bc = x, \qquad ad = y,$$

the inequality becomes

$$\frac{1}{xy-10(x+y)+55} + \frac{1}{xy+5(x+y)+25} \ge \frac{4}{3(25-xy)}.$$

Let $z = \dfrac{x+y}{2}$. Since $xy \le z^2$, it suffices to show that

$$\frac{1}{z^2-20z+55} + \frac{1}{z^2+10z+25} \ge \frac{4}{3(25-z^2)},$$

which is equivalent to

$$10 - 21z + 12z^2 - z^3 \ge 0.$$

$$(1-z)^2(10-z) \ge 0.$$

Since $2z = bc + ad < ab + ac + ad + bc + bd + cd = 6$, the latter inequality is obvious. The original inequality is an equality for $a = b = c = d = 1$.

Remark 1. The inequality

$$\frac{1}{ab+k} + \frac{1}{bc+k} + \frac{1}{cd+k} + \frac{1}{da+k} \ge \frac{4}{1+k}$$

does not hold for $k > 5$. By choosing $a = b$ and $c = d$, the constraint $ab+ac+ad+bc+bd+cd = 6$ becomes $a^2 + d^2 + 4ad = 6$, while the inequality can be written as follows:

$$\frac{1}{a^2+k} + \frac{1}{d^2+k} + \frac{2}{ad+k} \ge \frac{4}{1+k},$$

$$\frac{a^2+d^2+2k}{a^2d^2+k(a^2+d^2)+k^2} + \frac{2}{ad+k} \ge \frac{4}{1+k}.$$

Denoting $x = ad$, we have $6 = a^2 + d^2 + 4ad \ge 6ad = 6x$, hence $x \in [0, 1]$. Since $a^2 + d^2 = 6 - 4x$, the inequality becomes

$$\frac{3+k-2x}{x^2-4kx+k^2+6k} + \frac{1}{x+k} \ge \frac{2}{1+k},$$

which is equivalent to $(x - 1)P(x) \le 0$, where

$$P(x) = 2x^2 - (5k - 3)x + 9k - k^2.$$

Since $x - 1 \le 0$, the inequality is true if and only if $P(x) \ge 0$ for $x \in [0, 1]$. Letting $d \to 0$, we get the necessary condition $(5 - k)(1 + k) \ge 0$, that is $k \le 5$.

Remark 2. We claim that the following *open problem* is valid:

- *If a, b, c, d, e are nonnegative real numbers such that*

$$ab + ac + ad + ae + bc + bd + be + cd + ce + de = 10, \qquad a \ge b \ge c \ge d \ge e,$$

then

$$\frac{1}{ab+4} + \frac{1}{bc+4} + \frac{1}{cd+4} + \frac{1}{de+4} + \frac{1}{ea+4} \ge 1.$$

An interesting (open) particular case is for $a = b$ and $d = e$: *If b, c, d are nonnegative real numbers such that*

$$2(bc + cd + db) + (b + d)^2 = 10\ ,$$

then

$$\frac{1}{bc+4} + \frac{1}{cd+4} + \frac{1}{db+4} + \frac{1}{b^2+4} + \frac{1}{d^2+4} \ge 1.$$

Remark 3. We claim that the following *open problem* is valid:

• *If a, b, c, d, e are nonnegative real numbers such that*

$$ab + ac + ad + ae + bc + bd + be + cd + ce + de = 10,$$

then

$$\frac{1}{ab+3} + \frac{1}{bc+3} + \frac{1}{cd+3} + \frac{1}{de+3} + \frac{1}{ea+3} \geq \frac{5}{4}.$$

□

P 1.224. *If a, b, c, d are nonnegative real numbers such that*

$$ab + bc + cd + da = 4, \qquad a \geq b \geq c \geq d,$$

then

$$\frac{1}{ab+4} + \frac{1}{ac+4} + \frac{1}{ad+4} + \frac{1}{bc+4} + \frac{1}{bd+4} + \frac{1}{cd+4} \geq \frac{6}{5}.$$

(Vasile Cîrtoaje, GM-B, 1, 2024)

Solution. By the AM-GM inequality, we have

$$4 = ab + bc + cd + da \geq \sqrt[4]{a^2b^2c^2d^2} = \sqrt{abcd},$$

hence

$$p \leq 1, \qquad p = abcd.$$

Write the required inequality as follows:

$$\left(\frac{1}{ab+4} + \frac{1}{cd+4}\right) + \left(\frac{1}{ac+4} + \frac{1}{bd+4}\right) + \left(\frac{1}{ad+4} + \frac{1}{bc+4}\right) \geq \frac{6}{5},$$

$$\frac{ab+cd+8}{p+4(ab+cd)+16} + \frac{ac+bd+8}{p+4(ac+bd)+16} + \frac{ad+bc+8}{p+4(ad+bc)+16} \geq \frac{6}{5},$$

$$\frac{4(ab+cd)+32}{p+4(ab+cd)+16} + \frac{4(ac+bd)+32}{p+4(ac+bd)+16} + \frac{4(ad+bc)+32}{p+4(ad+bc)+16} \geq \frac{24}{5},$$

$$1 + \frac{16-p}{p+4(ab+cd)+16} + 1 + \frac{16-p}{p+4(ac+bd)+16} + 1 + \frac{16-p}{p+4(ad+bc)+16} \geq \frac{24}{5},$$

$$\frac{1}{p+4(ab+cd)+16} + \frac{1}{p+4(ac+bd)+16} + \frac{1}{p+4(ad+bc)+16} \geq \frac{9}{5(16-p)},$$

According to the AM-HM inequality, it is sufficient to show that

$$\frac{9}{[p+4(ab+cd)+16] + [p+4(ac+bd)+16] + [p+4(ad+bc)+16]} \geq \frac{9}{5(16-p)},$$

which is equivalent to

$$\frac{1}{3p+4(ab+cd+ac+bd+ad+bc)+48} \geq \frac{1}{5(16-p)},$$

$$\frac{1}{3p+4(ac+bd)+64} \geq \frac{1}{5(16-p)},$$

$$4 \geq 2p+ac+bd,$$

$$ab+bc+cd+da \geq 2p+ac+bd,$$

$$(a-d)(b-c)+bc+da \geq 2p.$$

It suffices to show that

$$bc+da \geq 2p.$$

Indeed, we have

$$bc+da \geq 2\sqrt{p} \geq 2p.$$

The proof is completed. The equality occurs for $a=b=c=d=1$.

□

P 1.225. *If a,b,c,d are nonnegative real numbers such that*

$$ab+bc+cd+da=4, \qquad a \geq b \geq c \geq d,$$

then

$$\frac{1}{ab+7}+\frac{1}{ac+7}+\frac{1}{ad+7}+\frac{1}{bc+7}+\frac{1}{bd+7}+\frac{1}{cd+7} \geq \frac{3}{4}.$$

(Vasile Cîrtoaje, SSMJ, 1, 2024)

Solution. Denote $p=abcd$ and write the desired inequality as follows:

$$\frac{ab+cd+14}{p+7(ab+cd)+49}+\frac{ac+bd+14}{p+7(ac+bd)+49}+\frac{ad+bc+14}{p+7(ad+bc)+49} \geq \frac{3}{4},$$

$$1+\frac{49-p}{p+7(ab+cd)+49}+1+\frac{49-p}{p+7(ac+bd)+49}+1+\frac{49-p}{p+7(ad+bc)+49} \geq \frac{21}{4},$$

$$\frac{1}{p+7(ab+cd)+49}+\frac{1}{p+7(ac+bd)+49}+\frac{1}{p+7(ad+bc)+49} \geq \frac{9}{4(49-p)}.$$

By the AM-HM inequality, it suffices to show that

$$\frac{4}{2p+7(ab+cd+ac+bd)+98}+\frac{1}{p+7(ad+bc)+49} \geq \frac{9}{4(49-p)}.$$

Since

$$ab+cd+ac+bd=2(ab+cd)-(a-d)(b-c) \leq 2(ab+cd)=2(4-ad-bc),$$

it suffices to prove that

$$\frac{2}{p-7(ad+bc)+77}+\frac{1}{p+7(ad+bc)+49}\geq\frac{9}{4(49-p)}.$$

Using the substitution

$$ad=x,\qquad bc=y,$$

the inequality becomes

$$\frac{2}{xy-7(x+y)+77}+\frac{1}{xy+7(x+y)+49}\geq\frac{9}{4(49-xy)}.$$

Let $z=\dfrac{x+y}{2}$. Since $xy\leq z^2$, it suffices to show that

$$\frac{2}{z^2-14z+77}+\frac{1}{z^2+14z+49}\geq\frac{9}{4(49-z^2)},$$

which is equivalent to

$$(z-1)^2(7-3z)\geq 0.$$

Since $z<\dfrac{ab+bc+cd+da}{2}=2$, the latter inequality is obvious. The original inequality is an equality for $a=b=c=d=1$.

Remark. The inequality

$$\frac{1}{ab+k}+\frac{1}{ac+k}+\frac{1}{ad+k}+\frac{1}{bc+k}+\frac{1}{bd+k}+\frac{1}{cd+k}\geq\frac{6}{1+k}.$$

does not hold true for $k>7$. By choosing $b=c=1$, the constraint $ab+bc+cd+da=4$ becomes $ad=3-2S$, where $S=\dfrac{a+d}{2}$, while the inequality can be written as follows:

$$\frac{2}{a+k}+\frac{2}{d+k}+\frac{1}{ad+k}\geq\frac{5}{1+k},$$

$$\frac{2(a+d+2k)}{ad+k(a+d)+k^2}+\frac{1}{ad+k}\geq\frac{5}{1+k},$$

$$\frac{4S+4k}{(2k-2)S+k^2+3}+\frac{1}{k+3-2S}\geq\frac{5}{1+k},$$

$$2(3k-7)S^2-(k^2+6k-35)S+k^2-21\geq 0,$$

$$(S-1)[2(3k-7)S-k^2+21]\geq 0.$$

From $3-2S=ad\leq S^2$, we get $S\geq 1$. Thus, the inequality is true if and only if $2(3k-7)S-k^2+21\geq 0$ for $S>1$. So, we get the necessary condition $2(3k-7)-k^2+21\geq 0$, i.e. $(7-k)(1+k)\geq 0$, $k\leq 7$.

□

P 1.226. *If a, b, c, d are nonnegative real numbers such that*

$$ab + bc + cd + da = 4,$$

then

$$\frac{4\sqrt{2}}{3} \le \frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} + \frac{1}{d^2+1} < 3.$$

(Vasile Cîrtoaje, Recreatii Matematice, no. 1, 2022)

Solution. The hypothesis is equivalent to

$$(a+c)(b+d) = 4.$$

(a) Write the right inequality as

$$\frac{a^2}{a^2+1} + \frac{b^2}{b^2+1} + \frac{c^2}{c^2+1} + \frac{d^2}{d^2+1} > 1.$$

Since

$$\frac{a^2}{a^2+1} + \frac{c^2}{c^2+1} \ge \frac{(a+c)^2}{(a^2+1)+(c^2+1)} \ge \frac{(a+c)^2}{(a+c)^2+2}$$

and

$$\frac{b^2}{b^2+1} + \frac{d^2}{d^2+1} \ge \frac{(b+d)^2}{(b+d)^2+2} = \frac{8}{(a+c)^2+8},$$

it suffices to show that

$$\frac{(a+c)^2}{(a+c)^2+2} + \frac{8}{(a+c)^2+8} > 1.$$

This is equivalent to the obvious inequality

$$\frac{(a+c)^2}{(a+c)^2+2} > \frac{(a+c)^2}{(a+c)^2+8}.$$

(b) To prove the left inequality, consider $a + c \ge 2 \ge b + d$ and first show that

$$\frac{1}{a^2+1} + \frac{1}{c^2+1} \ge \frac{8}{(a+c)^2+4}.$$

Write this inequality as follows

$$\frac{1}{a^2+1} - \frac{4}{(a+c)^2+4} + \frac{1}{c^2+1} - \frac{4}{(a+c)^2+4} \ge 0,$$

$$\frac{(c-a)(3a+c)}{a^2+1} + \frac{(a-c)(3c+a)}{c^2+1} \ge 0,$$

$$(a-c)^2(a^2+c^2+4ac-2) \ge 0.$$

The last inequality is true since

$$a^2 + c^2 + 4ac - 2 = (a+c)^2 + 2ac - 2 \geq 4 + 2ac - 2 > 0.$$

Thus, we need to show that $b + d \leq 2$ involves

$$\frac{8}{(a+c)^2+4} + \frac{1}{b^2+1} + \frac{1}{d^2+1} \geq \frac{4\sqrt{2}}{3},$$

that is

$$\frac{2s}{s+1} + \frac{1}{b^2+1} + \frac{1}{d^2+1} \geq \frac{4\sqrt{2}}{3},$$

where

$$s = \frac{1}{4}(b+d)^2, \qquad 0 \leq s \leq 1.$$

Denoting

$$p = bd, \qquad 0 \leq p \leq s \leq 1,$$

the inequality is equivalent to

$$\frac{s}{s+1} + \frac{2s+1-p}{p^2-2p+4s+1} \geq \frac{2\sqrt{2}}{3}.$$

We have two cases to consider.

Case 1: $2s \leq 1$. Since

$$1 - (2s+1)p \geq 1 - (2s+1)s = (1-2s)(1+s) \geq 0,$$

we have

$$\frac{2s+1-p}{p^2-2p+4s+1} - \frac{2s+1}{4s+1} = \frac{p[1-(2s+1)p]}{(4s+1)(p^2-2p+4s+1)} \geq 0.$$

Thus, it suffices to show that

$$\frac{s}{s+1} + \frac{2s+1}{4s+1} \geq \frac{2\sqrt{2}}{3}.$$

This inequality is equivalent to

$$\left(14s - 3\sqrt{2} + 2\right)^2 \geq 0.$$

Case 2: $2s \geq 1$. Since

$$\frac{2s+1-p}{p^2-2p+4s+1} - \frac{1}{s+1} = \frac{(s-p)(2s-1+p)}{(s+1)(p^2-2p+4s+1)} \geq 0,$$

it suffices to show that

$$\frac{s}{s+1} + \frac{1}{s+1} \geq \frac{2\sqrt{2}}{3},$$

which is obvious.

The equality holds for

$$a^2 = c^2 = 2 + 3\sqrt{2}\ , \qquad b^2 = \frac{2(3\sqrt{2}-2)}{7}\ , \qquad d = 0$$

(or any cyclic permutation).

□

P 1.227. *If a, b, c, d are nonnegative real numbers such that*

$$ab + bc + cd + da = 4, \qquad a \ge b \ge 1 \ge c \ge d,$$

then

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} + \frac{1}{d^2+1} \ge 2.$$

(Vasile Cîrtoaje, 2023)

Solution. From

$$4 = (a+c)(b+d) \le (a+c)^2,$$

we get

$$a + c \ge 2.$$

Write the inequality as

$$\frac{1}{a^2+1} + \frac{1}{c^2+1} \ge \frac{b^2}{b^2+1} + \frac{d^2}{d^2+1}.$$

Since

$$\frac{b^2}{b^2+1} \le \frac{b^2}{2b} = \frac{b}{2}, \qquad \frac{d^2}{d^2+1} \le \frac{d}{2},$$

it suffices to show that

$$\frac{1}{a^2+1} + \frac{1}{c^2+1} \ge \frac{b+d}{2},$$

which is equivalent to $E(a, c) \ge 0$, where

$$E(a, c) = \frac{1}{a^2+1} + \frac{1}{c^2+1} - \frac{2}{a+c}.$$

It is true because $a \ge 1 \ge c$ and $a + c \ge 2$ involve

$$E(a, 1) = \frac{1}{a^2+1} + \frac{1}{2} - \frac{2}{a+1} = \frac{(a-1)^3}{2(a+1)(a^2+1)} \ge 0$$

and

$$E(a, c) - E(a, 1) = \left(\frac{1}{c^2+1} - \frac{1}{2}\right) - \left(\frac{2}{a+c} - \frac{2}{a+1}\right) = \frac{1-c}{2} F(a, c),$$

where

$$F(a,c) = \frac{c+1}{c^2+1} - \frac{4}{(a+c)(a+1)} \ge \frac{c+1}{c^2+1} - \frac{2}{a+1}$$

$$\ge \frac{c+1}{c^2+1} - \frac{2}{3-c} = \frac{(1-c)(1+3c)}{(c^2+1)(3-c)} \ge 0.$$

The equality occurs for $a = b = c = d = 1$.

□

P 1.228. *If a, b, c, d are nonnegative real numbers such that*

$$ab + bc + cd + da = 4,$$

then

$$2 \le \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} < 3.$$

Solution. The hypothesis is equivalent to

$$(a+c)(b+d) = 4.$$

(a) Since

$$\frac{1}{a+1} + \frac{1}{c+1} = \frac{a+c+2}{ac+a+c+1} \le \frac{a+c+2}{a+c+1}$$

and

$$\frac{1}{b+1} + \frac{1}{d+1} = \frac{b+d+2}{bd+b+d+1} \le \frac{b+d+2}{b+d+1} = \frac{2(a+c+2)}{a+c+4},$$

the right inequality is true if

$$\frac{a+c+2}{a+c+1} + \frac{2(a+c+2)}{a+c+4} < 3,$$

which is equivalent to

$$a + c > 0.$$

The inequality is strict since the necessary equality condition $a + c = 0$ contradicts the constraint $(a+c)(b+d) = 4$.

(b) To prove the left inequality, we apply the AM-HM inequality as follows:

$$\frac{1}{a+1} + \frac{1}{c+1} \ge \frac{4}{a+c+2},$$

$$\frac{1}{b+1} + \frac{1}{d+1} \ge \frac{4}{b+d+2}.$$

So, we only need to show that

$$\frac{2}{a+c+2} + \frac{2}{b+d+2} \ge 1,$$

which is an identity. The equality holds for $a = c = \frac{1}{b} = \frac{1}{d}$.

□

P 1.229. *If a,b,c,d,e are nonnegative real numbers such that $ab+bc+cd+de+ea=1$, then*

$$3<\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}+\frac{1}{d+1}+\frac{1}{e+1}\le 4.$$

(Vasile Cîrtoaje, Math. Reflections, 3, 2023)

Solution. (a) To prove the left inequality, assume that $a=\min\{a,b,c,d,e\}<1/2$ and $c\le d$. For $a=c=0$, we need to show that $de=1$ involves

$$\frac{1}{b+1}+\frac{1}{d+1}+\frac{1}{e+1}>1,$$

which reduces to the obvious inequality

$$\frac{1}{b+1}>0.$$

Consider next that $a+c>0$. Since

$$(a+c)b+(a+d)e=1-cd, \qquad cd\le 1,$$

$$(a+c)(b+1)+(a+d)(e+1)=2a+c+d+1-cd,$$

by the Cauchy-Schwarz inequality we have

$$\frac{1}{b+1}+\frac{1}{e+1}\ge\frac{(\sqrt{a+c}+\sqrt{a+d})^2}{(a+c)(b+1)+(a+d)(e+1)}$$

$$=\frac{2a+c+d+2\sqrt{(a+c)(a+d)}}{2a+c+d+1-cd}\ge\frac{4a+c+d+2\sqrt{cd}}{2a+c+d+1-cd}.$$

Thus, it suffices to prove that

$$\frac{1}{a+1}+\frac{1}{c+1}+\frac{1}{d+1}+\frac{4a+c+d+2\sqrt{cd}}{2a+c+d+1-cd}>3 \tag{*}$$

for $a<1/2$, $a+c>0$ and $cd\le 1$. Denote

$$x=\frac{c+d}{2}, \qquad p=\sqrt{cd}.$$

From $c+d\ge 2\sqrt{cd}$ and $(a-c)(a-d)\ge 0$, we get $x\ge p$ and $a^2-2ax+p^2\ge 0$, therefore

$$p\le x\le\frac{a^2+p^2}{2a}.$$

We have $x=p$ for $c=d\le 1$, and $x=(a^2+p^2)/(2a)$ for $a=c\in(0,1/2)$. Write now the inequality (*) as

$$\frac{1}{a+1}+\frac{2(x+1)}{1+2x+p^2}+\frac{4a+2x+2p}{2a+2x+1-p^2}>3.$$

For fixed a and p, the inequality is equivalent to $f(x) > 0$, where f is a polynomial of second order with the expression

$$f(x) = -4ax^2 + B(a,p)x + C(a,p).$$

Since f is concave, it suffices to prove the inequality $f(x) > 0$ for $x = p$ and $x = \dfrac{a^2+p^2}{2a}$, therefore for $c = d \le 1$ and for $a = c \in (0, 1/2)$, respectively.

Case 1: $c = d \le 1$. The inequality (*) becomes

$$\frac{1}{a+1} + \frac{2}{c+1} + \frac{4a+4c}{2a+2c+1-c^2} > 3,$$

which is equivalent to the obvious inequality

$$2(1-c)a^2 + (3c^3 - c^2 - c + 3)a + 2c(c^2+1) > 0.$$

Case 2: $a = c \in (0, 1/2)$. The inequality (*) becomes

$$\frac{2}{c+1} + \frac{1}{d+1} + \frac{5c + d + 2\sqrt{cd}}{3c + d + 1 - cd} > 3,$$

$$2(c+1)(d+1)\sqrt{cd} + 3c^2d^2 - cd(2c+d) - c(c+2d) + 3c > 0.$$

Using the substitution

$$cd = y^2, \qquad 0 < y \le 1,$$

the inequality becomes

$$2y(c+1)\left(\frac{y^2}{c} + 1\right) + 3y^4 - y^2\left(2c + \frac{y^2}{c}\right) - c\left(c + \frac{2y^2}{c}\right) + 3c > 0,$$

$$\frac{y^3(2-y)}{c} - c^2 + (3 + 2y - 2y^2)c + y(2 - 2y + 2y^2 + 3y^3) > 0.$$

It is true if $-c^2 + (3 + 2y - 2y^2)c \ge 0$. Indeed, we have

$$-c^2 + (3 + 2y - 2y^2)c > -c + (3 + 2y - 2y^2)c = 2(1 + y - y^2)c > 0.$$

(b) To prove the right inequality, suppose that $(x_1, x_2, x_3, x_4, x_5)$ is a permutation of (a, b, c, d, e) such that $x_1 \ge x_2 \ge x_3 \ge x_4 \ge x_5$. Due to symmetry, the desired inequality is equivalent to

$$\frac{1}{x_1+1} + \frac{1}{x_2+1} + \frac{1}{x_3+1} + \frac{1}{x_4+1} + \frac{1}{x_5+1} \le 4,$$

that can be written in the form

$$\frac{x_1}{x_1+1} + \frac{x_2}{x_2+1} + \frac{x_3}{x_3+1} + \frac{x_4}{x_4+1} + \frac{x_5}{x_5+1} \ge 1.$$

For $x_1x_2 \ge 1$, the inequality is true because

$$\frac{x_1}{x_1+1}+\frac{x_2}{x_2+1}-1=\frac{x_1x_2-1}{(x_1+1)(x_2+1)} \ge 0.$$

Consider further $x_1x_2 \le 1$. For $x_3+x_4+x_5>0$, by the Cauchy-Schwarz inequality we have

$$\frac{x_3}{x_3+1}+\frac{x_4}{x_4+1}+\frac{x_5}{x_5+1} \ge \frac{(x_3+x_4+x_5)^2}{x_3(x_3+1)+x_4(x_4+1)+x_5(x_5+1)}$$

$$\ge \frac{(x_3+x_4+x_5)^2}{(x_3+x_4+x_5)^2+x_3+x_4+x_5}=\frac{x_3+x_4+x_5}{x_3+x_4+x_5+1}=1-\frac{1}{x_3+x_4+x_5+1}\ ,$$

hence

$$\frac{x_3}{x_3+1}+\frac{x_4}{x_4+1}+\frac{x_5}{x_5+1} \ge 1-\frac{1}{x_3+x_4+x_5+1}.$$

We can see that this inequality is also true for $x_3=x_4=x_5=0$. So, it suffices to prove that

$$\frac{x_1}{x_1+1}+\frac{x_2}{x_2+1}-\frac{1}{x_3+x_4+x_5+1} \ge 0.$$

By Lemma below, we have

$$x_3+x_4+x_5 \ge \frac{1-x_1x_2}{x_1+x_2},$$

hence

$$\frac{x_1}{x_1+1}+\frac{x_2}{x_2+1}-\frac{1}{x_3+x_4+x_5+1} \ge \frac{x_1}{x_1+1}+\frac{x_2}{x_2+1}-\frac{1}{\frac{1-x_1x_2}{x_1+x_2}+1}$$

$$=\frac{x_1}{x_1+1}+\frac{x_2}{x_2+1}-\frac{x_1+x_2}{1-x_1x_2+x_1+x_2}=\frac{2x_1x_2(1-x_1x_2)}{(x_1+1)(x_2+1)(1-x_1x_2+x_1+x_2)} \ge 0.$$

The proof is completed. The equality is an equality for $ab=1$ and $c=d=e=0$ (or any cyclic permutation).

Lemma. *Let a,b,c,d,e be nonnegative real numbers, and let (x_1,x_2,x_3,x_4,x_5) be a permutation of (a,b,c,d,e) such that $x_1 \ge x_2 \ge x_3 \ge x_4 \ge x_5$. Then,*

$$x_1x_2+(x_1+x_2)(x_3+x_4+x_5) \ge ab+bc+cd+de+ea.$$

Proof. Assume that $a=\max\{a,b,c,d,e\}$, hence $a=x_1$ and $x_2+x_3+x_4+x_5=b+c+d+e$. Since

$$x_1x_2+(x_1+x_2)(x_3+x_4+x_5)=x_1(x_2+x_3+x_4+x_5)+x_2(x_3+x_4+x_5)$$

$$=a(b+c+d+e)+x_2(x_3+x_4+x_5) \ge a(b+c+d+e)+x_2x_3,$$

it suffices to show that

$$a(b+c+d+e)+x_2x_3 \ge ab+bc+cd+de+ea,$$

that is

$$a(c+d)+x_2x_3 \geq bc+cd+de,$$

which is equivalent to the obvious inequality

$$c(a-b)+d(a-c)+(x_2x_3-de) \geq 0.$$

□

P 1.230. *If a,b,c,d,e,f are nonnegative real numbers such that*

$$ab+bc+cd+de+ef+fa=6,$$

then

$$(2a+1)^2+(2b+1)^2+(2c+1)^2+(2d+1)^2+(2e+1)^2+(2f+1)^2 \geq 54.$$

(Vasile Cîrtoaje, MATINF, 9-10, 2022)

Solution. Let

$$s=a+c+e, \qquad q=ac+ce+ea, \qquad 3q \leq s^2.$$

By the Cauchy-Schwarz inequality, we have:

$$\left[(a+c)^2+(c+e)^2+(e+a)^2\right]\left[(2b+1)^2+(2d+1)^2+(2f+1)^2\right] \geq$$

$$\geq [(a+c)(2b+1)+(c+e)(2d+1)+(e+a)(2f+1)]^2$$

$$=4(a+c+e+ab+bc+cd+de+ef+fa)^2=4(s+6)^2.$$

Since

$$(a+c)^2+(c+e)^2+(e+a)^2=2(a^2+c^2+e^2+ac+ce+ea)=2(s^2-q)>0,$$

we get

$$(2b+1)^2+(2d+1)^2+(2f+1)^2 \geq \frac{2(s+6)^2}{s^2-q}.$$

Thus, it suffices to prove that

$$(2a+1)^2+(2c+1)^2+(2e+1)^2+\frac{2(s+6)^2}{s^2-q} \geq 54.$$

Since

$$(2a+1)^2+(2c+1)^2+(2e+1)^2=4(a^2+c^2+e^2+a+c+e)+3=4(s^2+s-2q)+3,$$

we need to prove the inequality

$$4(s^2+s-2q)+\frac{2(s+6)^2}{s^2-q} \geq 51,$$

which is equivalent to $f(q) \ge 0$, where

$$f(q) = 8q^2 - (12s^2 + 4s - 51)q + 4s^4 + 4s^3 - 49s^2 + 24s + 72.$$

For $s \le 1$, we have

$$f(q) > 8q^2 - (12s^2 + 4s - 51)q \ge (51 - 4s - 12s^2)q \ge 0.$$

Consider now $s \ge 1$ and write $f(q)$ as

$$f(q) = 8\left(\frac{12s^2 + 4s - 51}{16} - q\right)^2 + \frac{g(s)}{32},$$

where

$$g(s) = -16s^4 + 32s^3 - 360s^2 + 1176s - 297.$$

For $1 \le s \le \frac{5}{2}$, we have $g(s) > 0$, therefore $f(q) > 0$. Indeed,

$$g(s) = 4s(2s+3)(s-1)(5-2s) - 404s^2 + 1236s - 297 \ge -404s^2 + 1236s - 297$$

$$> -420s^2 + 1200s - 375 = 15(-28s^2 + 80s - 25) = 15(5-2s)(14s-5) \ge 0.$$

For $s \ge \frac{5}{2}$, since

$$\frac{12s^2 + 4s - 51}{16} - q \ge \frac{12s^2 + 4s - 51}{16} - \frac{s^2}{3} = \frac{20s^2 + 12s - 153}{48}$$

$$> \frac{20s^2 + 12s - 155}{48} = \frac{(2s-5)(10s+31)}{48} \ge 0,$$

$f(q)$ is a decreasing function, hence

$$f(q) \ge f\left(\frac{s^2}{3}\right) = \frac{8(s^4 + 3s^3 - 36s^2 + 27s + 81)}{9} = \frac{8(s-3)^2(s^2 + 9s + 9)}{9} \ge 0.$$

The proof is completed. The equality occurs for $a = b = c = d = e = f = 1$.

□

P 1.231. *Prove that 4 is the largest positive value of the constant k such that*

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n \ge k(a_1 + a_2 + \cdots + a_n - n)$$

for all odd integers $n \ge 3$ and nonnegative real numbers a_i which satisfy $a_1a_2 + a_2a_3 + \cdots + a_na_1 = n$.

(Vasile Cîrtoaje, AMM, 3, 2025)

Solution. For $a_1 = a_3 = \cdots = a_{n-2} := x$, $a_2 = a_4 = \cdots = a_{n-1} := y$ and $a_n = 1$, the constraint becomes $(n-2)xy + x + y = n$, i.e.

$$2S = n - (n-2)p,$$

where $S = \dfrac{x+y}{2}$ and $p = xy$. From $n-(n-2)p = 2S \ge 2\sqrt{p}$, we get $p \le 1$. On the other hand, the inequality becomes as follows:

$$\frac{(n-1)(x^2+y^2)}{2} + 1 - n \ge k\left[\frac{(n-1)(x+y)}{2} + 1 - n\right],$$

$$x^2 + y^2 - 2 \ge k(x+y-2), \qquad 4S^2 - 2p - 2 \ge k(2S-2),$$

$$[n-(n-2)p]^2 - 2p - 2 \ge k[n-(n-2)p-2],$$

$$(1-p)[n^2 - 2 - (n-2)^2 p] \ge k(n-2)(1-p).$$

It is true for all $p \in [0,1]$ if and only if

$$n^2 - 2 - (n-2)^2 p \ge k(n-2).$$

For $p = 1$, we get

$$k \le 4 + \frac{2}{n-2}.$$

Clearly, this condition is true for all odd integer $n \ge 3$ if and only if $k \le 4$. To finish the proof, we need to prove the inequality

$$a_1^2 + a_2^2 + \cdots + a_n^2 + 3n \ge 4(a_1 + a_2 + \cdots + a_n),$$

which is equivalent to the obvious inequality

$$\sum_{cyc}(a_1 + a_2 - 2)^2 \ge 0.$$

For $k = 4$, the equality occurs when $a_1 = a_2 = \cdots = a_n = 1$.

Remark 1. From the proof above, it follows that the inequality

$$a_1^2 + a_2^2 + \cdots + a_n^2 + 3n \ge 4(a_1 + a_2 + \cdots + a_n)$$

holds for all integer $n \ge 2$ and all nonnegative real numbers $a_1, a_2, \ldots, a_n$ which satisfy $a_1a_2 + a_2a_3 + \cdots + a_na_1 = n$.

Remark 2. For a given even $n \ge 2$, the largest values of k_n is 4. Indeed, by choosing $a_1 = a_3 = \cdots = a_{n-1} := x$ and $a_2 = a_4 = \cdots = a_n := y$, the constraint becomes $xy = 1$, while the inequality becomes as follows:

$$x^2 + y^2 - 2 \ge k_n(x+y-2), \qquad \frac{(x^2-1)^2}{x^2} \ge \frac{k_n(x-1)^2}{x}.$$

The inequality is true for any positive x if and only if $k_n \le \dfrac{(x+1)^2}{x}$. For $x=1$, we get the necessary condition $k_n \le 4$.

Remark 3. For a given odd $n \ge 3$, the largest values of k_n (when a_i are real numbers) is $2+2\sec\left(\dfrac{\pi}{n}\right)$. Denoting $a_i = 1+x_i$ for $i=1,2,\dots,n$, and

$$X = x_1^2+x_2^2+\cdots+x_n^2, \qquad Y = x_1x_2+x_2x_3+\cdots+x_nx_1,$$

the constraint becomes

$$Y + 2(x_1+x_2+\cdots+x_n) = 0,$$

while the inequality becomes

$$X - (k-2)(x_1+x_2+\cdots+x_n) \ge 0,$$

hence

$$\frac{2X}{k-2} + Y \ge 0.$$

It is known that the least value of A_n such that $A_nX + Y \ge 0$ for any real $x_1,x_2,\dots,x_n$ is $A_n = \cos\left(\dfrac{\pi}{n}\right)$. So, from

$$\frac{2}{k-2} = \cos\left(\frac{\pi}{n}\right),$$

we obtain the largest value of k, that is $k_n = 2+2\sec\left(\dfrac{\pi}{n}\right)$.

□

P 1.232. *If a,b,c,d,e are positive real numbers such that*

$$ab+bc+cd+de+ea=5,$$

then

$$5\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right) \ge 4(a+b+c+d+e)+5.$$

(Vasile Cîrtoaje, RMM, 36, 2025)

Solution. Using Lemma in the proof of P 1.217, it suffices to consider

$$ae+ad+be+bc+cd=5, \qquad a \ge b \ge c \ge d \ge e.$$

Denote

$$x = \frac{a+b}{2}, \qquad y = \frac{d+e}{2}, \qquad a \ge x \ge b \ge c \ge d \ge y \ge e.$$

As shown at P 1.217, we have

$$4xy \le 5 - c^2, \qquad c < \sqrt{5},$$

and

$$4c(x+y) \geq 3c^2 + 5.$$

By the AM-HM inequality, we have

$$\frac{1}{a} + \frac{1}{b} \geq \frac{4}{a+b} = \frac{2}{x}, \qquad \frac{1}{d} + \frac{1}{e} \geq \frac{2}{y}.$$

Thus, it suffices to show that the conditions

$$4xy \leq 5 - c^2, \qquad 4c(x+y) \geq 3c^2 + 5, \qquad x \geq c \geq y > 0$$

involve

$$5\left(\frac{2}{x} + \frac{2}{y} + \frac{1}{c}\right) \geq 4(2x + 2y + c) + 5,$$

that is

$$2(x+y)\left(\frac{5}{xy} - 4\right) + \frac{5}{c} - 4c - 5 \geq 0.$$

Since

$$\frac{5}{xy} - 4 \geq \frac{20}{5 - c^2} - 4 = \frac{4c^2}{5 - c^2},$$

it suffices to show that

$$\frac{8(x+y)c^2}{5 - c^2} + \frac{5}{c} - 4c - 5 \geq 0.$$

This inequality is true if

$$\frac{2c(3c^2 + 5)}{5 - c^2} + \frac{5}{c} - 4c - 5 \geq 0,$$

which is equivalent to

$$2c^4 + c^3 - 3c^2 - 5c + 5 \geq 0,$$

$$(c-1)^2(2c^2 + 5c + 5) \geq 0.$$

The proof is completed. The equality occurs for $a = b = c = d = e = 1$.

Remark 1. The inequality

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} - 5 \geq k(a + b + c + d + e - 5)$$

is not valid for $k > \frac{4}{5}$. To prove this assert, consider the case

$$a = x^2, \quad b = e = \frac{m}{x^2}, \quad c = d = \frac{1}{x}, \qquad m, x > 0.$$

From the constraint $ab + bc + cd + de + ea = 5$, we get

$$2m + \frac{2m}{x^3} + \frac{1}{x^2} = 5, \qquad m = \frac{x(5x^2 - 1)}{2(x^3 + 1)},$$

The desired inequality becomes

$$\frac{1}{x^2}+\frac{2x^2}{m}+2x-5\ge k\left(x^2+\frac{2m}{x^2}+\frac{2}{x}-5\right),$$

$$\frac{1}{x^2}+\frac{4x(x^3+1)}{5x^2-1}+2x-5\ge k\left(x^2+\frac{5x^2-1}{x^4+x}+\frac{2}{x}-5\right).$$

For $x\to\infty$, this inequality leads to the necessary condition

$$\frac{4}{5}\ge k.$$

Remark 2. Since

$$\frac{a+b+c+d+e}{5}+\frac{5}{a+b+c+d+e}\ge 2,$$

the following inequality follows from P 1.232:

- *If a,b,c,d,e are positive real numbers such that*

$$ab+bc+cd+de+ea=5,$$

then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\frac{20}{a+b+c+d+e}\ge 9.$$

□

P 1.233. *If a,b,c,d,e are positive real numbers such that*

$$ab+bc+cd+de+ea=5,\qquad a\ge b\ge c\ge 1\ge d\ge e,$$

then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+10\ge 3(a+b+c+d+e).$$

(Vasile Cîrtoaje, Math. Reflections, 4, 2024)

Solution. Write the inequality as $E\ge 0$. For fixed c, d and e, we may consider that a and E are functions of b. Differentiating the equality constraint yields

$$(b+e)a'+a+c=0,\qquad -a'=\frac{a+c}{b+e}\ge\frac{a+c}{b+c}\ge\frac{a+b}{2b}.$$

Therefore,

$$E'(b)=-\left(\frac{1}{b^2}+3\right)-\left(\frac{1}{a^2}+3\right)a'\ge-\left(\frac{1}{b^2}+3\right)+\left(\frac{1}{a^2}+3\right)\frac{a+b}{2b}$$

$$= 3\left(\frac{a+b}{2b} - 1\right) - \left(\frac{1}{b^2} - \frac{a+b}{2a^2b}\right) = \frac{(a-b)(3a^2b - 2a - b)}{2a^2b^2}.$$

Since $ab \geq 1$, we have $3a^2b - 2a - b \geq 3a - 2a - b = a - b \geq 0$. Thus, $E'(b) \geq 0$, $E(b)$ is increasing and has the minimum value when b is minimum, hence when $b = c$. So, we need to show that $F \geq 0$ for

$$ac + c^2 + cd + de + ea = 5, \qquad a \geq c \geq 1 \geq d \geq e > 0,$$

where

$$F = \frac{1}{a} + \frac{2}{c} + \frac{1}{d} + \frac{1}{e} + 10 - 3(a + 2c + d + e).$$

For fixed a and e, we may consider that d and F are functions of c. Differentiating the equality constraint yields

$$(c+e)d' + a + 2c + d = 0, \qquad -d' = \frac{a + 2c + d}{c + e} \geq 2.$$

Therefore,

$$F'(c) = -2\left(\frac{1}{c^2} + 3\right) - \left(\frac{1}{d^2} + 3\right)d' \geq -2\left(\frac{1}{c^2} + 3\right) + 2\left(\frac{1}{d^2} + 3\right) \geq 0,$$

$F(c)$ is increasing and has the minimum value when c is minimum (d is maximum), hence when either $c = 1$ or $d = 1$. So, it suffices to consider these cases.

Case 1: $c = 1$. We have

$$F = \frac{1}{a} + \frac{1}{d} + \frac{1}{e} + 6 - 3(a + d + e)$$

and

$$a + d + de + ea = 4, \qquad a \geq 1 \geq d \geq e > 0.$$

Denoting $s = \dfrac{a+d}{2}$, we have

$$e = \frac{4 - a - d}{a + d} = \frac{2 - s}{s},$$

and from $(a-1)(d-1) \leq 0$, we get $ad \leq 2s - 1$. Therefore,

$$F = \frac{2s}{ad} + \frac{s}{2-s} + 6 - 6s - \frac{3(2-s)}{s} \geq \frac{2s}{2s-1} + \frac{s}{2-s} + 6 - 6s - \frac{3(2-s)}{s} = \frac{12(s-1)^4}{s(2s-1)(2-s)} \geq 0.$$

Case 2: $d = 1$. We have

$$F = \frac{1}{a} + \frac{2}{c} + \frac{1}{e} + 8 - 3(a + 2c + e)$$

and

$$ac + c^2 + c + e + ea = 5, \qquad a \geq c \geq 1 \geq e > 0.$$

Since $\dfrac{2}{c} \ge 4 - 2c$, it follows that

$$F \ge \frac{1}{a} - 3a - 8c + \frac{1}{e} - 3e + 12.$$

Since $c^2 \ge 2c - 1$, the constraint $ac + c^2 + c + e + ea = 5$ yields

$$ac + 3c + e + ea \le 6.$$

Thus, we have

$$c \le \frac{6 - (1+a)e}{3+a},$$

$$6 \ge ac + 3c + e + ea > a + 3 + 0 + 0, \qquad 1 \le a < 3,$$

$$6 \ge ac + 3c + e + ea \ge a + 3 + e + ea, \qquad 0 < e \le \frac{3-a}{1+a} := e_0.$$

Therefore, for fixed $a \in [1, 3)$, we have $F \ge f(e)$, where

$$f(e) = \frac{1}{a} - 3a - 8 \cdot \frac{6 - (1+a)e}{3+a} + \frac{1}{e} - 3e + 12, \qquad e \in (0, e_0].$$

Since

$$f'(e) = \frac{8(1+a)}{3+a} - \frac{1}{e^2} - 3 = \frac{5a-1}{3+a} - \frac{1}{e^2} \le \frac{5a-1}{3+a} - \left(\frac{1+a}{3-a}\right)^2$$

$$= \frac{4(a^3 - 9a^2 + 11a - 3)}{(3+a)(3-a)^2} = \frac{4(a-1)(a^2 - 8a + 3)}{(3+a)(3-a)^2} \le 0,$$

$f(e)$ is decreasing, therefore

$$f(e) \ge f(e_0) = \frac{1}{a} - 3a - 8 + \frac{1+a}{3-a} - \frac{3(3-a)}{1+a} + 12 = \frac{3(1-a)^4}{a(3-a)(1+a)} \ge 0.$$

The proof is completed. The equality occurs for $a = b = c = d = e = 1$.

□

P 1.234. *For given $n \ge 3$, prove that 3 is the largest positive value of the constant k such that*

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n \ge k(a_1 + a_2 + \cdots + a_n - n)$$

for any $a_1 \ge a_2 \ge \cdots \ge a_{n-1} \ge 1 \ge a_n > 0$ with $a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1 = n$.

(Vasile Cîrtoaje, RMM, 39, 2025)

Solution. Choosing $a_2 = \cdots = a_{n-1} = 1$, the inequality becomes

$$\frac{1}{a_1} + \frac{1}{a_n} - 2 \ge k(a_1 + a_n - 2),$$

where $a_1 \ge 1 \ge a_n > 0$ such that $a_1a_n + a_1 + a_n = 3$. Let $p = a_1a_n$. From

$$3 = a_1a_n + a_1 + a_n \ge p + 2\sqrt{p},$$

we get $p \in (0, 1]$. Write the inequality as follows:

$$\frac{3-p}{p} - 2 \ge k(1-p), \qquad (1-p)(3-kp) \ge 0.$$

It is true if and only if $3 - kp \ge 0$ for $p \in (0, 1)$. From the necessary condition

$$\lim_{p \to 1}(3 - kp) \ge 0,$$

we get $k \le 3$. To show that 3 is the largest positive value of the constant k , we need to prove the inequality

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + 2n \ge 3(a_1 + a_2 + \cdots + a_n).$$

By the AM-HM inequality, we have

$$\frac{1}{a_2} + \cdots + \frac{1}{a_{n-1}} \ge \frac{n-2}{S},$$

where $S = \dfrac{a_2 + \cdots + a_{n-1}}{n-2} \ge 1$. So, it suffices to show that $E \ge 0$, where

$$E = \frac{1}{a_1} + \frac{1}{a_n} + \frac{n-2}{S} + 2n - 3[a_1 + a_n + (n-2)S].$$

By Lemma in the proof of P 1.216, we have

$$(n-3)S^2 + (a_1 + a_n)S + a_1a_n \le n.$$

Since the expression E decreases when a_1 increases, we may increase a_1 to have

$$(n-3)S^2 + (a_1 + a_n)S + a_1a_n = n.$$

Denoting $x = \dfrac{a_1 + a_n}{2}$, we need to show that

$$\frac{2x}{n - 2Sx - (n-3)S^2} + \frac{n-2}{S} + 2n - 3[2x + (n-2)S] \ge 0$$

for

$$(n-3)S^2 + 2Sx + a_1a_n = n.$$

From $(S-a_1)(S-a_n)\le 0$, we obtain

$$2Sx \ge S^2 + a_1a_n = n - 2Sx - (n-4)S^2,$$

therefore

$$4Sx \ge n - (n-4)S^2.$$

For fixed S, the desired inequality is equivalent to $F(x)\ge 0$, where

$$F(x) = 12S^2x^2 + [6(2n-5)S^2 - 4nS - 8n + 6]Sx + [n-(n-3)S^2][n-2+2nS-3(n-2)S^2].$$

Since

$$F'(x) = 24S^2x + 6(2n-5)S^3 - 4nS^2 - (8n-6)S \ge 6S[n-(n-4)S^2] + 6(2n-5)S^3 - 4nS^2 - (8n-6)S$$

$$= 6(n-1)S^3 - 4nS^2 - (2n-6)S \ge 6(n-1)S^2 - 4nS^2 - (2n-6)S = n(n-3)S(S-1) \ge 0,$$

$F(x)$ is increasing, hence

$$F(x) \ge F\left(\frac{n-(n-4)S^2}{4S}\right) = \frac{n[3(n-2)S^4 - 4(n-2)S^3 - 2nS^2 + 4nS - n - 2]}{4}$$

$$= \frac{n(S-1)^2[3(n-2)S^2 + 2(n-2)S - n - 2]}{4} \ge 0.$$

The proof is completed. For $k=3$, the equality occurs when $a_1 = a_2 = \cdots = a_n = 1$.

Remark. Since

$$\frac{a_1+a_2+\cdots+a_n}{n} \ge 2 - \frac{n}{a_1+a_2+\cdots+a_n},$$

the following inequality follows from P 1.234:

- *If* $a_1 \ge a_2 \ge \cdots \ge a_{n-1} \ge 1 \ge a_n > 0$ *such that* $a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1 = n$, *then*

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} + \frac{3n^2}{a_1+a_2+\cdots+a_n} \ge 4n.$$

□

P 1.235. *If* a,b,c,d,e,f *are nonnegative real numbers such that*

$$ab+bc+cd+de+ef+fa = 6, \qquad a\ge b\ge c\ge d\ge e\ge f,$$

then

$$\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3} + \frac{1}{d+3} + \frac{1}{e+3} + \frac{1}{f+3} \ge \frac{3}{2}.$$

(Vasile Cîrtoaje, GMA, no. 1-2, 2022)

First Solution. Let $x = \dfrac{b+c+d+e}{4}$ and $y = \dfrac{a+f}{2}$. By the AM-HM inequality, we have

$$\frac{1}{b+3}+\frac{1}{c+3}+\frac{1}{d+3}+\frac{1}{e+3} \ge \frac{16}{(b+3)+(c+3)+(d+3)+(e+3)} = \frac{4}{x+3}.$$

So, it suffices to show that

$$\frac{1}{a+3}+\frac{1}{f+3}+\frac{4}{x+3} \ge \frac{3}{2},$$

i.e.

$$\frac{2y+6}{af+6y+9} - \frac{3x+1}{2(x+3)} \ge 0.$$

By Lemma in the proof of P 1.216, we have

$$3x^2+2xy+af \le 6, \qquad x < \sqrt{2}.$$

So, it is enough to show that

$$\frac{2y+6}{15-3x^2+2(3-x)y} - \frac{3x+1}{2(x+3)} \ge 0.$$

After multiplying by $3-x$, we get the equivalent inequalities

$$\left[\frac{(3-x)(2y+6)}{15-3x^2+2(3-x)y} - 1\right] + \left[1 - \frac{(3-x)(3x+1)}{2(x+3)}\right] \ge 0,$$

$$\frac{3(x-1)^2}{15-3x^2+2(3-x)y} + \frac{3(x-1)^2}{2(x+3)} \ge 0.$$

The equality holds for $b=c=d=e=1$ and $a+f+af=3$ $(a \ge 1 \ge f)$.

□

P 1.236. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that*

$$a_1a_2+a_2a_3+\cdots+a_na_1 = n, \qquad a_1 \ge a_2 \ge \cdots \ge a_n.$$

Prove that:

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n} \ge a_1+a_2+\cdots+a_n.$$

(Vasile Cîrtoaje, Crux Mathematicorum, 9, 2023)

Solution. For $n=2$, the inequality is an equality. Consider further $n \ge 3$. Let

$$y = \frac{a_2+\cdots+a_{n-1}}{n-2}, \qquad a_1 \ge y \ge a_n > 0.$$

By the AM-HM inequality,

$$\frac{1}{a_2}+\cdots+\frac{1}{a_{n-1}} \ge \frac{(n-2)^2}{a_2+\cdots+a_{n-1}} = \frac{n-2}{y}.$$

By Lemma in the proof of P 1.216, it suffices to show that $(n-3)y^2+(a_1+a_n)y+a_1a_n \le n$ involves

$$\frac{1}{a_1}+\frac{1}{a_n}+\frac{n-2}{y} \ge a_1+a_n+(n-2)y.$$

Denoting a_1 by x and a_n by z, it suffices to prove the homogeneous inequality

$$[(n-3)y^2+xy+yz+zx]\left(\frac{n-3}{y}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \ge n[(n-3)y+x+y+z],$$

which is equivalent to $A+(n-3)B \ge 0$, where

$$A = \frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}-(x+y+z), \qquad B = \frac{y^2}{x}+\frac{y^2}{z}+\frac{xz}{y}-3y.$$

Since

$$2A = \frac{x^2(y-z)^2+y^2(z-x)^2+z^2(x-y)^2}{xyz} \ge 0$$

and

$$B \ge 3\left(\frac{y^2}{x}\cdot\frac{y^2}{z}\cdot\frac{xz}{y}\right)^{1/3}-3y = 0,$$

we obtain the required inequality $A+(n-3)B \ge 0$. So, the proof is completed. The equality occurs for $a_1=a_2=\ldots=a_n=1$.

Remark. Since

$$\frac{a_1+a_2+\cdots+a_n}{n}+\frac{n}{a_1+a_2+\cdots+a_ne} \ge 2,$$

the following inequality follows from P 1.236:

- *If $a_1,a_2,\ldots,a_n$ are positive real numbers such that*

$$a_1a_2+a_2a_3+\cdots+a_na_1=n, \qquad a_1 \ge a_2 \ge \cdots \ge a_n,$$

then

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}+\frac{n^2}{a_1+a_2+\cdots+a_n} \ge 2n.$$

□

P 1.237. *If $n \ge 3$ and $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$, then*

$$\sqrt{\frac{1}{n}\sum_{cyclic} a_1a_2} \ge \sqrt[n-1]{\frac{1}{n}\sum_{cyclic} a_1a_2\cdots a_{n-1}}.$$

(Vasile Cîrtoaje, AMM, 1, 2024)

Solution. For $n = 3$, the inequality is an equality. Consider further $n \ge 4$. Due to homogeneity, we may set

$$\sum_{cyclic} a_1a_2 = n$$

to prove that

$$\sum_{cyclic} a_1a_2\cdots a_{n-1} \le n.$$

Write the desired inequality as

$$(a_1 + a_n)A + a_1a_nB \le n,$$

where

$$A = a_2a_3\cdots a_{n-1}, \qquad B = a_2a_3\cdots a_{n-2} + a_3a_4\cdots a_{n-1} + \cdots + a_{n-1}a_1\cdots a_{n-3}.$$

Let

$$S = \frac{a_1 + a_n}{2}, \qquad x = \frac{a_2 + \cdots + a_{n-1}}{n-2}, \qquad a_1 \ge x \ge a_n \ge 0.$$

Since $A \le x^{n-2}$ and $B \le (n-2)x^{n-3}$, it suffices to prove that

$$2Sx^{n-2} + (n-2)a_1a_nx^{n-3} \le n.$$

On the other hand, by Lemma in the proof of P 1.216, we have $(n-3)x^2 + 2Sx + a_1a_n \le n$. Since the left hand side of the desired inequality increases when a_1a_n increases, we may replace the inequality constraint with the equality constraint

$$(n-3)x^2 + 2Sx + a_1a_n = n.$$

So, the desired inequality is equivalent to

$$2Sx^{n-2} + (n-2)\left(n - (n-3)x^2 - 2Sx\right)x^{n-3} \le n,$$

that is

$$(n-2)(n-3)x^{n-1} + 2(n-3)x^{n-2}S + n \ge n(n-2)x^{n-3}.$$

From $n = (n-3)x^2 + 2Sx + a_1a_n \le (n-3)x^2 + 2Sx + S^2$, we get

$$S \ge \sqrt{n - (n-4)x^2} - x.$$

Also, from $2S = a_1 + a_n \ge a_1 \ge x$ and $n = (n-3)x^2 + 2Sx + a_1a_n \ge (n-3)x^2 + 2Sx \ge (n-2)x^2$, we get $x \le \sqrt{\dfrac{n}{n-2}}$. So, it suffices to prove that $x \le \sqrt{\dfrac{n}{n-2}}$ involves

$$(n-2)(n-3)x^{n-1} + 2(n-3)x^{n-2}\left(\sqrt{n-(n-4)x^2} - x\right) + n \ge n(n-2)x^{n-3},$$

that is $f(x) \ge 0$, where

$$f(x) = (n-3)(n-4)x^{n-1} + 2(n-3)x^{n-2}\sqrt{n-(n-4)x^2} + n - n(n-2)x^{n-3}.$$

We have $f'(x) = (n-3)x^{n-4}g(x)$, where

$$g(x) = [n(n-2) - (n-1)(n-4)x^2]\left(\frac{2x}{\sqrt{n-(n-4)x^2}} - 1\right)$$

$$= \frac{n(x^2-1)[n(n-2)-(n-1)(n-4)x^2]}{\left(2x+\sqrt{n-(n-4)x^2}\right)\sqrt{n-(n-4)x^2}}.$$

Since

$$n(n-2) - (n-1)(n-4)x^2 \ge n(n-2) - \frac{n(n-1)(n-4)}{n-2} = \frac{n^2}{n-2} > 0,$$

we have $g(x) \le 0$ for $x \in [0,1]$, and $g(x) \ge 0$ for $x \in \left[1, \sqrt{\frac{n}{n-2}}\right]$, therefore f is decreasing on $[0,1]$ and increasing on $\left[1, \sqrt{\frac{n}{n-2}}\right]$. As a consequence, $f(x) \ge f(1) = 0$.

For $n \ge 4$, the equality occurs for $a_1 = a_2 = \cdots = a_n$, and also for $a_1 \ge a_2 = \cdots = a_n = 0$.

Remark. We claim that the following generalization is valid:

- *Let $n \ge 3$ and $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$. If $k \in \{2,3,\ldots,n-2\}$, then*

$$\sqrt[k]{\frac{1}{n}\sum_{cyclic} a_1a_2\cdots a_k} \ge \sqrt[n-1]{\frac{1}{n}\sum_{cyclic} a_1a_2\cdots a_{n-1}}.$$

□

P 1.238. *Let a,b,c,d,e be nonnegative real numbers satisfying $ab+bc+cd+de+ea = 5$. Prove that:*

(a) $$(a+2)^2 + (b+2)^2 + (c+2)^2 + (d+2)^2 + (e+2)^2 \ge 45.$$

(b) $$a^{3/2} + b^{3/2} + c^{3/2} + d^{3/2} + e^{3/2} \ge 5.$$

(Vasile Cîrtoaje, Crux Mathematicorum, 5, 2024)

Solution. (a) Denote

$$A = \sqrt{\frac{\sum ab}{5}},$$

and write the inequality as follows:

$$\sum a^2 + 4\sum a \ge 25, \qquad \sum a^2 + 4\sum a \ge 5\sum ab,$$

$$\sum a^2 - \sum ab \ge 4\left(\sum ab - \sum a\right), \qquad \sum (a-b)^2 \ge 8\left(\sum ab - A\sum a\right),$$

$$\sum(a-b)^2 \ge 8A\left(\sqrt{5\sum ab}-\sum a\right), \quad \sum(a-b)^2 \ge \frac{8A}{\sqrt{5\sum ab}+\sum a}\left[5\sum ab-\left(\sum a\right)^2\right],$$

$$\frac{5A+\sum a}{8A}\sum(a-b)^2 \ge 5\sum ab-\left(\sum a\right)^2, \quad \frac{5A+\sum a}{4A}\sum(a-b)^2+\sum(a-b)^2 \ge 4\sum a(b-c),$$

$$\frac{9A+\sum a}{4A}\sum(a-b)^2 \ge 4\sum a(b-c).$$

From $\sum(a-2b+2c-d)^2 \ge 0$, we get

$$10\sum a^2-16\sum ab+6\sum ac \ge 0, \qquad 5\sum(a-b)^2 \ge 6\sum a(b-c).$$

Thus, it suffices to show that

$$\frac{9A+\sum a}{4A} \ge \frac{10}{3},$$

that is $3\sum a \ge 13A$. By Lemma 1 below, we have

$$3\sum a \ge 6\sqrt{ab+bc+cd+de+ea} = 6\sqrt{5}\ A > 13A.$$

The proof is completed. The equality occurs for $a=b=c=d=e=1$.

(b) Assume that $a=\max\{a,b,c,d,e\}$. Since the inequality is true for $a^{3/2} \ge 5$, we assume next $a < 5^{2/3}$, hence $a<3$, $b<3$, $c<3$, $d<3$ and $e<3$. Based on the inequality in Lemma 2 and the inequality in (a), we have

$$4\sum a^{3/2} \ge \sum(a+2)^2-25 \ge 45-25 \ge 20.$$

The equality occurs for $a=b=c=d=e=1$.

Lemma 1. *If a,b,c,d,e are nonnegative real numbers, then*

$$a+b+c+d+e \ge 2\sqrt{ab+bc+cd+de+ea}.$$

Proof. Due to cyclicity, we may assume that $a=\max\{a,b,c,d,e\}$ and $b \ge e$. We need to prove the homogeneous inequality

$$(a+b+c+d+e)^2 \ge 4(ab+bc+cd+de+ea),$$

which is equivalent to the obvious inequality

$$(a-b-e)^2+c^2+d^2+2c(a-b+e)+2d(a-c)+2d(b-e) \ge 0.$$

The equality occurs for $a=b+e$ and $c=d=0$ (or any cyclic permutation).

Lemma 2. *If $0 \le x \le 4$, then*

$$4x^{3/2} \ge (x+2)^2-5.$$

Proof. Denote $f(x) = 4x^{3/2} - (x+2)^2 + 5$. From

$$f'(x) = 6\sqrt{x} - 2(x+2) = 2(\sqrt{x}-1)(2-\sqrt{x}),$$

we get $f'(x) \le 0$ for $x \in [0,1]$ and $f'(x) \ge 0$ for $x \in [1,4]$. As a consequence, $f(x)$ is decreasing on $[0,1]$ and increasing on $[1,4]$, therefore $f(x) \ge f(1) = 0$.

Remark. Similarly, using the inequality in Lemma 1 and the inequality

$$\sum (a - mb + mc - d)^2 \ge 0$$

for $m = \dfrac{\sqrt{5}+1}{2}$, we have

$$(a+k)^2 + (b+k)^2 + (c+k)^2 + (d+k)^2 + (e+k)^2 \ge 5(1+k)^2$$

for $0 \le k \le 1 + \dfrac{\sqrt{5}}{2}$. So, from $\sum (a - mb + mc - d)^2 \ge 0$, we get

$$2(m^2+1)\sum a^2 - 2m(m+2)\sum ab + 2(2m-1)\sum ac \ge 0,$$

$$(m^2+1)\sum (a-b)^2 \ge 2(2m-1)\sum a(b-c).$$

For $m = \dfrac{\sqrt{5}+1}{2}$ (when this inequality is strongest), we obtain

$$\frac{\sqrt{5}+1}{4}\sum (a-b)^2 \ge \sum a(b-c).$$

On the other hand, the desired inequality is equivalent to

$$\frac{(2k+5)A + \sum a}{2kA}\sum (a-b)^2 \ge 4\sum a(b-c).$$

So, it suffices to show that

$$\frac{(2k+5)A + \sum a}{2kA} \ge \sqrt{5}+1,$$

which is true if

$$\frac{(2k+5)A + 2\sqrt{5}A}{2kA} \ge \sqrt{5}+1,$$

that is $k \le 1 + \dfrac{\sqrt{5}}{2} \approx 2.118$. Note that the computer calculations show that the inequality is true for $0 \le k \le k_0$, where $k_0 \approx 2.123535$.

□

P 1.239. *If a, b, c, d are nonnegative real numbers such that $ab + bc + cd + da \geq 4$, then*

$$(a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1) \geq (a + b + c + d)^2.$$

(Vasile Cîrtoaje, Crux Mathematicorum, 10, 2024)

Solution. Write the hypothesis as

$$(a + c)(b + d) \geq 4,$$

and assume that

$$a + c \geq b + d.$$

There are two cases to consider: $b + d \geq 2$ and $b + d \leq 2$.

Case 1: $b + d \geq 2$. Since

$$(a^2 + 1)(c^2 + 1) \geq (a + c)^2, \qquad (b^2 + 1)(d^2 + 1) \geq (b + d)^2,$$

it suffices to show that

$$(a + c)(b + d) \geq a + b + c + d.$$

Indeed, we have

$$(a + c)(b + d) - (a + b + c + d) \geq 2(a + c) - (a + b + c + d) = (a + c) - (b + d) \geq 0.$$

Case 2: $b + d \leq 2$. Let

$$x = \frac{a + c}{2}, \qquad y = \frac{b + d}{2}, \qquad x \geq 1 \geq y.$$

We have

$$xy \geq 1, \qquad bd \leq y^2.$$

Since

$$(a^2 + 1)(c^2 + 1) \geq (a + c)^2 = 4x^2$$

and

$$(b^2 + 1)(d^2 + 1) = (b + d)^2 + (1 - bd)^2 \geq 4y^2 + (1 - y^2)^2 = (1 + y^2)^2,$$

it suffices to show that

$$4x^2(1 + y^2)^2 \geq (2x + 2y)^2,$$

which is equivalent to

$$x(1 + y^2) \geq x + y, \qquad y(xy - 1) \geq 0.$$

The inequality is an equality for $ac = 1$ and $b = d = \dfrac{2}{a + c}$, or $bd = 1$ and $a = c = \dfrac{2}{b + d}$.

Remark 1. The inequality is also true for $abcd \geq 1$ (Pham Kim Hung, 2006). Indeed, if $abcd \geq 1$, then

$$ab + bc + cd + da \geq 4\sqrt{abcd} \geq 4,$$

Remark 2. The inequality is also true for $ab+ac+ad+bc+bd+cd \ge 6$ (see P 3.69, Volume 1). Indeed, if $ab+ac+ad+bc+bd+cd \ge 6$, then at least one of the inequalities

$$(a+b)(c+d) \ge 4, \qquad (a+c)(b+d) \ge 4, \qquad (a+d)(b+c) \ge 4$$

is true.

□

P 1.240. *Let a,b,c,d,e be real numbers such that $a \ge b \ge c \ge d \ge e \ge 0$ and $ab+bc+cd+de+ea=5$. Prove that*

$$a^{5/4}+b^{5/4}+c^{5/4}+d^{5/4}+e^{5/4} \ge 5.$$

(Vasile Cîrtoaje, GMA, no. 3-4, 2023)

Solution. Denote

$$x=\frac{a+b}{2}, \qquad y=\frac{d+e}{2}, \qquad x \ge c \ge y.$$

By Jensen's inequality for convex functions, we have

$$a^{5/4}+b^{5/4} \ge 2x^{5/4}, \qquad d^{5/4}+e^{5/4} \ge 2y^{5/4}.$$

Also, by Bernoulli's inequality, we have

$$c^{5/4}=(1+(c-1))^{5/4} \ge 1+\frac{5}{4}(c-1)=\frac{5c-1}{4}.$$

So, it suffices to show that

$$8(x^{5/4}+y^{5/4})+5c \ge 21.$$

We will first show that

$$x^2+y^2+xy+c(x+y) \ge 5.$$

Indeed, we have

$$4\left(x^2+y^2+xy+c(x+y)-5\right)=(a+b)^2+(d+e)^2+(a+b)(d+e)+2c(a+b+d+e)$$

$$-4(ab+bc+cd+de+ea)=(a-b)^2+(d-e)^2+a(d+2c-3e)+b(d+e-2c)+2c(e-d)$$

$$\ge b(d+2c-3e)+b(d+e-2c)+2c(e-d)=2b(d-e)+2c(e-d)=2(d-e)(b-c) \ge 0.$$

So, we only need to show that

$$8(x^{5/4}+y^{5/4})+\frac{5(5-x^2-y^2-xy)}{x+y} \ge 21.$$

Denoting $x=s+t$, $y=s-t$ and

$$f(t)=(s+t)^{5/4}+(s-t)^{5/4}, \qquad t \in [0,s],$$

we need to show that $g(t) \geq 0$, where

$$g(t) = f(t) - \frac{5t^2 + 15s^2 + 42s - 25}{16s}.$$

For even $j \quad (j \geq 2)$, we have

$$f^{(j)}(0) = 2k(k-1)\cdots(k-j+1)s^{k-j} > 0,$$

where $k = 5/4$. Thus, by the Maclaurin series expansion of the even function f, we have

$$f(t) = f(0) + \frac{f^{(2)}(0)t^2}{2!} + \frac{f^{(4)}(0)t^4}{4!} + \cdots \geq f(0) + \frac{f^{(2)}(0)t^2}{2!} + \frac{f^{(4)}(0)t^4}{4!}$$

$$= 2s^k + k(k-1)s^{k-2}t^2 + \frac{k(k-1)(k-2)(k-3)}{12}s^{k-4}t^4$$

$$= 2s^{5/4} + \frac{5}{16}s^{-3/4}t^2 + \frac{35}{1024}s^{-11/4}t^4 \geq 2s^{5/4} + \frac{5}{16}s^{-3/4}t^2 + \frac{1}{32}s^{-11/4}t^4.$$

Consequently, to prove that $g(t) \geq 0$, it suffices to show that

$$2s^{5/4} + \frac{5}{16}s^{-3/4}t^2 + \frac{1}{32}s^{-11/4}t^4 \geq \frac{5t^2 + 15s^2 + 42s - 25}{16s},$$

which is equivalent to

$$s^{-7/4}t^4 - 10(1 - s^{1/4})t^2 + 64s^{9/4} - 30s^2 - 84s + 50 \geq 0.$$

Substituting $r = s^{1/4}$, the inequality becomes

$$r^{-7}t^4 - 10(1-r)t^2 + 64r^9 - 30r^8 - 84r^4 - 50 \geq 0,$$

$$t^4 - 10r^7(1-r)t^2 + r^7(64r^9 - 30r^8 - 84r^4 - 50) \geq 0,$$

$$(t^2 - 5r^7 + 5r^8)^2 + r^7(39r^9 + 20r^8 - 25r^7 - 84r^4 + 50) \geq 0.$$

Since

$$39r^9 + 20r^8 - 25r^7 - 84r^4 + 50 = (r-1)^2E,$$

where

$$E = 39r^7 + 98r^6 + 132r^5 + 166r^4 + 200r^3 + 150r^2 + 100r + 50 > 0,$$

the proof is completed. The equality occurs for $a = b = c = d = e = 1$.

Remark. For $0 < k < \frac{5}{4}$, the inequality $a^k + b^k + c^k + d^k + e^k \geq 5$ does not hold. To prove this assert, suppose

$$a = b = 1 + x, \qquad c = 1 - x^2/2, \qquad d = e = 1 - x.$$

For $x \in [0,1]$, we have $a \geq b \geq c \geq d \geq e \geq 0$ and $ab + bc + cd + de + ea = 5$, while the inequality $a^k + b^k + c^k + d^k + e^k \geq 5$ is equivalent to $g(x) \geq 0$, where

$$g(x) = 2(1+x)^k + 2(1-x)^k + (1 - x^2/2)^k - 5.$$

We have $g(0) = 0$, $g'(0) = 0$ and $g''(0) = k(4k-5)$. Since $g''(0) < 0$ for $0 < k < 5/4$, the point $x = 0$ is a local maximum of g. In addition, since $g(0) = 0$, there is a neighbourhood V of 0 such that $g(x) < 0$ for $x \in V \cap (0, 1]$.

Open problem 1. *If $a_1, a_2, \ldots, a_7$ are real numbers such that*

$$a_1 \ge a_2 \ge \cdots \ge a_7 \ge 0, \qquad a_1a_2 + a_2a_3 + \cdots + a_7a_1 = 7,$$

then

$$a_1^{3/2} + a_2^{3/2} + \cdots + a_7^{3/2} \ge 7.$$

Open problem 2. *Let n $(n \ge 5)$ be an odd integer number and $k \ge k_0 = \dfrac{2n-5}{n-1}$. If $a_1, a_2, \ldots, a_n$ are real numbers such that*

$$a_1 \ge a_2 \ge \cdots \ge a_n \ge 0, \qquad a_1a_2 + a_2a_3 + \cdots + a_na_1 = n,$$

then

$$a_1^k + a_2^k + \cdots + a_n^k \ge n.$$

Note that for $0 < k < k_0$, the inequality $a_1^k + a_2^k + \cdots + a_n^k \ge n$ does not hold. To prove this claim, suppose

$$a_1 = a_2 = \cdots = a_j = 1 + x, \qquad a_{j+1} = 1 - (n-4)x^2/2, \qquad a_{j+2} = a_{j+3} = \cdots = a_n = 1 - x,$$

where $j = \dfrac{n-1}{2}$. For $x \in [0, 1]$, we have $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$ and $a_1a_2+a_2a_3+\cdots+a_na_1 = n$. The inequality $a_1^k + a_2^k + \cdots + a_n^k \ge n$ is equivalent to $g(x) \ge 0$, where

$$g(x) = j\left((1+x)^k + (1-x)^k\right) + \left(1 - \frac{(n-4)x^2}{2}\right)^k - n.$$

We have $g(0) = 0$, $g'(0) = 0$ and $\dfrac{1}{k}g''(0) = (n-1)k - 2n + 5$. Since $g''(0) < 0$ for $0 < k < k_0$, the point $x = 0$ is a local maximum of g. In addition, since $g(0) = 0$, there is a neighbourhood V of 0 such that $g(x) < 0$ for $x \in V \cap (0, 1]$.

□

P 1.241. *If $a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n \ge 0$ such that $a_1 + a_2 + \cdots + a_n = n$, then*

$$a_1a_2 + a_2a_3 + \cdots + a_na_1 \le n.$$

(Vasile Cîrtoaje, RMM, 38, 2025)

Solution. For $n = 2$, the inequality reduces to $(a_1 - a_2)^2 \geq 0$. Consider next $n \geq 3$ and write the desired inequality in the homogeneous form

$$(a_1 + a_2 + \cdots + a_n)^2 - n(a_1a_2 + a_2a_3 + \cdots + a_na_1) \geq 0.$$

From

$$na_2 \leq n = a_1 + a_2 + \cdots + a_n,$$

we get

$$a_1 \geq (n-1)a_2 - a_3 - \cdots - a_n.$$

For fixed $a_2, a_3, \ldots, a_n$, the homogeneous inequality is equivalent to $f(a_1) \geq 0$, where

$$f(a_1) = (a_1 + a_2 + \cdots + a_n)^2 - n(a_2a_3 + \cdots + a_{n-1}a_n) - n(a_2 + a_n)a_1.$$

Since

$$f'(a_1) = 2(a_1+a_2+\cdots+a_n) - n(a_2+a_n) = (a_1+a_2+\cdots+a_n-na_2)+(a_1+a_2+\cdots+a_n-na_n)$$

$$= n(1-a_2) + (a_1 + a_2 + \cdots + a_n - na_n) \geq 0,$$

$f(a_1)$ is increasing, hence

$$f(a_1) \geq f((n-1)a_2 - a_3 - \cdots - a_n).$$

Thus, it suffices to show that $f((n-1)a_2 - a_3 - \cdots - a_n) \geq 0$, that is

$$na_2^2 - (a_2a_3 + \cdots + a_{n-1}a_n) - (a_2 + a_n)[(n-1)a_2 - a_3 - \cdots - a_n] \geq 0.$$

For $n = 3$, the inequality reduces to $(a_2 - a_3)^2 \geq 0$, while for $n \geq 4$, the inequality is equivalent to

$$na_2^2 - (a_2a_3 + \cdots + a_{n-1}a_n) - n(a_2 + a_n)a_2 + (a_2 + a_n)(a_2 + a_3 + \cdots + a_n) \geq 0,$$

$$-(a_2a_3 + a_3a_4 + \cdots + a_{n-1}a_n + a_na_2) - (n-1)a_2a_n + (a_2 + a_n)(a_2 + a_3 + \cdots + a_n) \geq 0,$$

$$a_2(a_2+a_3+\cdots+a_n) - (a_2a_3+a_3a_4+\cdots+a_{n-1}a_n+a_na_2) + a_n[a_2+a_3+\cdots+a_n-(n-1)a_2] \geq 0,$$

$$a_2(a_2-a_3) + a_3(a_2-a_4) + \cdots + a_{n-1}(a_2-a_n) - a_n[(a_2-a_3)+(a_2-a_4)+\cdots+(a_2-a_n)] \geq 0,$$

$$(a_2 - a_3)(a_2 - a_n) + (a_2 - a_4)(a_3 - a_n) + \cdots + (a_2 - a_n)(a_{n-1} - a_n) \geq 0.$$

The last inequality is clearly true. The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. The following statement is also valid:

- *If* $a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n \geq 0$ *such that* $a_1a_2 + a_2a_3 + \cdots + a_na_1 = n$, *then*

$$a_1 + a_2 + \cdots + a_n \geq n.$$

□

P 1.242. *If* $0 \le a_1 \le 1 \le a_2 \le \cdots \le a_n$ *such that* $a_1 + a_2 + \cdots + a_n = n$, *then*

$$a_1a_2 + a_2a_3 + \cdots + a_na_1 \le n.$$

(Vasile Cîrtoaje, Math. Reflections, 3, 2024)

Solution. For $n = 2$, the inequality reduces to $(a_1 - a_2)^2 \ge 0$. Consider next $n \ge 3$ and write the desired inequality in the homogeneous form

$$(a_1 + a_2 + \cdots + a_n)^2 - n(a_1a_2 + a_2a_3 + \cdots + a_na_1) \ge 0.$$

From

$$na_2 \ge n = a_1 + a_2 + \cdots + a_n,$$

we get

$$a_1 \le (n-1)a_2 - a_3 - \cdots - a_n.$$

For fixed $a_2, a_3, \ldots, a_n$, the desired inequality is equivalent to $f(a_1) \ge 0$, where

$$f(a_1) = (a_1 + a_2 + \cdots + a_n)^2 - n(a_2a_3 + \cdots + a_{n-1}a_n) - n(a_2 + a_n)a_1.$$

Since

$$f'(a_1) = 2(a_1+a_2+\cdots+a_n)-n(a_2+a_n) = (a_1+a_2+\cdots+a_n-na_2)+(a_1+a_2+\cdots+a_n-na_n) \le 0,$$

$f(a_1)$ is decreasing, hence

$$f(a_1) \ge f((n-1)a_2 - a_3 - \cdots - a_n).$$

Thus, it suffices to show that $f((n-1)a_2 - a_3 - \cdots - a_n) \ge 0$, that is

$$na_2^2 - (a_2a_3 + \cdots + a_{n-1}a_n) - (a_2 + a_n)[(n-1)a_2 - a_3 - \cdots - a_n] \ge 0.$$

For $n = 3$, the inequality reduces to $(a_2 - a_3)^2 \ge 0$, while for $n \ge 4$, the inequality is equivalent to

$$(a_2 + a_n)(a_2 + a_3 + \cdots + a_n) - (a_2a_3 + a_3a_4 + \cdots + a_{n-1}a_n + a_na_2) - (n-1)a_2a_n \ge 0,$$

$$[a_2(a_2+a_3+\cdots+a_n)-(a_2a_3+a_3a_4+\cdots+a_{n-1}a_n+a_na_2)]+a_n[a_2+a_3+\cdots+a_n-(n-1)a_2] \ge 0,$$

$$[a_2(a_2-a_3)+a_3(a_2-a_4)+\cdots+a_{n-1}(a_2-a_n)]-a_n[(a_2-a_3)+(a_2-a_4)+\cdots+(a_2-a_n)] \ge 0,$$

$$(a_2 - a_3)(a_2 - a_n) + (a_2 - a_4)(a_3 - a_n) + \cdots + (a_2 - a_n)(a_{n-1} - a_n) \ge 0.$$

The last inequality is clearly true. The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. The following statement is also valid.

- *If* $0 \le a_1 \le 1 \le a_2 \le \cdots \le a_n$ *such that* $a_1a_2 + a_2a_3 + \cdots + a_na_1 = n$, *then*

$$a_1 + a_2 + \cdots + a_n \ge n.$$

□

P 1.243. *Suppose $n \ge 4$ and $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$. If $a_1 = a_2$ and $a_{n-1} = a_n$, then*

$$n(a_1a_2 + a_2a_3 + \cdots + a_na_1) \ge (a_1 + a_2 + \cdots + a_n)^2.$$

(Vasile Cîrtoaje, Math. Reflections, 1, 2024)

Solution. For $n = 4$, the inequality reduces to an identity. Consider next $n > 4$, denote

$$S = \frac{a_2 + a_{n-1}}{2}, \qquad s = \frac{a_3 + \cdots + a_{n-2}}{n-4},$$

and write the inequality as follows:

$$n[a_2^2 + a_{n-1}^2 + a_2a_{n-1} + (a_2a_3 + \cdots + a_{n-2}a_{n-1})] \ge [2(a_2 + a_{n-1}) + (a_3 + \cdots + a_{n-2})]^2,$$

$$n[4S^2 - a_2a_{n-1} + (a_2a_3 + \cdots + a_{n-2}a_{n-1})] \ge [4S + (n-4)s]^2.$$

Since the sequences $(a_2, \dots, a_{n-2})$ and $(a_3, \dots, a_{n-1})$ are decreasing, by Chebyshev's inequality we have

$$(n-3)(a_2a_3 + \cdots + a_{n-2}a_{n-1}) \ge (a_2 + \cdots + a_{n-2})(a_3 + \cdots + a_{n-1}),$$

$$(n-3)(a_2a_3 + \cdots + a_{n-2}a_{n-1}) \ge [a_2 + (n-4)s][a_{n-1} + (n-4)s],$$

$$(n-3)(a_2a_3 + \cdots + a_{n-2}a_{n-1}) \ge a_2a_{n-1} + 2(n-4)sS + (n-4)^2s^2.$$

So, it suffices to show that

$$n\left[4S^2 - a_2a_{n-1} + \frac{a_2a_{n-1} + 2(n-4)sS + (n-4)^2s^2}{n-3}\right] \ge [4S + (n-4)s]^2,$$

which is equivalent to

$$4(n-3)S^2 - 6(n-4)sS + 3(n-4)s^2 \ge na_2a_{n-1}.$$

Since $a_2 \ge s \ge a_{n-1}$, we have

$$(s - a_2)(s - a_{n-1}) \le 0, \qquad a_2a_{n-1} \le 2sS - s^2.$$

Therefore, it suffices to show that

$$4(n-3)S^2 - 6(n-4)sS + 3(n-4)s^2 \ge n(2sS - s^2),$$

that is equivalent to

$$(n-3)(S-s)^2 \ge 0.$$

The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.244. *Let $a \ge b \ge c \ge d \ge e \ge 0$ such that $ab+bc+cd+de+ea=5$. Prove that*

$$a^2+b^2+c^2+d^2+e^2+5(a+b+c+d+e) \ge 30.$$

(Vasile Cîrtoaje, Math. Reflections, 6, 2023)

Solution. Denote

$$x=\frac{a+b}{2}, \quad y=\frac{d+e}{2}, \qquad x \ge c \ge y.$$

Since

$$a^2+b^2 \ge 2x^2, \qquad d^2+e^2 \ge 2y^2,$$

it suffices to show that

$$2(x^2+y^2)+10(x+y)+c^2+5c \ge 30.$$

Moreover, since $c^2 \ge 2c-1$, it suffices to show that

$$2(x^2+y^2)+10(x+y)+7c \ge 31.$$

We will first show that

$$x^2+y^2+xy+c(x+y) \ge 5.$$

Indeed, we have

$$4[x^2+y^2+xy+c(x+y)-5]=(a+b)^2+(d+e)^2+(a+b)(d+e)$$

$$+2c(a+b+d+e)-4(ab+bc+cd+de+ea)$$

$$=(a-b)^2+(d-e)^2+a(d+2c-3e)+b(d+e-2c)+2c(e-d)$$

$$\ge b(d+2c-3e)+b(d+e-2c)+2c(e-d)=2b(d-e)+2c(e-d)=2(d-e)(b-c) \ge 0.$$

So, it suffices to show that

$$2(x^2+y^2)+10(x+y)+\frac{7(5-x^2-y^2-xy)}{x+y} \ge 31.$$

Denoting

$$s=\frac{x+y}{2}, \qquad p=xy \qquad (p \le s^2),$$

the desired inequality becomes

$$8s^2-4p+20s+\frac{7(5-4s^2+p)}{2s} \ge 31,$$

$$16s^3+12s^2-62s+35 \ge p(8s-7).$$

For $8s-7 \le 0$, it suffices to show that $16s^3+12s^2-62s+35 \ge 0$. Indeed,

$$16s^3+12s^2-62s+35=s(4s-3)^2+(1-s)(35-36s)>0.$$

Also, for $8s-7 \ge 0$, we have

$$16s^3+12s^2-62s+35-p(8s-7) \ge 16s^3+12s^2-62s+35-s^2(8s-7)$$

$$=8s^3+19s^2-62s+35=(s-1)^2(8s+35) \ge 0.$$

The proof is completed. The equality occurs for $a=b=c=d=e=1$.

Remarks. Similarly, we can prove the stronger inequality

$$2(a^2+b^2+c^2+d^2+e^2)+11(a+b+c+d+e) \ge 65.$$

It suffices to show that

$$4(x^2+y^2)+22(x+y)+2c^2+11c \ge 65$$

for

$$x^2+y^2+xy+c(x+y) \ge 5.$$

Since $c^2 \ge 2c-1$, it suffices to show that

$$4(x^2+y^2)+22(x+y)+15c \ge 67.$$

Denoting

$$s=\frac{x+y}{2}, \qquad p=xy,$$

we need to show that

$$16s^2-8p+44s+15c \ge 67$$

for

$$2cs \ge 5+p-4s^2.$$

It suffices to show that

$$16s^2-8p+44s+\frac{15(5+p-4s^2)}{2s} \ge 67,$$

i.e.

$$32s^3+28s^2-134s+75 \ge p(16s-15).$$

Since

$$32s^3+28s^2-134s+75=32s(s-1)^2+92s^2-166s+75 \ge 92s^2-166s+75>0,$$

the inequality is true if $16s-15 \le 0$. For $16s-15 \ge 0$, since $p \le s^2$, it suffices to prove that

$$32s^3+28s^2-134s+75 \ge 2(16s-15),$$

i.e.

$$16s^3+43s^2-134s+75 \ge 0,$$

$$(s-1)^2(16s+75) \ge 0.$$

□

P 1.245. *If $a \ge b \ge 1 \ge c \ge d \ge e \ge f \ge 0$ such that $ab+bc+cd+de+ef+fa=6$, then*

$$(2a+3)^2+(2b+3)^2+(2c+3)^2+(2d+3)^2+(2e+3)^2+(2f+3)^2 \ge 150.$$

(Vasile Cîrtoaje, RMM, 38, 2025)

Solution. Denote by E the left hand side of the inequality. For fixed c,d,e,f, we may assume that b and E are functions of a. By differentiating the equality constraint, we get

$$(a+c)b'+b+f=0, \qquad b'=\frac{-(b+f)}{a+c} \ge -1.$$

Since

$$\frac{E'(a)}{4}=2a+3+(2b+3)b' \ge 2a+3-(2b+3)=2(a-b) \ge 0,$$

$E(a)$ is increasing and has the minimum value when a is minimum, hence when $a=b$. Similarly, for fixed a,b,c,d, assume that e and E are functions of f. By differentiating the equality constraint, we get

$$(d+f)e'+a+e=0, \qquad e'=\frac{-(a+e)}{d+f} \le -1.$$

Since

$$\frac{E'(f)}{4}=2f+3+(2e+3)e' \le 2f+3-(2e+3)=2(f-e) \le 0,$$

$E(f)$ is decreasing and has the minimum value when f is maximum, hence when $f=e$. So, it suffices to consider $a=b$ and $f=e$, when we need to show that $F \ge 150$ for $b \ge 1 \ge c \ge d \ge e \ge 0$ such that $b^2+bc+cd+de+e^2+be=6$, where

$$F=2(2b+3)^2+(2c+3)^2+(2d+3)^2+2(2e+3)^2.$$

Now, for fixed d and e, assume that b and F are functions of c. By differentiating the equality constraint, we get

$$(2b+c+e)b'+b+d=0, \qquad b'=\frac{-(b+d)}{2b+c+e} \le \frac{-(b+d)}{2b+c+d},$$

hence

$$\frac{F'(c)}{4}=2c+3+2(2b+3)b' \le 2c+3-\frac{2(2b+3)(b+d)}{2b+c+d} \le 5-\frac{2(2b+3)(b+d)}{2b+1+d}=\frac{5+4b-4b^2-(4b+1)d}{2b+1+d}.$$

We will show that $F'(c) \le 0$, that is

$$4b^2-4b-5+(4b+1)d \ge 0.$$

From

$$6=b^2+bc+cd+de+e^2+be \le b^2+bc+cd+2d^2+bd \le b^2+b+3d+bd$$

we get

$$d \geq \frac{6-b-b^2}{b+3},$$

therefore

$$4b^2-4b-5+(4b+1)d \geq 4b^2-4b-5+\frac{(4b+1)(6-b-b^2)}{b+3} = \frac{3(b-1)(b+3)}{b+3} \geq 0.$$

Since $F'(c) \leq 0$, $F(c)$ is decreasing and has the minimum value when c is maximum, hence when $c=1$. So, it sufficesed to consider this case, when we need to show that $G \geq 125$ for $b \geq 1 \geq d \geq e \geq 0$ such that $b^2+b+d+de+e^2+be=6$, where

$$G = 2(2b+3)^2+(2d+3)^2+2(2e+3)^2.$$

For fixed b, we may assume that d is a function of e. By differentiating the equality constraint, we get

$$(1+e)d'+b+d+2e=0,$$

hence

$$\frac{G(e)}{4} = 2(2e+3)+(2d+3)d' = 2(2e+3)-\frac{(2d+3)(b+d+2e)}{1+e} \leq 2(2e+3)-\frac{(2e+3)(b+d+2e)}{1+e}$$

$$= \frac{(2e+3)(2-b-d)}{1+e}.$$

From

$$6 = b^2+b+d+de+e^2+be \leq b^2+b+d+2d^2+bd \leq (b+d)^2+(b+d),$$

we get $b+d \geq 2$, therefore $G'(e) \leq 0$, $G(e)$ is decreasing and has the minimum value when e is maximum, hence when $e=d$. So, it suffices to consider $e=d$, when we need to show that if $b \geq 1 \geq d$ such that

$$b^2+b+d+2d^2+bd=6,$$

then $2(2b+3)^2+3(2d+3)^2 \geq 125$, i.e.

$$2b^2+3d^2+6b+9d \geq 20, \qquad 2b(2-d) \geq d^2-7d+8.$$

Since $2b = -d-1+\sqrt{25-2d-7d^2}$, we need to show that

$$(-d-1+\sqrt{25-2d-7d^2})(2-d) \geq d^2-7d+8,$$

i.e.

$$(2-d)\sqrt{25-2d-7d^2} \geq 10-6d.$$

This is true if

$$(2-d)^2(25-2d-7d^2) \geq (10-6d)^2,$$

which is equivalent to the obvious inequality

$$d(d-1)^2(12-7d) \geq 0.$$

The equality occurs for $a = b = c = d = e = f = 1$, and also for $a = b = 2$, $c = 1$ and $d = e = f = 0$.

Remark. Note that $\frac{3}{2}$ is the largest positive value of k such that the inequality

$$(a+k)^2 + (b+k)^2 + (c+k)^2 + (d+k)^2 + (e+k)^2 + (f+k)^2 \geq 6(1+k)^2$$

holds for all nonnegative numbers a, b, c, d, e, f satisfying

$$ab + bc + cd + de + ef + fa = 6, \qquad a \geq b \geq 1 \geq c \geq d \geq e \geq f.$$

Indeed, assuming $a = b = 2$, $c = 1$ and $d = e = f = 0$, the equality constraint is satisfied, while the desired inequality becomes

$$2(2+k)^2 + 3k^2 \geq 5(1+k)^2,$$

which is equivalent to $2k \leq 3$.

□

P 1.246. *If $a \geq b \geq c \geq d \geq e \geq 0$, then*

$$\sqrt{\frac{ab + bc + cd + de + ea}{5}} \geq \sqrt[3]{\frac{abc + bcd + cde + dea + eab}{5}}.$$

(Vasile Cîrtoaje, Mathproblems, 4, 2023)

Solution. For $c = 0$, the right side of the inequality is zero, therefore the inequality is true. Consider further $c > 0$. Due to homogeneity, we may assume that the right hand side of the inequality is 1. So, we need to show that

$$ab + bc + cd + de + ea \geq 5$$

for

$$abc + bcd + cde + dea + eab = 5.$$

By Lemma below, it suffices to consider the case when $a = b = c$, and the case when $b = c$ and $d = e$.

Case 1: $a = b = c$. We need to show that

$$2c^2 + cd + de + ce \geq 5$$

for

$$2c^3 + c^2d + 2cde + c^2e = 5, \qquad c \geq d \geq e.$$

For fixed c, we may consider that d is a function of e. From the equality constraint, we get

$$(c+2e)d' + c + 2d = 0, \qquad d' = \frac{-(c+2d)}{c+2e}.$$

Writing the desired inequality as $F(e) \le 5$, we have

$$F'(e) = c + d + (c+e)d' = c + d - \frac{(c+e)(c+2d)}{c+2e} = \frac{-c(d-e)}{c+2e} \le 0,$$

$F(e)$ is decreasing and it is minimum when e is maximum, hence when $e = d$. So, we need to show that

$$2c^2 + 2cd + d^2 \ge 5$$

for

$$c^3 + 2c^2d + 2cd^2 = 5, \qquad c \ge d.$$

Write the desired inequality in the homogeneous form

$$(2c^2 + 2cd + d^2)^3 \ge 5(c^3 + 2c^2d + 2cd^2)^2.$$

Due to homogeneity, we may set $c = 1$. So, we need to show that $f(d) \ge 0$, where

$$f(d) = 3\ln(d^2 + 2d + 2) - \ln 5 - 2\ln(2d^2 + 2d + 1), \qquad d \in [0, 1].$$

We have

$$f'(d) = \frac{6(d+1)}{d^2 + 2d + 2} - \frac{4(2d+1)}{2d^2 + 2d + 1} = \frac{2(2d^3 + 2d^2 - 3d - 1)}{(d^2 + 2d + 2)(2d^2 + 2d + 1)}$$

$$= \frac{2(d-1)(2d^2 + 4d + 1)}{(d^2 + 2d + 2)(2d^2 + 2d + 1)} \le 0,$$

$f(d)$ is decreasing, hence $f(d) \ge f(1) = 0$.

Case 2: $b = c$ and $d = e$. We need to show that

$$ab + b^2 + bd + d^2 + ad \ge 5$$

for

$$ab^2 + b^2d + bd^2 + ad^2 + abd = 5, \qquad a \ge b \ge d.$$

For fixed b, we may consider that a is a function of d. From the equality constraint, we get

$$(b^2 + bd + d^2)a' + ab + b^2 + 2ad + 2bd = 0.$$

Writing the desired inequality as $F(d) \ge 5$, we have

$$F'(d) = a + b + 2d + (b+d)a' = a + b + 2d - \frac{(b+d)(ab + b^2 + 2ad + 2bd)}{b^2 + bd + d^2} = \frac{A(a,b,d)}{b^2 + bd + d^2},$$

where

$$A(a,b,d) = (a+b+2d)(b^2+bd+d^2) - (b+d)(ab+b^2+2ad+2bd).$$

We will show that

$$A(a,b,d) \le A(b,b,d) \le 0.$$

Indeed,

$$A(a,b,d) - A(b,b,d) = -d(2b+d)(a-b) \le 0,$$
$$A(b,b,d) = -2d(b^2-d^2) \le 0.$$

Since $F'(d) \le 0$, $F(d)$ is decreasing and it is minimum when d is maximum, hence when $d = b$. So, we need to show that

$$2ab + 3b^2 \ge 5$$

for

$$3ab^2 + 2b^3 = 5, \qquad a \ge 1 \ge b.$$

We have

$$3(2ab+3b^2-5) \ge \frac{2(5-2b^3)}{b} + 9b^2 - 15 = \frac{5(b^3-3b+2)}{b} = \frac{5(b-1)^2(b+2)}{b} \ge 0.$$

The proof is completed. The equality occurs for $a = b = c = d = e = 1$, and also for $b = c = d = e = 0$.

Lemma. *If $a \ge b \ge c \ge d \ge e \ge 0$ such that $abc+bcd+cde+dea+eab = 5$, then the expression*

$$E = ab + bc + cd + de + ea$$

is minimum when $a = b = c$, or when $b = c$ and $d = e$.

Proof. For fixed a, d and e, we may consider that b is a function of c. From the equality constraint, we get

$$(ac+cd+ea)b' + ab + bd + de = 0.$$

So,

$$E'(c) = b + d + (a+c)b' = b + d - \frac{(a+c)(ab+bd+de)}{ac+cd+ea} = \frac{-(a+d-e)(ab-cd)}{ac+cd+ea} \le 0,$$

hence $E(c)$ is decreasing and is minimum when c is maximum, hence when $c = b$.

Similarly, for fixed b, c and d, we may consider that a is a decreasing function of e. From the equality constraint, we get

$$(bc+de+eb)a' + cd + da + ab = 0,$$

$$E'(e) = a + d - (b+e)a' = a + d - \frac{(b+e)(cd+da+ab)}{bc+de+eb} = \frac{-(b-c+d)(ab-de)}{bc+de+eb} \le 0,$$

hence $E(e)$ is decreasing and is minimum when e is maximum (a is minimum), i.e. when $e = d$ or $a = b$.

Finally, we conclude that E is minimum when either $b = c$ and $d = e$, or $b = c$ and $a = b$ (i.e. $a = b = c$).

□

P 1.247. *Let a, b, c, d be nonnegative real numbers such that*

$$\frac{1}{a+3}+\frac{1}{b+3}+\frac{1}{c+3}+\frac{1}{d+3}=1.$$

Prove that there is a permutation (x_1, x_2, x_3, x_4) of the sequence (a, b, c, d) such that

$$x_1x_2+x_2x_3+x_3x_4+x_4x_1 \geq 4.$$

(Vasile Cîrtoaje, 2023)

Solution. Assume that $a \geq b \geq c \geq d \geq 0$. Since

$$(a+d)(b+c)-(a+c)(b+d)=(a-b)(c-d) \geq 0$$

and

$$(a+d)(b+c)-(a+b)(c+d)=(a-c)(b-d) \geq 0,$$

the sum

$$S=ab+bd+dc+ca=(a+d)(b+c)$$

is the largest cyclic sum of this form. So, we will show that the sequence

$$(x_1, x_2, x_3, x_4)=(a, b, d, c)$$

satisfies the requirement $S \geq 4$. Denoting

$$x=\frac{a+d}{2}, \qquad y=\frac{b+c}{2},$$

we need to show that

$$\frac{1}{a+3}+\frac{1}{d+3}+\frac{1}{b+3}+\frac{1}{c+3}=1$$

involves $xy \geq 1$. Since

$$\frac{1}{b+3}+\frac{1}{c+3} \geq \frac{2}{y+3}$$

(from the AM-HM inequality or Jensen's inequality), we have

$$\frac{1}{a+3}+\frac{1}{d+3}+\frac{2}{y+3} \leq 1,$$

$$\frac{2(x+3)}{ad+6x+9} \leq \frac{y+1}{y+3}.$$

From $(y-a)(y-d) \leq 0$, we get $ad \leq 2xy-y^2$, therefore

$$\frac{2(x+3)}{2xy-y^2+6x+9} \leq \frac{y+1}{y+3},$$

$$\frac{2(x+3)}{(2x-y+3)(y+3)} \leq \frac{y+1}{y+3},$$

$$\frac{2(x+3)}{2x-y+3} \le y+1,$$

$$2xy \ge y^2 - 2y + 3,$$

hence

$$2(xy-1) \ge (y-1)^2 \ge 0.$$

Remark. The following generalization is valid:

- *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that*

$$\frac{1}{a_1+n-1} + \frac{1}{a_2+n-1} + \cdots + \frac{1}{a_n+n-1} = 1,$$

then there is a permutation $X = (x_1, x_2, \ldots, x_n)$ of the sequence $A = (a_1, a_2, \ldots, a_n)$ such that

$$x_1x_2 + x_2x_3 + \cdots + x_nx_1 \ge n.$$

To prove this, it suffices to show that $\sum\limits_{1\le i<j\le n} a_ia_j \ge n$. Using the contradiction method, we need to show that $\sum\limits_{1\le i<j\le n} a_ia_j < n$ involves

$$\frac{1}{a_1+n-1} + \frac{1}{a_2+n-1} + \cdots + \frac{1}{a_n+n-1} > 1.$$

This is true if $\sum\limits_{1\le i<j\le n} a_ia_j = n$ involves

$$\frac{1}{a_1+n-1} + \frac{1}{a_2+n-1} + \cdots + \frac{1}{a_n+n-1} \ge 1,$$

which is just P 1.208 in Volume 2.

□

P 1.248. *Let $a_1 \ge a_2 \ge \cdots \ge a_9 \ge 0$ such that $a_1 + a_2 + \cdots + a_9 = 2$. Prove that*

$$a_1a_2 + a_2a_3 + \cdots + a_9a_1 \le 1.$$

(Vasile Cîrtoaje, Math. Reflections, 1, 2023)

Solution. Write the inequality as $F(a_1, a_2, \ldots, a_9) \ge 0$, where

$$F(a_1, a_2, \ldots, a_9) = (a_1 + a_2 + \cdots + a_9)^2 - 4(a_1a_2 + a_2a_3 + \cdots + a_9a_1).$$

We will show that

$$F(a_1, a_2, a_3, \ldots, a_9) \ge F(a_2, a_2, a_3 \ldots, a_9) \ge 0.$$

The left inequality is equivalent to

$$(a_1+a_2+a_3+\cdots+a_9)^2-(2a_2+a_3+\cdots+a_9)^2 \ge 4(a_1a_2+a_2a_3+\cdots+a_9a_1)-4(a_2^2+a_2a_3+\cdots+a_9a_2),$$

$$(a_1-a_2)(a_1+3a_2+2a_3+\cdots+2a_9) \ge 4(a_1-a_2)(a_2+a_9),$$

$$(a_1-a_2)(a_1-a_2+2a_3+\cdots+2a_8-2a_9) \ge 0,$$

while the right inequality is equivalent to $G(a_2,a_3,\ldots,a_8,a_9) \ge 0$, where

$$G(a_2,a_3,\ldots,a_8,a_9) = (2a_2+a_3+\cdots+a_8+a_9)^2 - 4(a_2^2+a_2a_3+\cdots+a_8a_9+a_9a_2).$$

We will show that

$$G(a_2,a_3,\ldots,a_8,a_9) \ge G(a_2,a_3,\ldots,a_8,0) \ge \cdots \ge G(a_2,0,\ldots,0,0) = 0.$$

We have

$$G(a_2,a_3,\ldots,a_8,a_9) - G(a_2,a_3,\ldots,a_8,0) =$$

$$= (2a_2+a_3+\cdots+a_8+a_9)^2 - (2a_2+a_3+\cdots+a_8)^2 - 4(a_2^2+a_2a_3+\cdots+a_8a_9+a_9a_2)$$

$$+4(a_2^2+a_2a_3+\cdots+a_7a_8)$$

$$= a_9(4a_2+2a_3+\cdots+2a_8+a_9) - 4a_9(a_8+a_2) = a_9(2a_3+\cdots+2a_7-2a_8+a_9) \ge 0,$$

$$G(a_2,a_3,\ldots,a_7,a_8,0) - G(a_2,a_3,\ldots,a_7,0,0) =$$

$$= (2a_2+a_3+\cdots+a_8)^2 - (2a_2+a_3+\cdots+a_7)^2 - 4(a_2^2+a_2a_3+\cdots+a_7a_8)$$

$$+4(a_2^2+a_2a_3+\cdots+a_6a_7) = a_8(4a_2+2a_3+\cdots+2a_7+a_8) - 4a_7a_8$$

$$= a_8(4a_2+2a_3\cdots+2a_6-2a_7+a_8) \ge 0$$

and, similarly,

$$G(a_2,a_3,\ldots,a_7,0,0) - G(a_2,a_3,\ldots,a_6,0,0,0) = a_7(4a_2+2a_3+2a_4+2a_5-2a_6+a_7) \ge 0,$$

$$G(a_2,a_3,a_4,a_5,a_6,0,0,0) - G(a_2,a_3,a_4,a_5,0,0,0,0) = a_6(4a_2+2a_3+2a_4-2a_5+a_6) \ge 0,$$

$$G(a_2,a_3,a_4,a_5,0,0,0,0) - G(a_2,a_3,a_4,0,0,0,0,0) = a_5(4a_2+2a_3-2a_4+a_5) \ge 0,$$

$$G(a_2,a_3,a_4,0,0,0,0,0) - G(a_2,a_3,0,0,0,0,0,0) = a_4(4a_2-2a_3+a_4) \ge 0,$$

$$G(a_2,a_3,0,0,0,0,0,0) - G(a_2,0,0,0,0,0,0,0) = a_3^2 \ge 0.$$

The proof is completed. The equality occurs for $a_1 = a_2 = 1$ and $a_3 = \cdots = a_9 = 0$.

□

P 1.249. *Let n be a natural number, $n \ge 3$. Prove that there is a real number $q_n > 1$ such that*

$$\frac{a_1}{a_2+a_3}+\frac{a_2}{a_3+a_4}+\cdots+\frac{a_n}{a_1+a_2}\ge\frac{n}{2}$$

for any real numbers $a_1, a_2, \dots, a_n \in [1/q_n, q_n]$.

(Vasile Cîrtoaje, Crux Mathematicorum, 8, 2006)

Solution. Write the inequality as

$$\sum_{i=1}^{n}\frac{2q_n^2a_i-a_{i+1}-a_{i+2}}{a_{i+1}+a_{i+2}}\ge n(q_n^2-1),$$

where $a_{n+1}=a_1$ and $a_{n+2}=a_2$. Since

$$2q_n^2a_i-a_{i+1}-a_{i+2}=(q_n^2a_i-a_{i+1})+(q_n^2a_i-a_{i+2})\ge 0,$$

the Cauchy-Schwarz inequality may be applied to get

$$\sum_{i=1}^{n}(a_{i+1}+a_{i+2})(2q_n^2a_i-a_{i+1}-a_{i+2})\cdot\sum_{i=1}^{n}\frac{2q_n^2a_i-a_{i+1}-a_{i+2}}{a_{i+1}+a_{i+2}}\ge\left(\sum_{i=1}^{n}(2q_n^2a_i-a_{i+1}-a_{i+2})\right)^2.$$

Thus, to obtain the desired inequality, it suffices to prove that

$$\left(\sum_{i=1}^{n}(2q_n^2a_i-a_{i+1}-a_{i+2})\right)^2\ge n(q_n^2-1)\sum_{i=1}^{n}(a_{i+1}+a_{i+2})(2q_n^2a_i-a_{i+1}-a_{i+2}).$$

Since

$$\sum_{i=1}^{n}(2q_n^2a_i-a_{i+1}-a_{i+2})=2(q_n^2-1)\sum_{i=1}^{n}a_i$$

and

$$\sum_{i=1}^{n}(a_{i+1}+a_{i+2})(2q_n^2a_i-a_{i+1}-a_{i+2})=2q_n^2\sum_{i=1}^{n}a_i(a_{i+1}+a_{i+2})-\sum_{i=1}^{n}(a_i+a_{i+1})^2,$$

the inequality becomes

$$\frac{4}{n}(q_n^2-1)\left(\sum_{i=1}^{n}a_i\right)^2\ge 2q_n^2\sum_{i=1}^{n}a_i(a_{i+1}+a_{i+2})-\sum_{i=1}^{n}(a_i+a_{i+1})^2,$$

i.e.

$$\frac{4}{n}(q_n^2-1)\left(\sum_{i=1}^{n}a_i\right)^2\ge 2q_n^2\sum_{i=1}^{n}(a_i+a_{i+1})(a_{i+1}+a_{i+2})-(q_n^2+1)\sum_{i=1}^{n}(a_i+a_{i+1})^2.$$

Using the substitution $b_i = a_i + a_{i+1}$ for $i = 1, 2, \ldots, n$, the inequality reduces to

$$\frac{1}{n}(q_n^2 - 1)\left(\sum_{i=1}^n b_i\right)^2 \geq 2q_n^2 \sum_{i=1}^n b_i b_{i+1} - (q_n^2 + 1)\sum_{i=1}^n b_i^2.$$

Since

$$\left(\sum_{i=1}^n b_i\right)^2 = n\sum_{i=1}^n b_i^2 - \sum_{j<k}(b_j - b_k)^2,$$

the inequality is equivalent to

$$2q_n^2\left(\sum_{i=1}^n b_i^2 - \sum_{i=1}^n b_i b_{i+1}\right) \geq \frac{1}{n}(q_n^2 - 1)\sum_{j<k}(b_j - b_k)^2,$$

i.e.

$$n\sum_{i=1}^n (b_i - b_{i+1})^2 \geq \left(1 - \frac{1}{q_n^2}\right)\sum_{j<k}(b_j - b_k)^2. \tag{*}$$

But, for $j < k$, we have

$$\sum_{i=1}^n (b_i - b_{i+1})^2 \geq \sum_{i=j}^{k-1}(b_i - b_{i+1})^2 \geq \frac{1}{k-j}\left(\sum_{i=j}^{k-1}(b_i - b_{i+1})\right)^2 \geq \frac{1}{n-1}(b_j - b_k)^2.$$

Summing over j and k with $j < k$ yields

$$\frac{n(n-1)}{2}\sum_{i=1}^n (b_i - b_{i+1})^2 \geq \frac{1}{n-1}\sum_{j<k}(b_j - b_k)^2,$$

i.e.

$$n\sum_{i=1}^n (b_i - b_{i+1})^2 \geq \frac{2}{(n-1)^2}\sum_{j<k}(b_j - b_k)^2.$$

Comparing this inequality with (*), we see that (*) is true by choosing

$$1 - \frac{1}{q_n^2} = \frac{2}{(n-1)^2},$$

that is

$$q_n = \frac{1}{\sqrt{1 - 2/(n-1)^2}} = \frac{n-1}{\sqrt{n^2 - 2n - 1}}.$$

Since $q_n > \dfrac{1}{\sqrt{1 - 2/n^2}} = \dfrac{n}{\sqrt{n^2 - 2}}$, we can also choose

$$q_n = \frac{n}{\sqrt{n^2 - 2}}.$$

Open problem. *Does there exist a real constant $q > 1$ such that*

$$\frac{a_1}{a_2+a_3}+\frac{a_2}{a_3+a_4}+\frac{a_n}{a_1+a_2}\ge\frac{n}{2}$$

for any natural number $n \ge 3$ and for any real numbers $a_1, a_2, \dots, a_n \in [1/q, q]$?

□

P 1.250. *If a, b, c, d are positive real numbers and $0 \le x \le 1$, then*

$$\sum_{cyclic}\frac{a}{a+(3-x)b+xc}\ge 1.$$

(Vasile Cîrtoaje, Crux Mathematicorum, 2006,1)

Solution. Let $y = 3 - x$. Writing

$$\frac{a}{a+by+cx}+\frac{c}{c+dy+ax}=\frac{A}{A+B}$$

and

$$\frac{b}{b+cy+dx}+\frac{d}{d+ay+bx}=\frac{C}{C+D},$$

we need to show that

$$AC \ge BD,$$

where

$$A=(a^2+c^2)x+(ad+bc)y+2ac,$$
$$B=(ab+cd)xy-ac(1-x^2)+bdy^2,$$
$$C=(b^2+d^2)x+(ab+cd)y+2bd,$$
$$D=(ad+bc)xy-bd(1-x^2)+acy^2.$$

Using the substitution

$$p=ac,\quad q=bd,\quad r=ab+cd,\quad s=ad+bc,\quad u=a^2+c^2,\quad v=b^2+d^2,$$

we find

$$A=ux+sy+2p,\qquad B=rxy-p(1-x^2)+qy^2,$$
$$C=vx+ry+2q,\qquad D=sxy-q(1-x^2)+py^2,$$

and

$$AC=uvx^2+4pq+rsy^2+2(qu+pv)x+(ru+sv)xy+2(pr+qs)y,$$
$$BD=rsx^2y^2+pq(1-2x^2+x^4+y^4)-(ps+qr)x(1-x^2)y+(pr+qs)xy^3-(p^2+q^2)(1-x^2)y^2$$
$$=rsx^2y^2+pq(x^2+y^2-1)^2-(ps+qr)x(1-x^2)y+(pr+qs)xy^3-(p-q)^2(1-x^2)y^2.$$

Since $u \geq 2p$ and $v \geq 2q$, we have $qu + pv \geq 4pq$ and $ru + sv \geq 2(pr + qs)$, hence

$$AC \geq E,$$

where

$$E = uvx^2 + 4pq + rsy^2 + 8pqx + 2(pr + qs)(x + 1)y.$$

So,

$$AC - BD \geq E - BD = E_1 + E_2,$$

where

$$E_1 = uvx^2 + (p - q)^2(1 - x^2)y^2 + (pr + qs)(2 + 2x - xy^2)y$$
$$= uvx^2 + (p - q)^2(1 - x^2)y^2 + (pr + qs)(1 - x)(2 - 5x + x^2)y$$

and

$$E_2 = rs(1 - x^2)y^2 + (ps + qr)x(1 - x^2)y + pq[4 + 8x - (x^2 + y^2 - 1)^2].$$

Since $r \geq 2\sqrt{pq}$ and $s \geq 2\sqrt{pq}$, we have $rs \geq 4pq$ and

$$ps + qr \geq 2(p + q)\sqrt{pq} \geq 4pq,$$

hence

$$E_2 \geq 4pq(1 - x^2)y^2 + 4pqx(1 - x^2)y + pq[4 + 8x - (x^2 + y^2 - 1)^2] = pq(E_3 - 4x^2),$$

where

$$E_3 = 4(1 - x^2)(x + y)y + 4(x + 1)^2 - (x^2 + y^2 - 1)^2.$$

Since

$$4(1 - x^2)(x + y)y = 12(1 - x^2)y$$

and

$$4(x + 1)^2 - (x^2 + y^2 - 1)^2 = (2x + 3 - x^2 - y^2)(2x + 1 + x^2 + y^2)$$
$$= -4(1 - x)(3 - x)(x^2 - 2x + 5) = -4(1 - x)(x^2 - 2x + 5)y,$$

we have

$$E_3 = 4(1 - x)y[3(1 + x) - (x^2 - 2x + 5)] = -4(1 - x)(2 - 5x + x^2)y.$$

Thus,

$$AC - BD \geq E_1 + E_2 \geq E_1 + pq(E_3 - 4x^2) = F,$$

where

$$F = (uv - 4pq)x^2 + (p - q)^2(1 - x^2)y^2 + (pr + qs - 4pq)(1 - x)(2 - 5x + x^2)y.$$

It suffices to show that $F \geq 0$. Since $uv \geq 4pq$ and $pr + qs \geq 2(p + q)\sqrt{pq} \geq 4pq$, we have clearly $F \geq 0$ for $(1 - x)(2 - 5x + x^2) \geq 0$, that is for $x = 1$ and for $0 \leq x \leq (5 - \sqrt{17})/2 \approx 0.438$. Next, we claim that

$$uv + 2p^2 - 8pq + 2q^2 \geq 2(pr + qs - 4pq). \tag{1}$$

Indeed,

$$uv + 2p^2 - 8pq + 2q^2 - 2(pr + qs - 4pq) = (ab - p)^2 + (cd - p)^2 + (bc - q)^2 + (ad - q)^2 \ge 0.$$

We distinguish two cases, $y^2 - x^2y^2 - 2x^2 \ge 0$ and $2x^2 + x^2y^2 - y^2 \ge 0$, which are equivalent to $x \in [0, x_1]$ and $x \in [x_1, 1]$, respectively, where $x_1 \approx 0.837$ is the positive root of the equation

$$x^4 - 6x^3 + 10x^2 + 6x - 9 = 0.$$

Case 1: $y^2 - x^2y^2 - 2x^2 \ge 0$. Using (1), we have

$$F = (uv + 2p^2 - 8pq + 2q^2)x^2 + (pr + qs - 4pq)(1 - x)(2 - 5x + x^2)y + (p - q)^2(y^2 - x^2y^2 - 2x^2)$$

$$\ge (uv + 2p^2 - 8pq + 2q^2)x^2 + (pr + qs - 4pq)(1 - x)(2 - 5x + x^2)y$$

$$\ge 2(pr + qs - 4pq)x^2 + (pr + qs - 4pq)(1 - x)(2 - 5x + x^2)y$$

$$= (pr + qs - 4pq)\left[2x^2 + (1 - x)(2 - 5x + x^2)y\right].$$

Since $pr + qs - 4pq \ge 0$ and

$$2x^2 + (1 - x)(2 - 5x + x^2)y = 6 - 23x + 27x^2 - 9x^3 + x^4 = (1 - x)^4 + 5 - 19x + 21x^2 - 5x^3$$

$$= (1 - x)^4 + \frac{(3 - x)(45 - 156x + 137x^2) + 2x^3}{27} > 0,$$

we have $F \ge 0$.

Case 2: $2x^2 + x^2y^2 - y^2 \ge 0$. Using (1), we have

$$2F = (uv + 2p^2 - 8pq + 2q^2)(1 - x^2)y^2 + 2(pr + qs - 4pq)(1 - x)(2 - 5x + x^2)y + (uv - 4pq)(2x^2 + x^2y^2 - y^2)$$

$$\ge (uv + 2p^2 - 8pq + 2q^2)(1 - x^2)y^2 + 2(pr + qs - 4pq)(1 - x)(2 - 5x + x^2)y$$

$$\ge 2(pr + qs - 4pq)(1 - x^2)y^2 + 2(pr + qs - 4pq)(1 - x)(2 - 5x + x^2)y$$

$$= 2(pr + qs - 4pq)(1 - x)(y + xy + 2 - 5x + x^2)y$$

$$= 2(pr + qs - 4pq)(1 - x)(5 - 3x)y \ge 0.$$

The equality occurs for $a = b = c = d$.

□

P 1.251. *Prove that* 18 *is the largest positive value of the constant k such that*

$$\frac{1}{ab^2 + k} + \frac{1}{bc^2 + k} + \frac{1}{ca^2 + k} \ge \frac{3}{1 + k}$$

for all $a \ge b \ge c \ge 0$ such that $a + b + c = 3$.

(Vasile Cîrtoaje, Math. Reflections, 6, 2024)

Solution. Setting $a = b = \dfrac{3}{2}$ and $c = 0$, the inequality leads to $k \le 18$. We will further show that the inequality is true for $k = 18$. Let us denote $p = a + b + c$, $q = ab + bc + ca$, $r = abc$ and

$$A = a^2b + b^2c + c^2a, \qquad B = ab^2 + bc^2 + ca^2.$$

Since $p = 3$, we have $q \le p^2/3 = 3$ and $r \le p^3/27 = 1$. By expanding, the inequality can be restated as follows:

$$3k^2 \ge 3r^3 + (2k-1)rA + k(k-2)B,$$

$$6k^2 \ge 6r^3 + [(2k-1)r + k(k-2)](A+B) + [(2k-1)r - k(k-2)](A-B).$$

Since $(2k-1)r - k(k-2) \le 2k - 1 - k(k-2) = -k^2 + 4k - 1 < 0$ and $A - B = (a-b)(b-c)(a-c) \ge 0$, it suffices to show that

$$6k^2 \ge 6r^3 + [(2k-1)r + k(k-2)](A+B),$$

i.e.

$$6k^2 \ge 6r^3 + [(2k-1)r + k(k-2)](pq - 3r),$$

$$648 \ge 2r^3 + (35r + 288)(q - r).$$

Case 1: $0 \le q \le 9/4$. It suffices to show that

$$648 \ge 2r^3 + (35r + 288)\left(\frac{9}{4} - r\right),$$

which is equivalent to

$$r(837 + 140r - 8r^2) \ge 0.$$

Case 2: $9/4 \le q \le 3$. Let $z = q/3 \in [3/4, 1]$. For fixed z, we need to show that $648 \ge f(r)$, where $f(r) = 2r^3 + (35r + 288)(3z - r)$. By the fourth degree Schur's inequality, we have

$$r \le \frac{(p^2 - q)(4q - p^2)}{6p} = \frac{(3-z)(4z-3)}{2} := r_0.$$

Since $f'(r) = 6r^2 - 70r + 105z - 288 \le 6 - 70r + 105 - 288 < 0$, $f(r)$ is decreasing. So, we only need to show that $648 \ge f(r_0)$, i.e.

$$648 - 864z \ge 2r_0^3 - 35r_0^2 - (288 - 105z)r_0.$$

Since

$$648 - 864z = -216(4z - 3) = \frac{-432r_0}{3 - z},$$

we need to show that

$$\frac{-432}{3 - z} \ge 2r_0^2 - 35r_0 - (288 - 105z),$$

that is

$$\frac{-864}{3 - z} \ge 16z^4 - 120z^3 + 437z^2 - 585z - 180,$$

$$16z^5-168z^4+797z^3-1896z^2+1575z-324\ge 0,$$

$$(z-1)g(z)\ge 0,$$

where

$$g(z)=16z^4-152z^3+645z^2-1251z+324.$$

The inequality holds if $g(z)\le 0$. Indeed,

$$g(z)<645z^2-1251z+324=-645z(1-z)-151(4z-3)-2z-129<0.$$

For $k=18$, the equality occurs when $a=b=c=1$, and also when $a=b=\frac{3}{2}$ and $c=0$. □

P 1.252. *Let $a=b\ge c\ge d\ge 0$ such that $ab+bc+cd+da=4$. Prove that*

$$a^2+b^2+c^2+d^2+28\ge 8(a+b+c+d).$$

(Vasile Cîrtoaje, Math. Reflections, 5, 2024)

Solution. We need to show that

$$2b^2+c^2+d^2-8(2b+c+d)+28\ge 0$$

for

$$(b+c)(b+d)=4,\qquad b\ge c\ge d\ge 0.$$

Denote

$$x=\frac{2b+c}{3},\qquad b\ge x\ge c\ge d\ge 0,\quad x\ge 1.$$

Since $2b^2+c^2\ge 3x^2$, it suffices to prove that

$$3x^2+d^2-8(3x+d)+28\ge 0,$$

i.e.

$$3x^2-24x+12+(4-d)^2\ge 0.$$

From

$$16=(2b+2c)2b+2d)=(3x+c)(3x+2d-c)=(3x+d+c-d)(3x+d+d-c)$$

$$=(3x+d)^2-(c-d)^2\ge(3x+d)^2-(x-d)^2=8(x^2+dx),$$

we get

$$d\le\frac{2-x^2}{x},\qquad 4-d\ge 4-\frac{2-x^2}{x}=\frac{x^2+4x-2}{x},$$

therefore

$$3x^2-24x+12+(4-d)^2 \geq 3x^2-24x+12+\frac{(x^2+4x-2)^2}{x^2}=\frac{4(x-1)^4}{x^2} \geq 0.$$

Thus, the proof is completed. The equality occurs for $a=b=c=d=1$.

Remark 1. Note that 8 is the largest positive value of k such that

$$a^2+b^2+c^2+d^2-4 \geq k(a+b+c+d-4)$$

whenever $a=b \geq c \geq d \geq 0$ satisfying $ab+bc+cd+da=4$. To prove this assert, we assume $a=b=c$. The equality constraint becomes $c^2+cd=2$ where $c \in [1,\sqrt{2}]$, while the inequality becomes as follows:

$$3c^2+d^2-4 \geq k(3c+d-4), \quad 3c^2+\frac{(2-c^2)^2}{c^2}-4 \geq k\left(3c+\frac{2-c^2}{c}-4\right), \quad \frac{4(c^2-1)^2}{c^2} \geq \frac{2k(c-1)^2}{c}.$$

It is true for all $c \in (1,\sqrt{2}]$ if and only if $\frac{2(c+1)^2}{c} \geq k$. Setting $c \to 1$, we get the necessary condition $k \leq 8$.

Remark 2. Since

$$\frac{a+b+c+d}{4}+\frac{4}{a+b+c+d} \geq 2,$$

the following inequality follows from P 1.252:

- *If a,b,c,d are nonnegative real numbers such that*

$$ab+bc+cd+da=4, \qquad a=b \geq c \geq d,$$

then

$$a^2+b^2+c^2+d^2+\frac{128}{a+b+c+d} \geq 36.$$

□

P 1.253. *If x_1,x_2,x_3,x_4,x_5 are positive real numbers such that*

$$x_1x_2+x_2x_3+x_3x_4+x_4x_5+x_5x_1=5,$$

then

$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}+\frac{1}{x_5}+\frac{25}{x_1+x_2+x_3+x_4+x_5} \geq 10.$$

(Vasile Cîrtoaje, Recreatii Matematice, 1, 2025)

Solution. By Lemma from P 1.217, it suffices to show that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\frac{25}{a+b+c+d+e}-10 \ge 0$$

for

$$ae+ad+be+bc+cd=5, \qquad a \ge b \ge c \ge d \ge e > 0.$$

Denote

$$x=\frac{a+b}{2}, \qquad y=\frac{d+e}{2}, \qquad a \ge x \ge b \ge c \ge d \ge y \ge e.$$

Replacing a and e with $2x-b$ and $2y-d$, respectively, we have

$$5=a(d+e)+be+bc+cd=2(2x-b)y+b(2y-d)+bc+cd=4xy+bc-(b-c)d.$$

From this, we get

$$5 \ge 4xy+bc-(b-c)c=4xy+c^2,$$

hence

$$4xy \le 5-c^2, \qquad c<\sqrt{5},$$

and

$$5=4xy+bc-(b-c)d \le 4xy+bc-(b-c)y=4xy+b(c-y)+cy$$

$$\le 4xy+x(c-y)+cy=3xy+c(x+y) \le \frac{3}{4}(5-c^2)+c(x+y),$$

hence

$$4c(x+y) \ge 3c^2+5.$$

By the AM-HM inequality, we have

$$\frac{1}{a}+\frac{1}{b} \ge \frac{4}{a+b}=\frac{2}{x}, \qquad \frac{1}{d}+\frac{1}{e} \ge \frac{2}{y}.$$

Thus, it suffices to show that

$$\frac{2}{x}+\frac{2}{y}+\frac{1}{c}+\frac{25}{2x+2y+c} \ge 10$$

for $x \ge c \ge y > 0$ such that $4xy \le 5-c^2$ and $4c(x+y) \ge 3c^2+5$. Denoting $S=\dfrac{x+y}{2}$, we need to show that

$$\frac{4S}{xy}+\frac{1}{c}+\frac{25}{4S+c} \ge 10$$

for

$$xy \le \frac{5-c^2}{4}, \qquad 8cS \ge 3c^2+5.$$

The inequality is true if

$$\frac{8S}{5-c^2}+\frac{1}{c}+\frac{25}{4S+c} \ge 10,$$

which is equivalent to

$$32cS^2 + 2(10c^3 + 3c^2 - 50c + 5)S + c(5c^3 - 13c^2 - 25c + 65) \geq 0,$$

that is

$$(32cS + A)^2 + 25B \geq 0,$$

where

$$A = 10c^3 + 3c^2 - 50c + 5,$$

$$B = -4c^6 + 4c^5 + 23c^4 - 24c^3 - 18c^2 + 20c - 1 = (c-1)^2(c+1)(-4c^3 + 19c - 1).$$

Case 1: $c \in \left[\frac{1}{15}, 1\right]$. Since

$$-4c^3 + 19c - 1 \geq -4c + 19c - 1 = 15c - 1 \geq 0,$$

we have $B \geq 0$, therefore $(32cS + A)^2 + 25B \geq 0$.

Case 2: $c \in \left(0, \frac{1}{15}\right] \cup [1, \sqrt{5})$. Since

$$32cS + A \geq 4(3c^2 + 5) + A = 5(2c^3 + 3c^2 - 10c + 5) = 5(c-1)(2c^2 + 5c - 5) \geq 0,$$

we have

$$(32cS + A)^2 + 25B \geq 25(c-1)^2(2c^2 + 5c - 5)^2 + 25(c-1)^2(c+1)(-4c^3 + 19c - 1)$$

$$= 200(c-1)^2(2c^3 + 3c^2 - 4c + 3) \geq 200(c-1)^2(2c^2 - 4c + 2) = 400(c-1)^4 \geq 0.$$

The proof is completed. The equality occurs for $x_1 = x_2 = x_3 = x_4 = x_5 = 1$.

□

P 1.254. *Prove that* $\frac{7}{6}$ *is the least positive value of the power exponent* k *such that*

$$x_1^k + x_2^k + x_3^k + x_4^k + x_5^k \geq 5$$

for any nonnegative real numbers x_i *with at most one* $x_i < 1$ *and* $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 = 5$.

(Vasile Cîrtoaje, Arhimede Math. J., No. 1, 2024)

Solution. Assuming $x_1 = x_2 := x$, $x_3 = x_5 = 1$ and $x_4 = \dfrac{5-2x-x^2}{2}$, the constraints are satisfied for $x \in [1, \sqrt{6}-1]$, while the inequality becomes $f(x) \ge 0$, where $f(x) = 2x^k + \left(\dfrac{5-2x-x^2}{2}\right)^k - 3$. From

$$\frac{1}{k}f'(x) = 2x^{k-1} - (x+1)\left(\frac{5-2x-x^2}{2}\right)^{k-1},$$

$$\frac{1}{k}f''(x) = 2(k-1)x^{k-2} - \left(\frac{5-2x-x^2}{2}\right)^{k-1} + (k-1)(x+1)^2\left(\frac{5-2x-x^2}{2}\right)^{k-2},$$

we find $f(1) = f'(1) = 0$ and $f''(1) = k(6k-7)$. From the necessary condition $f''(1) \ge 0$, we get $k \ge 7/6$. To show that $7/6$ is the least positive value of k, we need to prove the required inequality for $k = 7/6$. By Lemma below, it suffices to show that $E(a,b,c,d,e) \ge 0$ for $a \ge b \ge c \ge d \ge 1 \ge e \ge 0$ such that $ab + ac + bd + ce + de = 5$, where

$$E(a,b,c,d,e) = a^k + b^k + c^k + d^k + e^k - 5.$$

For fixed b, c and e, we may assume that a and E are functions of d. By differentiating the equality constraint, we get

$$(b+c)a' + b + e = 0, \qquad a' = \frac{-(b+e)}{b+c} \ge \frac{-(b+e)}{b+d} = \frac{d-e}{b+d} - 1 \ge \frac{d-e}{a+d} - 1 = \frac{-(a+e)}{a+d}.$$

Denoting $E(a,b,c,d,e)$ by $f(d)$, we have

$$\frac{6f'(d)}{7} = d^{1/6} + a^{1/6}a' \ge d^{1/6} - \frac{a^{1/6}(a+e)}{a+d}.$$

We claim that $f'(d) \ge 0$. To prove this, it suffices to show that $\dfrac{a+d}{a+e} \ge \left(\dfrac{a}{d}\right)^{1/6}$. By Bernoulli's inequality,

$$\left(\frac{a}{d}\right)^{1/6} = \left(1 + \frac{a-d}{d}\right)^{1/6} \le 1 + \frac{a-d}{6d} = \frac{a+5d}{6d}.$$

So, it is enough to show that $\dfrac{a+d}{a+e} \ge \dfrac{a+5d}{6d}$. From $5 = ab + ac + bd + ce + de \ge ad + ad + d^2 + de + de$, we get $e \le \dfrac{5-2ad-d^2}{2d}$ and $a + e \le \dfrac{5-d^2}{2d}$, therefore

$$\frac{a+d}{a+e} - \frac{a+5d}{6d} \ge \frac{2d(a+d)}{5-d^2} - \frac{a+5d}{6d} = \frac{a(13d^2-5) + d(17d^2-25)}{6d(5-d^2)}$$

$$\ge \frac{d(13d^2-5) + d(17d^2-25)}{6d(5-d^2)} = \frac{5(d^2-1)}{5-d^2} \ge 0.$$

Since $f'(d) \geq 0$, $f(d)$ is increasing and has the minimum value when d is minimum, hence when $d = 1$. So, we need to show that

$$a^{7/6} + b^{7/6} + c^{7/6} + e^{7/6} \geq 4$$

for $a \geq b \geq c \geq 1 \geq e \geq 0$ such that $ab + ac + b + ce + e = 5$.

For fixed a and e, we may assume that b is a decreasing function of c. By differentiating the equality constraint, we get $(a+1)b' + a + e = 0$. Denoting the left side of the desired inequality by $g(c)$, we have

$$\frac{6g'(c)}{7} = c^{1/6} + b^{1/6}b' = c^{1/6} - \frac{b^{1/6}(a+e)}{a+1} \geq 1 - \frac{a^{1/6}(a+e)}{a+1}.$$

We claim that $g'(d) \geq 0$. To prove this, it suffices to show that $\dfrac{a+1}{a+e} \geq a^{1/6}$. By Bernoulli's inequality,

$$a^{1/6} = [1 + (a-1)]^{1/6} \leq 1 + \frac{a-1}{6} = \frac{a+5}{6}.$$

So, it suffices to show that $\dfrac{a+1}{a+e} \geq \dfrac{a+5}{6}$. From $5 = ab + ac + b + ce + e \geq a + a + 1 + e + e$, we get $a + e \leq 2$, therefore

$$\frac{a+1}{a+e} - \frac{a+5}{6} \geq \frac{a+1}{2} - \frac{a+5}{6} = \frac{a-1}{3} \geq 0.$$

Since $g'(c) \geq 0$, $g(c)$ is increasing and has the minimum value when c is minimum (b is maximum), that is when $c = 1$ or $b = a$. Consider now these cases.

Case 1: $c = 1$. We need to show that $a^{7/6} + b^{7/6} + e^{7/6} \geq 3$ for $a \geq b \geq 1 \geq e \geq 0$ such that $ab + a + b + 2e = 5$. Let $x = \dfrac{a+b}{2} \geq 1$. Since, by Jensen's inequality and Bernoulli's inequality,

$$a^{7/6} + b^{7/6} \geq 2x^{7/6} = 2[1 + (x-1)]^{7/6} \geq 2\left[1 + \frac{7(x-1)}{6}\right] = \frac{7x-1}{3},$$

we have

$$a^{7/6} + b^{7/6} + e^{7/6} - 3 \geq \frac{7x-1}{3} - 3 = \frac{7x-10}{3} \geq 0$$

for $x \geq 10/7$. For $x \in [1, 10/7]$, since $e = \dfrac{5 - 2x - ab}{2} \geq \dfrac{5 - 2x - x^2}{2} > 0$, we have

$$a^{7/6} + b^{7/6} + e^{7/6} - 3 \geq 2x^{7/6} + \left(\frac{5 - 2x - x^2}{2}\right)^{7/6} - 3 := G(x).$$

If $G'(x) \geq 0$, then $G(x)$ is increasing, therefore $G(x) \geq G(1) = 0$. Since

$$G'(x) = \frac{7}{3}x^{1/6} - \frac{7}{6}(x+1)\left(\frac{5 - 2x - x^2}{2}\right)^{1/6} = \frac{7}{6}x^{1/6}(x+1)\left[\frac{2}{x+1} - \left(\frac{5 - 2x - x^2}{2x}\right)^{1/6}\right],$$

we need to show that $H(x) \ge 0$, where $H(x) = \left(\frac{2}{x+1}\right)^6 - \frac{5-2x-x^2}{2x}$. Indeed, since $\left(\frac{2}{x+1}\right)^6 \ge 2\left(\frac{2}{x+1}\right)^3 - 1$, we have

$$H(x) \ge \frac{16}{(x+1)^3} - 1 - \frac{5-2x-x^2}{2x} = \frac{16}{(x+1)^3} - \frac{5-x^2}{2x}$$

$$= \frac{x^5+3x^4-2x^3-14x^2+17x-5}{2x(x+1)^3} = \frac{(x-1)^2(x^3+5x^2+7x-5)}{2x(x+1)^3} \ge 0.$$

Case 2: $b = a$. We need to show that $2a^{7/6} + c^{7/6} + e^{7/6} \ge 4$ for $a \ge c \ge 1 \ge e \ge 0$ such that $a^2 + ac + a + ce + e = 5$. For fixed e, we may assume that a is a function of c. By differentiating the equality constraint, we get

$$(2a+c+1)a' + a + e = 0, \qquad a' = \frac{-(a+e)}{2a+c+1} \ge \frac{-(a+e)}{2(a+1)}.$$

Denoting the left side of the desired inequality by $h(c)$, we have

$$\frac{6h'(c)}{7} = c^{1/6} + 2a^{1/6}a' = c^{1/6} - \frac{a^{1/6}(a+e)}{a+1} \ge 1 - \frac{a^{1/6}(a+e)}{a+1} \ge 0.$$

The last inequality was proved before. Since $h'(c) \ge 0$, $h(c)$ is increasing and has the minimum value when c is minimum, hence when $c = 1$. So, we need to show that $2a^{7/6}+e^{7/6} \ge 3$ for $a \ge 1 \ge e \ge 0$ such that $a^2 + 2a + 2e = 5$. If $a \ge 10/7$, then

$$2a^{7/6} + e^{7/6} - 3 \ge 2a^{7/6} - 3 > 0,$$

and if $a \in [1, 10/7]$, then

$$2a^{7/6} + e^{7/6} - 3 = 2a^{7/6} + \left(\frac{5-2a-a^2}{2}\right)^{7/6} - 3 \ge 0.$$

The latter inequality was proved at Case 1.

The proof is completed. The equality occurs for $x_1 = x_2 = x_3 = x_4 = x_5 = 1$.

Lemma. *Let x_1, x_2, x_3, x_4, x_5 be nonnegative real numbers such that at most one of them is less than 1 and $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 = 5$, and let $E(x_1, x_2, x_3, x_4, x_5)$ be a symmetric and increasing function with respect to each variable. If $E(a, b, c, d, e) \ge 0$ for any $a \ge b \ge c \ge d \ge 1 \ge e \ge 0$ such that $ab+ac+bd+ce+de = 5$, then $E(x_1, x_2, x_3, x_4, x_5) \ge 0$.*

Proof. Let $T = (T_1, T_2, T_3, T_4, T_5)$ and $t = (t_1, t_2, t_3, t_4, t_5)$ be two decreasing sequences of positive real numbers. By Karamata majorization inequality applied to the convex function $f(x) = e^x$, if $T_1 \cdots T_j \ge t_1 \cdots t_j$ for $j = 1, 2, 3, 4, 5$, then

$$T_1 + T_2 + T_3 + T_4 + T_5 \ge t_1 + t_2 + t_3 + t_4 + t_5.$$

If (a,b,c,d,e) is a permutation of (x_1,x_2,x_3,x_4,x_5) such that $a \ge b \ge c \ge d \ge 1 \ge e \ge 0$, then

$$E(a,b,c,d,e) = E(x_1,x_2,x_3,x_4,x_5).$$

Let $T = (ab, ac, bd, ce, de)$ be a decreasing sequence, and t a decreasing permutation of the sequence $(x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1)$. Since $T_1 \cdots T_j \ge t_1 \cdots t_j$ for $j = 1,2,3,4,5$, by Karamata's inequality we have

$$ab + ac + bd + ce + de \ge x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 = 5.$$

In the case $ab + ac + bd + ce + de > 5$, by decreasing the numbers a,b,c,d,e to have $ab+ac+bd+ce+de = 5$ and to keep the constraint $a \ge b \ge c \ge d \ge 1 \ge e \ge 0$, the function $E(a,b,c,d,e)$ decreases, therefore

$$E(a,b,c,d,e) \le E(x_1,x_2,x_3,x_4,x_5).$$

On the other hand, by hypothesis, $E(a,b,c,d,e) \ge 0$. So, we have

$$E(x_1,x_2,x_3,x_4,x_5) \ge E(a,b,c,d,e) \ge 0.$$

□

P 1.255. *Let a,b,c,d be nonnegative real numbers such that at most one of them is larger than 1 and $ab + bc + cd + da \le 4$. Prove that*

$$a^2 + b^2 + c^2 + d^2 + 16 \ge 5(a + b + c + d).$$

(Vasile Cîrtoaje, 2024)

Solution. Without loss of generality, assume that $a \ge 1$ and $b,c,d \le 1$. Since $(a+c)(b+d) \le 4$, let us denote

$$x = \frac{a+c}{2}, \qquad y = \frac{b+d}{2}.$$

We have

$$xy \le 1, \qquad y \le 1.$$

Consider next two cases: $x \le 1$ and $x \ge 1$.

Case 1: $x \le 1$. Let $S = \dfrac{a+b+c+d}{4}$. We have

$$S = \frac{x+y}{2} \le 1$$

and

$$a^2 + b^2 + c^2 + d^2 \ge \frac{(a+b+c+d)^2}{4} = 4S^2,$$

therefore

$$a^2+b^2+c^2+d^2+16-5(a+b+c+d) \ge 4S^2+16-20S = 4(1-S)(4-S) \ge 0.$$

Case 2: $x \ge 1$. From $(a-1)(c-1) \le 0$, we get $ac \le 2x-1$. In addition, $bd \le y^2$. So, we have

$$a^2+b^2+c^2+d^2+16-5(a+b+c+d) = 4x^2-2ac+4y^2-2bd+16-10(x+y)$$

$$\ge 4x^2-2(2x-1)+4y^2-2y^2+16-10(x+y) = 4x^2-14x+\frac{11}{2}+2\left(\frac{5}{2}-y\right)^2$$

$$\ge 4x^2-14x+\frac{11}{2}+2\left(\frac{5}{2}-\frac{1}{x}\right)^2 = \frac{2(2x^4-7x^3+9x^2-5x+1)}{x^2} = \frac{2(x-1)^3(2x-1)}{x^2} \ge 0.$$

The proof is completed. The equality occurs for $a=b=c=d=1$.

□

P 1.256. *Prove that* $[-32,17]$ *is the range of values of the real constant* k *such that*

$$(a+b+c+d)^4+4k(a+b+c+d) \ge (16+k)(a+b)^2(c+d)^2$$

for all nonnegative real numbers a,b,c,d *with* $a \ge b \ge c \ge d$ *and* $abc+bcd+cda+dab=4.$

(Leonard Giugiuc and Vasile Cîrtoaje, Recreatii Matematice, 2, 2024)

Solution. Write the inequality in the homogeneous form

$$(a+b+c+d)^4+k(a+b+c+d)(abc+bcd+cda+dab) \ge (16+k)(a+b)^2(c+d)^2.$$

For $a=b=c=1$ and $d=0$, the inequality becomes $k \le 17$, and for $a=b:=x \ge 1$ and $c=d=1$, the inequality becomes

$$(x-1)^2[4(x^2+6x+1)+kx] \ge 0.$$

It is true for $x \ge 1$ if and only if $4(x^2+6x+1)+kx \ge 0$ for all $x>1$. From

$$\lim_{x\to 1}\,[4(x^2+6x+1)+kx] \ge 0,$$

we get the necessary condition $k \ge -32$. To finish the proof, we need to prove the inequality for $k \in [-32,17]$. For fixed a,b,c,d, the inequality has the form $f(k) \ge 0$. Since $f(k)$ is a linear function, it has the minimum value when $k=-32$ or $k=17$. Thus, it suffices to consider these two cases.

Case 1: $k=-32$. We need to show that

$$(a+b+c+d)^4+16(a+b)^2(c+d)^2 \ge 32(a+b+c+d)(abc+bcd+cda+dab).$$

Let $S = \dfrac{a+b}{2}$ and $s = \dfrac{c+d}{2}$. Since

$$abc + bcd + cda + dab = ab(c+d) + cd(a+b) \le 2S^2 s + 2s^2 S = 2Ss(S+s),$$

it suffices to show that

$$(S+s)^4 + 16S^2s^2 \ge 8Ss(S+s)^2,$$

which is equivalent to

$$(S-s)^4 \ge 0.$$

Case 2: $k = 17$. We need to show that

$$(a+b+c+d)^4 + 17(a+b+c+d)(abc+bcd+cda+dab) \ge 33(a+b)^2(c+d)^2.$$

Let $s = c+d$ and $x = cd$. For fixed a, b and s, we may write the inequality as $f(x) \ge 0$, where

$$f(x) = (a+b+s)^4 + 17(a+b+s)[abs + (a+b)x] - 33(a+b)^2 s^2.$$

Since $f(x)$ is increasing, it has the minimum value when $x = 0$, hence when $d = 0$. So, it suffices to prove the inequality for $d = 0$, that is

$$(a+b+c)^4 + 17(a+b+c)abc \ge 33(a+b)^2c^2.$$

Since

$$3(a+b)c \le 2(ab+bc+ca),$$

it suffices to show that

$$3p^4 + 51pr \ge 44q^2,$$

where $p = a+b+c$, $q = ab+bc+ca$, $r = abc$. If $p^2 \ge 4q$, then

$$3p^4 + 51pr - 44q^2 \ge 48q^2 + 51pr - 44q^2 > 0.$$

Consider now the case $3q \le p^2 < 4q$. By Schur's inequality, we have $p^3 + 9r \ge 4pq$. Thus,

$$3(3p^4 + 51pr - 44q^2) \ge 9p^4 + 17p(4pq - p^3) - 132q^2 = 4(-2p^4 + 17p^2q - 33q^2)$$

$$= 4(p^2 - 3q)(11q - 2p^2) \ge 4(p^2 - 3q)(8q - 2p^2) \ge 0.$$

The proof is completed. For $k \in [-32, 17]$, the equality occurs when $a = b = c = d = 1$. Moreover, for $k = 17$, the equality also occurs when $a = b = c = \sqrt[3]{4}$ and $d = 0$.

□

Chapter 2

Noncyclic Inequalities

2.1 Applications

2.1. If a, b are positive real numbers, then

$$\frac{1}{4a^2+b^2}+\frac{3}{b^2+4ab}\ge\frac{16}{5(a+b)^2}.$$

2.2. If a, b are positive real numbers, then

$$3a\sqrt{3a}+3b\sqrt{6a+3b}\ge 5(a+b)\sqrt{a+b}.$$

2.3. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$(ab+c)(ac+b)\le 4.$$

2.4. If a, b, c are nonnegative real numbers, then

$$a^3+b^3+c^3-3abc\ge\frac{1}{4}(b+c-2a)^3.$$

2.5. If a, b, c are nonnegative real numbers such that

$$c=\min\{a,b,c\},\qquad a^2+b^2+c^2=3,$$

then

(a) $$5b+2c\le 9;$$

(b) $$5(b+c)\le 9+3a.$$

2.6. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{(a+b)^2}+\frac{1}{(a+c)^2}+\frac{16}{(b+c)^2} \ge \frac{6}{ab+bc+ca}.$$

2.7. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{(a+b)^2}+\frac{1}{(a+c)^2}+\frac{2}{(b+c)^2} \ge \frac{5}{2(ab+bc+ca)}.$$

2.8. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{(a+b)^2}+\frac{1}{(a+c)^2}+\frac{25}{(b+c)^2} \ge \frac{8}{ab+bc+ca}.$$

2.9. If a, b, c are positive real numbers, then

$$(a+b)^3(a+c)^3 \ge 4a^2bc(2a+b+c)^2.$$

2.10. If a, b, c are positive real numbers such that $abc = 1$, then

(a) $$\frac{a}{b}+\frac{b}{c}+\frac{1}{a} \ge a+b+1;$$

(b) $$\frac{a}{b}+\frac{b}{c}+\frac{1}{a} \ge \sqrt{3(a^2+b^2+1)}.$$

2.11. If a, b, c are positive real numbers such that $abc \ge 1$, then

$$a^{\frac{a}{b}}b^{\frac{b}{c}}c^c \ge 1.$$

2.12. If a, b, c are positive real numbers such that $ab+bc+ca = 3$, then

$$ab^2c^3 < 4.$$

2.13. If a, b, c are positive real numbers such that $ab+bc+ca = \dfrac{5}{3}$, then

$$ab^2c^2 \le \frac{1}{3}.$$

2.14. Let a, b, c be positive real numbers such that

$$a \le b \le c, \qquad ab + bc + ca = 3.$$

Prove that

(a)
$$ab^2c \le \frac{9}{8};$$

(b)
$$ab^4c \le 2;$$

(c)
$$a^2b^3c \le 2.$$

2.15. Let a, b, c be positive real numbers such that

$$a \le b \le c, \qquad a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Prove that

$$b \ge \frac{1}{a + c - 1}.$$

2.16. Let a, b, c be positive real numbers such that

$$a \le b \le c, \qquad a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Prove that

$$ab^2c^3 \ge 1.$$

2.17. Let a, b, c be positive real numbers such that

$$a \le b \le c, \qquad a + b + c = abc + 2.$$

Prove that

$$(1 - b)(1 - ab^3c) \ge 0.$$

2.18. Let a, b, c be real numbers, no two of which are zero. Prove that

(a)
$$\frac{(a-b)^2}{a^2+b^2} + \frac{(a-c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{2(b^2+c^2)};$$

(b)
$$\frac{(a+b)^2}{a^2+b^2} + \frac{(a+c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{2(b^2+c^2)}.$$

2.19. Let a, b, c be real numbers, no two of which are zero. If $bc \geq 0$, then

(a) $$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2} \geq \frac{(b-c)^2}{(b+c)^2};$$

(b) $$\frac{(a+b)^2}{a^2+b^2}+\frac{(a+c)^2}{a^2+c^2} \geq \frac{(b-c)^2}{(b+c)^2}.$$

2.20. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{|a-b|^3}{a^3+b^3}+\frac{|a-c|^3}{a^3+c^3} \geq \frac{|b-c|^3}{(b+c)^3}.$$

2.21. Let a, b, c be positive real numbers, $b \neq c$. Prove that

$$\frac{ab}{(a+b)^2}+\frac{ac}{(a+c)^2} \leq \frac{(b+c)^2}{4(b-c)^2}.$$

2.22. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{3bc+a^2}{b^2+c^2} \geq \frac{3ab-c^2}{a^2+b^2}+\frac{3ac-b^2}{a^2+c^2}.$$

2.23. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$ab^2+bc^2+2ca^2 \leq 8.$$

2.24. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$ab^2+bc^2+\frac{3}{2}abc \leq 4.$$

2.25. Let a, b, c be nonnegative real numbers such that $a+b+c=5$. Prove that

$$ab^2+bc^2+2abc \leq 20.$$

2.26. If a, b, c are nonnegative real numbers, then

$$a^3+b^3+c^3-a^2b-b^2c-c^2a \geq \frac{8}{9}(a-b)(b-c)^2.$$

2.27. If a, b, c are positive real numbers, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3+\frac{(a-c)^2}{ab+bc+ca}.$$

2.28. If a, b, c are positive real numbers, then

(a)
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3+\frac{4(a-c)^2}{(a+b+c)^2};$$

(b)
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3+\frac{5(a-c)^2}{(a+b+c)^2}.$$

2.29. If $a \geq b \geq c > 0$, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3+\frac{3(b-c)^2}{ab+bc+ca}.$$

2.30. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

(a) if $a \geq b \geq 1 \geq c$, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3+\frac{2(a-b)^2}{ab};$$

(b) if $a \geq 1 \geq b \geq c$, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3+\frac{2(b-c)^2}{bc}.$$

2.31. Let a, b, c be positive real numbers such that

$$a \geq 1 \geq b \geq c, \qquad abc = 1.$$

prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3+\frac{9(b-c)^2}{ab+bc+ca}.$$

2.32. Let a, b, c be positive real numbers such that

$$a \geq 1 \geq b \geq c, \qquad a+b+c = 3.$$

prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3+\frac{4(b-c)^2}{b^2+c^2}.$$

2.33. Let a,b,c be positive real numbers such that

$$a \ge b \ge 1 \ge c, \qquad a+b+c=3.$$

Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3+\frac{3(a-b)^2}{ab}.$$

2.34. If a,b,c are positive real numbers, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3+\frac{2(a-c)^2}{(a+c)^2}.$$

2.35. If a,b,c are positive real numbers, then

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \ge a+b+c+\frac{4(a-c)^2}{a+b+c}.$$

2.36. If $a \ge b \ge c > 0$, then

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \ge a+b+c+\frac{6(b-c)^2}{a+b+c}.$$

2.37. If $a \ge b \ge c > 0$, then

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} > 5(a-b).$$

2.38. Let a,b,c be positive real numbers such that

$$a \ge b \ge 1 \ge c, \qquad a+b+c=3.$$

Prove that

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \ge 3+\frac{11(a-c)^2}{4(a+c)}.$$

2.39. If a,b,c are positive real numbers, then

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \ge \frac{3}{2}+\frac{27(b-c)^2}{16(a+b+c)^2}.$$

2.40. Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2} + \frac{9(b-c)^2}{4(a+b+c)^2}.$$

2.41. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2} + \frac{(b-c)^2}{2(b+c)^2}.$$

2.42. Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2} + \frac{(b-c)^2}{4bc}.$$

2.43. Let a, b, c be positive real numbers such that

$$a \le 1 \le b \le c, \qquad a + b + c = 3,$$

then

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2} + \frac{3(b-c)^2}{4bc}.$$

2.44. Let a, b, c be nonnegative real numbers such that

$$a \ge 1 \ge b \ge c, \qquad a + b + c = 3,$$

then

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2} + \frac{(b-c)^2}{(b+c)^2}.$$

2.45. Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$. Prove that

(a) $$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{2(b-c)^2}{3(b^2+c^2)} \le 1;$$

(b) $$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{(b-c)^2}{b^2+bc+c^2} \le 1;$$

(c) $$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{(a-b)^2}{2(a^2+b^2)} \le 1.$$

2.46. Let a, b, c be positive real numbers such that

$$a \le 1 \le b \le c, \qquad a+b+c=3,$$

then

$$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(b-c)^2}{bc} \le 1.$$

2.47. Let a, b, c be nonnegative real numbers such that $a = \max\{a, b, c\}$ and $b+c > 0$. Prove that

(a) $$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(b-c)^2}{2(ab+bc+ca)} \le 1;$$

(b) $$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{2(b-c)^2}{(a+b+c)^2} \le 1.$$

2.48. Let a, b, c be positive real numbers. Prove that

(a) if $a \ge b \ge c$, then

$$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(a-c)^2}{a^2-ac+c^2} \ge 1;$$

(b) if $a \ge 1 \ge b \ge c$ and $abc = 1$, then

$$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(b-c)^2}{b^2-bc+c^2} \le 1.$$

2.49. Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$. Prove that

(a) $$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1+\frac{4(b-c)^2}{3(b+c)^2};$$

(b) $$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1+\frac{(a-b)^2}{(a+b)^2}.$$

2.50. If a, b, c are positive real numbers, then

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1+\frac{9(a-c)^2}{4(a+b+c)^2}.$$

2.51. Let a, b, c be nonnegative real numbers, no two of which are zero. If $a = \min\{a, b, c\}$, then

$$\frac{1}{\sqrt{a^2 - ab + b^2}} + \frac{1}{\sqrt{b^2 - bc + c^2}} + \frac{1}{\sqrt{c^2 - ca + a^2}} \geq \frac{6}{b + c}.$$

2.52. If $a \geq 1 \geq b \geq c \geq 0$ such that

$$ab + bc + ca = abc + 2,$$

then

$$ac \leq 4 - 2\sqrt{2}.$$

2.53. If a, b, c are nonnegative real numbers such that

$$ab + bc + ca = 3, \qquad a \leq 1 \leq b \leq c,$$

then

(a) $$a + b + c \leq 4;$$

(b) $$2a + b + c \leq 4.$$

2.54. Let a, b, c be nonnegative real numbers such that $a \leq b \leq c$. Prove that

(a) if $a + b + c = 3$, then

$$a^4(b^4 + c^4) \leq 2;$$

(b) if $a + b + c = 2$, then

$$c^4(a^4 + b^4) \leq 1.$$

2.55. Let a, b, c be nonnegative real numbers such that

$$a \leq b \leq c, \qquad a + b + c = 3.$$

Find the greatest real number k such that

$$\sqrt{(56b^2 + 25)(56c^2 + 25)} + k(b - c)^2 \leq 14(b + c)^2 + 25.$$

2.56. If $a \geq b \geq c > 0$ such that $abc = 1$, then

$$3(a + b + c) \leq 8 + \frac{a}{c}.$$

2.57. If $a \ge b \ge c > 0$, then

$$(a+b-c)(a^2b-b^2c+c^2a) \ge (ab-bc+ca)^2.$$

2.58. If $a \ge b \ge c > 0$, then

$$\frac{ab+bc}{a^2+b^2+c^2} \le \frac{1+\sqrt{3}}{4}.$$

2.59. If $a \ge b \ge c \ge d > 0$, then

$$\frac{ab+bc+cd}{a^2+b^2+c^2+d^2} \le \frac{2+\sqrt{7}}{6}.$$

2.60. If

$$a \ge 1 \ge b \ge c \ge d \ge 0, \qquad a+b+c+d=4,$$

then

$$ab+bc+cd \le 3.$$

2.61. Let k and a,b,c be positive real numbers, and let

$$E=(ka+b+c)\left(\frac{k}{a}+\frac{1}{b}+\frac{1}{c}\right), \qquad F=(ka^2+b^2+c^2)\left(\frac{k}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right).$$

(a) If $k \ge 1$, then

$$\sqrt{\frac{F-(k-2)^2}{2k}}+2 \ge \frac{E-(k-2)^2}{2k};$$

(b) If $0 < k \le 1$, then

$$\sqrt{\frac{F-k^2}{k+1}}+2 \ge \frac{E-k^2}{k+1}.$$

2.62. If a,b,c are positive real numbers, then

$$\frac{a}{2b+6c}+\frac{b}{7c+a}+\frac{25c}{9a+8b} > 1.$$

2.63. If a, b, c are positive real numbers such that

$$\frac{1}{a} \geq \frac{1}{b} + \frac{1}{c},$$

then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{55}{12(a+b+c)}.$$

2.64. If a, b, c are positive real numbers such that

$$\frac{1}{a} \geq \frac{1}{b} + \frac{1}{c},$$

then

$$\frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} \geq \frac{189}{40(a^2+b^2+c^2)}.$$

2.65. Find the best real numbers k, m, n such that

$$(\sqrt{a} + \sqrt{b} + \sqrt{c})\sqrt{a+b+c} \geq ka + mb + nc$$

for all $a \geq b \geq c \geq 0$.

2.66. Let $a, b \in (0, 1]$, $a \leq b$.

(a) If $a \leq \frac{1}{e}$, then

$$2a^a \geq a^b + b^a;$$

(b) If $b \geq \frac{1}{e}$, then

$$2b^b \geq a^b + b^a.$$

2.67. If $0 \leq a \leq b$ and $b \geq \frac{1}{2}$, then

$$2b^{2b} \geq a^{2b} + b^{2a}.$$

2.68. If $a \geq b \geq 0$, then

(a)
$$a^{b-a} \leq 1 + \frac{a-b}{\sqrt{a}};$$

(b)
$$a^{a-b} \geq 1 - \frac{3(a-b)}{4\sqrt{a}}.$$

2.69. If a, b, c are positive real numbers such that

$$a \geq b \geq c, \qquad ab^2c^3 = 1,$$

then

$$a + 2b + 3c \geq \frac{1}{a} + \frac{2}{b} + \frac{3}{c}.$$

2.70. If a, b, c are positive real numbers such that

$$a + b + c = 3, \qquad a \leq b \leq c,$$

then

$$\frac{1}{a} + \frac{2}{b} \geq a^2 + b^2 + c^2.$$

2.71. If a, b, c are positive real numbers such that

$$a + b + c = 3, \qquad a \leq b \leq c,$$

then

$$\frac{2}{a} + \frac{3}{b} + \frac{1}{c} \geq 2(a^2 + b^2 + c^2).$$

2.72. If a, b, c are positive real numbers such that

$$a + b + c = 3, \qquad a \leq b \leq c,$$

then

$$\frac{31}{a} + \frac{25}{b} + \frac{25}{c} \geq 27(a^2 + b^2 + c^2).$$

2.73. If a, b, c are the lengths of the sides of a triangle, then

$$a^3(b + c) + bc(b^2 + c^2) \geq a(b^3 + c^3).$$

2.74. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{(a + b)^2}{2ab + c^2} + \frac{(a + c)^2}{2ac + b^2} \geq \frac{(b + c)^2}{2bc + a^2}.$$

2.75. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{a+b}{ab+c^2} + \frac{a+c}{ac+b^2} \geq \frac{b+c}{bc+a^2}.$$

2.76. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{b(a+c)}{ac+b^2} + \frac{c(a+b)}{ab+c^2} \geq \frac{a(b+c)}{bc+a^2}.$$

2.77. If a, b, c, d are positive real numbers such that

$$a \geq b \geq c \geq d, \qquad ab^2c^3d^6 = 1,$$

then

$$a + 2b + 3c + 6d \geq \frac{1}{a} + \frac{2}{b} + \frac{3}{c} + \frac{6}{d}.$$

2.78. If a, b, c, d are positive real numbers such that

$$a \geq b \geq c \geq d, \qquad abc^2d^4 \geq 1,$$

then

$$a + b + 2c + 4d \geq \frac{1}{a} + \frac{1}{b} + \frac{2}{c} + \frac{4}{d}.$$

2.79. If a, b, c, d, e, f are positive real numbers such that

$$abcdef \geq 1, \qquad a \geq b \geq c \geq d \geq e \geq f, \qquad af \geq be \geq cd,$$

then

$$a + b + c + d + e + f \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{1}{f}.$$

2.80. Let a, b, c, d be nonnegative real numbers such that

$$a^2 - ab + b^2 = c^2 - cd + d^2.$$

Prove that

$$(a+b)(c+d) \geq 2(ab+cd).$$

2.81. Let a, b, c, d be nonnegative real numbers such that

$$a^2 - ab + b^2 = c^2 - cd + d^2.$$

Prove that

$$\frac{1}{a^2 + ab + b^2} + \frac{1}{c^2 + cd + d^2} \le \frac{8}{3(a+b)(c+d)}.$$

2.82. Let a, b, c, d be nonnegative real numbers such that

$$a^2 - ab + b^2 = c^2 - cd + d^2.$$

Prove that

$$\frac{1}{a^2 + ab + b^2} + \frac{1}{c^2 + cd + d^2} \le \frac{8}{3(a+b)(c+d)}.$$

2.83. Let a, b, c, d be nonnegative real numbers such that

$$a^2 - ab + b^2 = c^2 - cd + d^2.$$

Prove that

$$\frac{1}{(ac+bd)^4} + \frac{1}{(ad+bc)^4} \le \frac{2}{(ab+cd)^4}.$$

2.84. Let a, b, c, d be nonnegative real numbers such that $a \ge b \ge c \ge d$ and

$$a + b + c + d = 13, \qquad a^2 + b^2 + c^2 + d^2 = 43.$$

Prove that

$$ab \ge cd + 3.$$

2.85. Let a, b, c, d be nonnegative real numbers such that $a \ge b \ge c \ge d$ and

$$a + b + c + d = 13, \qquad a^2 + b^2 + c^2 + d^2 = 43.$$

Prove that

$$\frac{83}{4} \le ac + bd \le \frac{169}{8}.$$

2.86. If a, b, c, d are positive real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 4, \qquad a \le b \le c \le d,$$

then

$$\frac{1}{a} + a + b + c + d \ge 5.$$

2.87. If a, b, c, d are real numbers, then

$$6(a^2 + b^2 + c^2 + d^2) + (a + b + c + d)^2 \geq 12(ab + bc + cd).$$

2.88. If a, b, c, d are positive real numbers, then

$$\frac{1}{a^2 + ab} + \frac{1}{b^2 + bc} + \frac{1}{c^2 + cd} + \frac{1}{d^2 + da} \geq \frac{4}{ac + bd}.$$

2.89. If a, b, c, d are positive real numbers, then

$$\frac{1}{a(1 + b)} + \frac{1}{b(1 + a)} + \frac{1}{c(1 + d)} + \frac{1}{d(1 + c)} \geq \frac{16}{1 + 8\sqrt{abcd}}.$$

2.90. If a, b, c, d are positive real numbers such that $a \geq b \geq c \geq d$ and

$$a + b + c + d = 4,$$

then

$$ac + bd \leq 2.$$

2.91. If a, b, c, d are positive real numbers such that $a \geq b \geq c \geq d$ and

$$a + b + c + d = 4,$$

then

$$2\left(\frac{1}{b} + \frac{1}{d}\right) \geq a^2 + b^2 + c^2 + d^2.$$

2.92. Let a, b, c, d be positive real numbers such that $a \geq b \geq c \geq d$ and

$$ab + bc + cd + da = 3.$$

Prove that

$$a^3bcd < 4.$$

2.93. Let a, b, c, d be positive real numbers such that $a \geq b \geq c \geq d$ and

$$ab + bc + cd + da = 6.$$

Prove that

$$acd \leq 2.$$

2.94. Let a, b, c, d be positive real numbers such that $a \geq b \geq c \geq d$ and

$$ab + bc + cd + da = 9.$$

Prove that

$$abd \leq 4.$$

2.95. Let a, b, c, d be positive real numbers such that $a \geq b \geq c \geq d$ and

$$a^2 + b^2 + c^2 + d^2 = 10.$$

Prove that

$$2b + 4d \leq 3c + 5.$$

2.96. Let a, b, c, d be positive real numbers such that $a \leq b \leq c \leq d$ and

$$abcd = 1.$$

Prove that

$$4 + \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 2(a + b)(c + d).$$

2.97. Let a, b, c, d be positive real numbers such that $a \geq b \geq c \geq d$ and

$$3(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2.$$

Prove that

(a) $$\frac{a + d}{b + c} \leq 2;$$

(b) $$\frac{a + c}{b + d} \leq \frac{7 + 2\sqrt{6}}{5};$$

(c) $$\frac{a + c}{c + d} \leq \frac{3 + \sqrt{5}}{2}.$$

2.98. Let a, b, c, d be nonnegative real numbers such that $a \geq b \geq c \geq d$ and

$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2.$$

Prove that

$$a \geq b + 3c + (2\sqrt{3} - 1)d.$$

2.99. If a, b, c, d, e are real numbers, then

$$\frac{ab + bc + cd + de}{a^2 + b^2 + c^2 + d^2 + e^2} \le \frac{\sqrt{3}}{2}.$$

2.100. If a, b, c, d, e are positive real numbers, then

$$\frac{a^2b^2}{bd + ce} + \frac{b^2c^2}{cd + ae} + \frac{c^2a^2}{ad + be} \ge \frac{3abc}{d + e}.$$

2.101. Let a, b, c and x, y, z be positive real numbers such that

$$x + y + z = a + b + c.$$

Prove that

$$ax^2 + by^2 + cz^2 + xyz \ge 4abc.$$

2.102. Let a, b, c and x, y, z be positive real numbers such that

$$x + y + z = a + b + c.$$

Prove that

$$\frac{x(3x + a)}{bc} + \frac{y(3y + b)}{ca} + \frac{z(3z + c)}{ab} \ge 12.$$

2.103. Let a, b, c be given positive numbers. Find the minimum value $F(a, b, c)$ of

$$E(x, y, z) = \frac{ax}{y + z} + \frac{by}{z + x} + \frac{cz}{x + y},$$

where x, y, z are nonnegative real numbers, no two of which are zero.

2.104. Let a, b, c and x, y, z be positive real numbers such that

$$\frac{a}{yz} + \frac{b}{zx} + \frac{c}{xy} = 1.$$

Prove that

(a) $$x + y + z \ge \sqrt{4(a + b + c + \sqrt{ab} + \sqrt{bc} + \sqrt{ca}\ ) + 3\sqrt[3]{abc}};$$

(b) $$x + y + z > \sqrt{a + b} + \sqrt{b + c} + \sqrt{c + a}.$$

2.105. If a,b,c and x,y,z are nonnegative real numbers, then

$$\frac{2}{(a+b)(x+y)}+\frac{2}{(b+c)(y+z)}+\frac{2}{(c+a)(z+x)}\ge\frac{9}{(b+c)x+(c+a)y+(a+b)z}.$$

2.106. Let a,b,c be the lengths of the sides of a triangle. If x,y,z are real numbers, then

$$(ya^2+zb^2+xc^2)(za^2+xb^2+yc^2)\ge(xy+yz+zx)(a^2b^2+b^2c^2+c^2a^2).$$

2.107. If a,b,c are nonnegative real numbers such that

$$2(a+b+c)+ab+bc+ca=9,$$

then

$$(a+1)bc+3(b+c)\le\frac{16}{a+1}.$$

2.108. If a,b,c are nonnegative real numbers such that

$$2(a+b+c)+ab+bc+ca=9,$$

then

$$\frac{1}{ab+4}+\frac{1}{ac+4}+\frac{1}{b+4}+\frac{1}{c+4}\ge\frac{4}{5}.$$

2.109. If a,b,c are nonnegative real numbers such that

$$6a^2+4a(b+c)+bc=15,$$

then

$$\frac{4}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\ge 3.$$

2.110. Let $a_1,a_2,\ldots,a_n$ be positive real numbers such that $a_1\ge 2a_2$. Prove that

$$(5n-1)(a_1^2+a_2^2+\cdots+a_n^2)\ge 5(a_1+a_2+\cdots+a_n)^2.$$

2.111. If $a_1,a_2,\ldots,a_n$ are positive real numbers such that $a_1\ge 4a_2$, then

$$(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)\ge\left(n+\frac{1}{2}\right)^2.$$

2.112. Suppose $n \ge 3$ and $a_1, a_2, \dots, a_n$ are nonnegative real numbers such that $a_1 \le a_2 \le \cdots \le a_n$.

(a) Prove that

$$\frac{a_1a_2 + a_2a_3 + \cdots + a_na_1}{n} \ge \left(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}\right)^2;$$

(b) If $k \ge k_1 = \dfrac{2}{1 + \sqrt{\dfrac{n}{n-2}}}$, prove that

$$\frac{a_1a_2 + a_2a_3 + \cdots + a_na_1}{n} \ge \left(\frac{ka_1 + a_2 + \cdots + a_{n-1}}{n-2+k}\right)^2.$$

(c) If $0 \le k \le k_2 = 1 + \dfrac{1}{1 + \sqrt{\dfrac{n}{n-2}}}$, prove that

$$\frac{a_1a_2 + a_2a_3 + \cdots + a_na_1}{n} \ge \left(\frac{a_1 + \cdots + a_{n-2} + ka_{n-1}}{n-2+k}\right)^2.$$

2.113. If $k \ge k_0 = 7 - 2\sqrt{6} \approx 2.101$ and $a \ge b \ge c \ge d \ge e \ge f \ge 0$, then

$$\left(\frac{ka + b + c + d + e + f}{k+5}\right)^2 \ge \frac{ab + bc + cd + de + ef + fa}{6}.$$

2.114. If $a_1 \ge a_2 \ge \cdots \ge a_9 \ge 0$, then

$$\left(\frac{4a_1 + a_2 + \cdots + a_9}{12}\right)^2 \ge \frac{a_1a_2 + a_2a_3 + \cdots + a_9a_1}{9}.$$

2.115. Prove that $\dfrac{3}{4}$ is the least positive value of k such that

$$\left(\frac{ka + b + c + d}{k+3}\right)^2 \ge \frac{ab + bc + cd + de + ea}{5}$$

whenever $a \ge b \ge c \ge d \ge e \ge 0$.

2.116. If $a_1 \ge a_2 \ge \cdots \ge a_8 \ge 0$, then

$$(2a_1 + a_2 + \cdots + a_7)^2 \ge 8(a_1a_2 + a_2a_3 + \cdots + a_8a_1).$$

2.117. Let a, b, c, d be nonnegative real numbers such that $ab + bc + cd = 7$. Prove that

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} \ge \frac{3}{2}.$$

2.2 Solutions

P 2.1. *If a, b are positive real numbers, then*

$$\frac{1}{4a^2+b^2}+\frac{3}{b^2+4ab}\ge\frac{16}{5(a+b)^2}.$$

Solution. Using the Cauchy-Schwarz inequality gives

$$\frac{1}{4a^2+b^2}+\frac{3}{b^2+4ab}\ge\frac{(1+3)^2}{(4a^2+b^2)+3(b^2+4ab)}=\frac{4}{a^2+b^2+3ab}.$$

Thus, we only need to show that

$$\frac{1}{a^2+b^2+3ab}\ge\frac{4}{5(a+b)^2},$$

which reduces to $(a-b)^2\ge 0$. The equality holds for $a=b$.

□

P 2.2. *If a, b are positive real numbers, then*

$$3a\sqrt{3a}+3b\sqrt{6a+3b}\ge 5(a+b)\sqrt{a+b}.$$

Solution. Due to homogeneity, we may assume that $a+b=3$. Thus, we need to show that

$$a\sqrt{a}+(3-a)\sqrt{3+a}\ge 5$$

for $0<a<3$. Substituting

$$\sqrt{a}=x,\qquad 0<x<\sqrt{3},$$

the inequality becomes

$$(3-x^2)\sqrt{3+x^2}\ge 5-x^3.$$

For $\sqrt[3]{5}\le x<\sqrt{3}$, the inequality is trivial. For $0<x<\sqrt[3]{5}$, squaring both sides of the inequality gives

$$(3-x^2)(9-x^4)\ge(5-x^3)^2,$$

$$3x^4-10x^3+9x^2-2\le 0,$$

$$(x-1)^2(3x^2-4x-2)\le 0.$$

Since $3x^2-4x-2\le 0$ for $\dfrac{2-\sqrt{10}}{3}\le x\le\dfrac{2+\sqrt{10}}{3}$, we only need to prove that

$$\sqrt[3]{5}\le\frac{2+\sqrt{10}}{3}.$$

Indeed, we have

$$\left(\frac{2+\sqrt{10}}{3}\right)^3 - 5 = \frac{22\sqrt{10}-67}{27} > 0.$$

The equality holds for $a = b/2$.

□

P 2.3. *If a, b, c are nonnegative real numbers such that $a+b+c=3$, then*

$$(ab+c)(ac+b) \le 4.$$

Solution. By the AM-GM inequality, we have

$$(ab+c)(ac+b) \le \left[\frac{(ab+c)+(ac+b)}{2}\right]^2 = \frac{(a+1)^2(b+c)^2}{4}.$$

Therefore, it suffices to show that

$$(a+1)(b+c) \le 4.$$

Indeed,

$$(a+1)(b+c) \le \left[\frac{(a+1)+(b+c)}{2}\right]^2 = 4.$$

The equality holds for $a=b=c=1$, for $a=1$, $b=0$, $c=2$, and for $a=1$, $b=2$, $c=0$.

□

P 2.4. *If a, b, c are nonnegative real numbers, then*

$$a^3+b^3+c^3-3abc \ge \frac{1}{4}(b+c-2a)^3.$$

Solution. Write the inequality as

$$2(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2] \ge (b+c-2a)^3.$$

Consider the non-trivial case $b+c-2a \ge 0$. Since $(b-c)^2 \ge 0$ and

$$a+b+c \ge b+c-a,$$

it suffices to show that

$$2(a-b)^2+2(c-a)^2 \ge (b+c-2a)^2.$$

Indeed, we have

$$2(a-b)^2+2(c-a)^2-(b+c-2a)^2 = (b-c)^2 \ge 0.$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$.

□

P 2.5. *If a, b, c are nonnegative real numbers such that*

$$c = \min\{a, b, c\}, \qquad a^2 + b^2 + c^2 = 3,$$

then

(a) $$5b + 2c \le 9;$$

(b) $$5(b + c) \le 9 + 3a.$$

Solution. (a) It suffices to show that

$$5b + 2c + (a - c) \le 9;$$

that is,

$$9 \ge a + 5b + c.$$

This follows from the Cauchy-Schwarz inequality

$$(1 + 25 + 1)(a^2 + b^2 + c^2) \ge (a + 5b + c)^2.$$

The equality holds for $a = c = \dfrac{1}{3}$ and $b = \dfrac{5}{3}$.

(b) It suffices to show that

$$5(b + c) + 4(a - c) \le 9 + 3a;$$

that is,

$$9 \ge a + 5b + c.$$

As we have shown at (a), this follows from the Cauchy-Schwarz inequality

$$(1 + 25 + 1)(a^2 + b^2 + c^2) \ge (a + 5b + c)^2.$$

The equality holds for $a = c = \dfrac{1}{3}$ and $b = \dfrac{5}{3}$.

□

P 2.6. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} + \frac{16}{(b+c)^2} \ge \frac{6}{ab + bc + ca}.$$

(Vasile Cîrtoaje, 2014)

Solution *(by Nguyen Van Quy).* Since the equality holds for $a = 0$ and $b = c$, we write the desired inequality in the form

$$\frac{16}{(b+c)^2} + \left(\frac{1}{a+b} + \frac{1}{a+c}\right)^2 \ge \frac{6}{ab+bc+ca} + \frac{2}{(a+b)(a+c)}$$

and apply then the AM-GM inequality

$$\frac{16}{(b+c)^2} + \left(\frac{1}{a+b} + \frac{1}{a+c}\right)^2 \ge \frac{8}{b+c}\left(\frac{1}{a+b} + \frac{1}{a+c}\right).$$

Therefore, it suffices to show that

$$\frac{8}{b+c}\left(\frac{1}{a+b} + \frac{1}{a+c}\right) \ge \frac{6}{ab+bc+ca} + \frac{2}{(a+b)(a+c)}.$$

Since $(a+b)(a+c) \ge ab+bc+ca$, it is enough to show that

$$\frac{8}{b+c}\left(\frac{1}{a+b} + \frac{1}{a+c}\right) \ge \frac{8}{ab+bc+ca},$$

which is equivalent to

$$(2a+b+c)(ab+bc+ca) \ge (a+b)(b+c)(c+a).$$

We have

$$\begin{aligned}(2a+b+c)(ab+bc+ca) &\ge (a+b+c)(ab+bc+ca)\\ &\ge (a+b)(b+c)(c+a).\end{aligned}$$

This completes the proof. The equality holds for $a = 0$ and $b = c$.

□

P 2.7. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} + \frac{2}{(b+c)^2} \ge \frac{5}{2(ab+bc+ca)}.$$

Solution. This inequality follows from Iran 1996 inequality (see P 1.72 in Volume 2, for $k = 2$), namely

$$\frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} + \frac{1}{(b+c)^2} \ge \frac{9}{4(ab+bc+ca)},$$

and the inequality in P 2.6, namely

$$\frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} + \frac{16}{(b+c)^2} \ge \frac{6}{ab+bc+ca}.$$

Indeed, summing the first inequality multiplied by 14 and the second inequality, we get the desired inequality. The equality holds for $a = 0$ and $b = c$.

□

P 2.8. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{(a+b)^2}+\frac{1}{(a+c)^2}+\frac{25}{(b+c)^2}\ge\frac{8}{ab+bc+ca}.$$

(Vasile Cîrtoaje, 2014)

Solution. Write the inequality as

$$\left(\frac{1}{a+b}+\frac{1}{a+c}\right)^2+\frac{25}{(b+c)^2}\ge\frac{8}{ab+bc+ca}+\frac{2}{(a+b)(a+c)}.$$

By the AM-GM inequality, we have

$$\left(\frac{1}{a+b}+\frac{1}{a+c}\right)^2+\frac{25}{(b+c)^2}\ge\frac{10}{b+c}\left(\frac{1}{a+b}+\frac{1}{a+c}\right).$$

Therefore, it suffices to show that

$$\frac{10}{b+c}\left(\frac{1}{a+b}+\frac{1}{a+c}\right)\ge\frac{8}{ab+bc+ca}+\frac{2}{(a+b)(a+c)}.$$

Since $(a+b)(a+c)\ge ab+bc+ca$, it is enough to show that

$$\frac{10}{b+c}\left(\frac{1}{a+b}+\frac{1}{a+c}\right)\ge\frac{10}{ab+bc+ca},$$

which is equivalent to

$$(2a+b+c)(ab+bc+ca)\ge(a+b)(b+c)(c+a).$$

Indeed,

$$\begin{aligned}(2a+b+c)(ab+bc+ca)&\ge(a+b+c)(ab+bc+ca)\\&\ge(a+b)(b+c)(c+a).\end{aligned}$$

This completes the proof. The equality holds for $a=0$ and $\dfrac{b}{c}+\dfrac{c}{b}=3$.

□

P 2.9. *If a, b, c are positive real numbers, then*

$$(a+b)^3(a+c)^3\ge4a^2bc(2a+b+c)^2.$$

(XZLBQ, 2014)

Solution *(by Nguyen Van Quy).* Write the inequality as

$$\frac{(a+b)^2(a+c)^2}{4a^2bc} \ge \frac{(2a+b+c)^2}{(a+b)(a+c)}.$$

Since

$$\begin{aligned}(a+b)^2(a+c)^2 &= [(a-b)^2+4ab][(a-c)^2+4ac]\\ &\ge 4ac(a-b)^2+4ab(a-c)^2+16a^2bc,\end{aligned}$$

it suffices to show that

$$\frac{(a-b)^2}{ab}+\frac{(a-c)^2}{ac}+4 \ge \frac{(2a+b+c)^2}{(a+b)(a+c)},$$

which is equivalent to

$$\frac{(a-b)^2}{ab}+\frac{(a-c)^2}{ac} \ge \frac{(b-c)^2}{(a+b)(a+c)}.$$

Indeed, by the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{ab}+\frac{(a-c)^2}{ac} \ge \frac{(a-b-a+c)^2}{ab+ac} \ge \frac{(b-c)^2}{(a+b)(a+c)}.$$

The equality holds for $a=b=c$.

□

P 2.10. *If a,b,c are positive real numbers such that $abc=1$, then*

(a) $$\frac{a}{b}+\frac{b}{c}+\frac{1}{a} \ge a+b+1;$$

(b) $$\frac{a}{b}+\frac{b}{c}+\frac{1}{a} \ge \sqrt{3(a^2+b^2+1)}.$$

(Vasile Cîrtoaje, 2007)

Solution. (a) ***First Solution.*** Write the inequality as

$$\left(2\frac{a}{b}+\frac{b}{c}\right)+\left(\frac{b}{c}+\frac{1}{a}\right)+\left(\frac{1}{a}+a\right) \ge 3a+2b+2.$$

By the AM-GM inequality, we have

$$\left(2\frac{a}{b}+\frac{b}{c}\right)+\left(\frac{b}{c}+\frac{1}{a}\right)+\left(\frac{1}{a}+a\right) \ge 3\sqrt[3]{\frac{a^2}{bc}}+2\sqrt{\frac{b}{ca}}+2 = 3a+2b+2.$$

The equality holds for $a=b=c=1$.

Second Solution. Since $c = \dfrac{1}{ab}$, the inequality becomes as follows:

$$\frac{a}{b} + ab^2 + \frac{1}{a} \ge a + b + 1,$$

$$\frac{1}{b} + b^2 + \frac{1}{a^2} \ge 1 + \frac{b}{a} + \frac{1}{a},$$

$$\frac{1}{a^2} - (b+1)\frac{1}{a} + b^2 + \frac{1}{b} - 1 \ge 0,$$

$$\left(\frac{1}{a} - \frac{b+1}{2}\right)^2 + \frac{(b-1)^2(3b+4)}{4b} \ge 0.$$

(b) Write the inequality as

$$a\left(\frac{1}{b} + b^2\right) + \frac{1}{a} \ge \sqrt{3(a^2+b^2+1)}.$$

By squaring, this inequality becomes

$$a^2\left(b^4 + 2b - 3 + \frac{1}{b^2}\right) + \frac{1}{a^2} \ge b^2 + 3 - \frac{2}{b}.$$

Since

$$b^4 + 2b - 3 + \frac{1}{b^2} > 2b - 3 + \frac{1}{b^2} = \frac{(b-1)^2(2b+1)}{b^2} \ge 0,$$

by the AM-GM inequality, we have

$$a^2\left(b^4 + 2b - 3 + \frac{1}{b^2}\right) + \frac{1}{a^2} \ge 2\sqrt{b^4 + 2b - 3 + \frac{1}{b^2}}.$$

Thus, it suffices to prove that

$$2\sqrt{b^4 + 2b - 3 + \frac{1}{b^2}} \ge b^2 + 3 - \frac{2}{b}.$$

Squaring again, we get the inequality

$$b^5 - 2b^3 + 4b^2 - 7b + 4 \ge 0,$$

which is equivalent to the obvious inequality

$$b(b^2-1)^2 + 4(b-1)^2 \ge 0.$$

The equality holds for $a = b = c = 1$.

$\square$

P 2.11. *If a, b, c are positive real numbers such that $abc \ge 1$, then*

$$a^{\frac{a}{b}} b^{\frac{b}{c}} c^c \ge 1.$$

(Vasile Cîrtoaje, 2011)

Solution. Write the inequality as

$$\frac{a}{b}\ln a + \frac{b}{c}\ln b + c\ln c \ge 0.$$

Since $f(x) = x\ln x$ is a convex function on $(0, \infty)$, apply Jensen's inequality to get

$$pa\ln a + qb\ln b + rc\ln c \ge (p+q+r)\left(\frac{pa+qb+rc}{p+q+r}\right)\ln\left(\frac{pa+qb+rc}{p+q+r}\right)$$

$$= (pa+qb+rc)\ln\left(\frac{pa+qb+rc}{p+q+r}\right),$$

where $p, q, r > 0$. Choosing

$$p = \frac{1}{b}, \quad q = \frac{1}{c}, \quad r = 1,$$

we get

$$\frac{a}{b}\ln a + \frac{b}{c}\ln b + c\ln c \ge (\frac{a}{b} + \frac{b}{c} + c)\ln\left(\frac{\frac{a}{b} + \frac{b}{c} + c}{\frac{1}{b} + \frac{1}{c} + 1}\right).$$

Thus, it suffices to show that

$$\frac{a}{b} + \frac{b}{c} + c \ge \frac{1}{b} + \frac{1}{c} + 1.$$

Since $a \ge \frac{1}{bc}$, we need to show that

$$\frac{1}{b^2c} + \frac{b}{c} + c \ge \frac{1}{b} + \frac{1}{c} + 1.$$

This is equivalent to

$$\frac{1}{b^2} + b + c^2 \ge \frac{c}{b} + 1 + c,$$

$$c^2 - \left(1 + \frac{1}{b}\right)c + b - 1 + \frac{1}{b^2} \ge 0,$$

$$\left(c - \frac{b+1}{2b}\right)^2 + \frac{(b-1)^2(4b+3)}{4b^2} \ge 0.$$

The equality holds for $a = b = c = 1$.

□

P 2.12. *If a, b, c are positive real numbers such that $ab + bc + ca = 3$, then*

$$ab^2c^3 < 4.$$

(Vasile Cîrtoaje, 2012)

Solution. From $ab + bc + ca = 3$, we get

$$c = \frac{3 - ab}{a + b} < \frac{3}{a + b}.$$

Therefore,

$$\begin{aligned} (a+b)^3(4 - ab^2c^3) &> 4(a+b)^3 - 27ab^2 \\ &= 4a^3 + 12a^2b - 15ab^2 + 4b^3 \\ &= (a + 4b)(2a - b)^2 \geq 0. \end{aligned}$$

□

P 2.13. *If a, b, c are positive real numbers such that $ab + bc + ca = \frac{5}{3}$, then*

$$ab^2c^2 \leq \frac{1}{3}.$$

(Vasile Cîrtoaje, 2012)

Solution. By the AM-GM inequality, we have

$$ab + ca \geq 2a\sqrt{bc}.$$

Thus, from $ab + bc + ca = \frac{5}{3}$, we get

$$2a\sqrt{bc} + bc \leq \frac{5}{3}.$$

Therefore, it suffices to show that

$$\frac{(5 - 3bc)b^2c^2}{6\sqrt{bc}} \leq \frac{1}{3}.$$

Setting $\sqrt{bc} = t$, this inequality becomes

$$3t^5 - 5t^3 + 2 \geq 0.$$

Indeed, be the AM-GM inequality, we have

$$3t^5 + 2 = t^5 + t^5 + t^5 + 1 + 1 \geq 5\sqrt[5]{t^5 \cdot t^5 \cdot t^5 \cdot 1 \cdot 1} = 5t^3.$$

The equality holds for $a = \frac{1}{3}$ and $b = c = 1$.

□

P 2.14. *Let a, b, c be positive real numbers such that*

$$a \le b \le c, \qquad ab + bc + ca = 3.$$

Prove that

(a) $$ab^2c \le \frac{9}{8};$$

(b) $$ab^4c \le 2;$$

(c) $$ab^3c^2 \le 2.$$

(Vasile Cîrtoaje, 2012)

Solution. From $(b-a)(b-c) \le 0$, we get

$$b^2 + ac \le b(a+c),$$

$$b^2 + ac \le 3 - ac,$$

$$b^2 + 2ac \le 3.$$

(a) We have

$$9 - 8ab^2c \ge 9 - 4b^2(3-b^2) = (2b^2-3)^2 \ge 0.$$

The equality holds for $a = \frac{1}{2}\sqrt{\frac{3}{2}}$ and $b = c = \sqrt{\frac{3}{2}}$.

(b) We have

$$4 - 2ab^4c \ge 4 - b^4(3-b^2) = (b^2-2)^2(b^2+1) \ge 0.$$

The equality holds for $a = \frac{\sqrt{2}}{4}$ and $b = c = \sqrt{2}$.

(c) Write the desired inequality as follows:

$$2(ab+bc+ca)^3 \ge 27ab^3c^2,$$

$$2\left(a + c + \frac{ca}{b}\right)^3 \ge 27ac^2.$$

Since $ca/b \ge a$, it suffices to show that

$$2(2a+c)^3 \ge 27ac^2,$$

which is equivalent to the obvious inequality

$$(a+2c)(4a-c)^2 \ge 0.$$

The equality holds for $a = \frac{\sqrt{2}}{4}$ and $b = c = \sqrt{2}$.

□

P 2.15. *Let a, b, c be positive real numbers such that*

$$a \le b \le c, \qquad a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Prove that

$$b \ge \frac{1}{a+c-1}.$$

(Vasile Cîrtoaje, 2007)

Solution. Let us show that

$$a \le 1, \qquad c \ge 1.$$

From $a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ and

$$a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-6 = \frac{(a-1)^2}{a}+\frac{(b-1)^2}{b}+\frac{(c-1)^2}{c} \ge 0,$$

we get

$$a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge 3.$$

Then,

$$\frac{1}{a} \ge \frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge 1, \qquad c \ge \frac{a+b+c}{3} \ge 1.$$

Further, consider the following two cases.

Case 1: $abc \ge 1$. Write the desired inequality as

$$a+c-1-\frac{1}{b} \ge 0.$$

We have

$$a+c-1-\frac{1}{b} = (1-a)(c-1)+\frac{abc-1}{b} \ge 0.$$

Case 2: $abc \le 1$. Since

$$a+c-1-\frac{1}{b} = \frac{1}{a}+\frac{1}{c}-1-b,$$

the desired inequality is equivalent to

$$\frac{1}{a}+\frac{1}{c}-1-b \ge 0.$$

We have

$$\frac{1}{a}+\frac{1}{c}-1-b = \left(\frac{1}{a}-1\right)\left(1-\frac{1}{c}\right)+\frac{1-abc}{ac} \ge 0.$$

This completes the proof. The equality holds for $a=b=c=1$.

$\square$

P 2.16. *Let a, b, c be positive real numbers such that*

$$a \le b \le c, \qquad a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Prove that

$$ab^2c^3 \ge 1.$$

(Vasile Cîrtoaje, 1998)

First Solution. Write the inequality in the homogeneous form

$$ab^2c^3 \ge \left[\frac{abc(a+b+c)}{ab+bc+ca}\right]^3,$$

which is equivalent to

$$(ab+bc+ca)^3 \ge a^2b(a+b+c)^3.$$

Since

$$(ab+bc+ca)^2 \ge 3abc(a+b+c),$$

it suffices to show that

$$3c(ab+bc+ca) \ge a(a+b+c)^2.$$

Indeed,

$$3c(ab+bc+ca) - a(a+b+c)^2 \ge (a+b+c)(ab+bc+ca) - a(a+b+c)^2$$

$$= (a+b+c)(bc-a^2) \ge 0.$$

The equality holds for $a = b = c = 1$.

Second Solution. Let us show that

$$a \le 1, \qquad bc \ge 1.$$

Indeed, if $a > 1$, then $1 < a \le b \le c$ and

$$a+b+c-\frac{1}{a}-\frac{1}{b}-\frac{1}{c} = \frac{1-a^2}{a}+\frac{1-b^2}{b}+\frac{1-c^2}{c} < 0,$$

which is false. On the other hand, from $a \le 1$ and

$$a-\frac{1}{a} = (b+c)\left(\frac{1}{bc}-1\right),$$

we get $bc \ge 1$. Similarly, we can prove that

$$c \ge 1, \qquad ab \le 1.$$

Since $bc \ge 1$, it suffices to show that

$$abc^2 \ge 1.$$

Taking account of $ab \le 1$, we have

$$c - \frac{1}{c} = (a+b)\left(\frac{1}{ab} - 1\right) \ge 2\sqrt{ab}\left(\frac{1}{ab} - 1\right) = 2\left(\frac{1}{\sqrt{ab}} - \sqrt{ab}\right) \ge \frac{1}{\sqrt{ab}} - \sqrt{ab},$$

hence

$$\left(c - \frac{1}{\sqrt{ab}}\right)\left(1 + \frac{\sqrt{ab}}{c}\right) \ge 0.$$

The last inequality involves

$$abc^2 \ge 0.$$

□

P 2.17. *Let a, b, c be positive real numbers such that*

$$a \le b \le c, \qquad a + b + c = abc + 2.$$

Prove that

$$(1-b)(1-ab^3c) \ge 0.$$

(Vasile Cîrtoaje, 1999)

Solution. Let us show that

$$a \le 1, \qquad c \ge 1.$$

To do this, we write the hypothesis $a + b + c = abc + 2$ in the equivalent form

$$(1-a)(1-c) + (1-b)(1-ac) = 0, \tag{*}$$

If $a > 1$, then $1 < a \le b \le c$, which contradicts (*). Similarly, if $c < 1$, then $a \le b \le c < 1$, which also contradicts (*). Therefore, we have $a \le 1$ and $c \ge 1$. According to (*), we get

$$(1-b)(1-ac) = (1-a)(c-1) \ge 0. \tag{**}$$

There are two cases to consider.

Case 1: $b \ge 1$. According to (**), we have $ac \ge 1$. Therefore,

$$ab^3c = ac \cdot b^3 \ge 1,$$

hence $(1-b)(1-ab^3c) \ge 0$.

Case 2: $b \le 1$. According to (**), we have $ac \le 1$. Therefore,

$$ab^3c = ac \cdot b^3 \le 1,$$

and hence

$$(1-b)(1-ab^3c) \ge 0.$$

This completes the proof. The equality holds for $a = b = 1 \le c$ or $a \le 1 = b = c$.

□

P 2.18. *Let a, b, c be real numbers, no two of which are zero. Prove that*

(a) $$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{2(b^2+c^2)};$$

(b) $$\frac{(a+b)^2}{a^2+b^2}+\frac{(a+c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{2(b^2+c^2)}.$$

Solution. (a) Consider two cases.
Case 1: $2a^2 \le b^2+c^2$. By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2)}{a^2+c^2} \ge \frac{[(b-a)+(a-c)]^2}{(a^2+b^2)+(a^2+c^2)} = \frac{(b-c)^2}{2a^2+b^2+c^2}.$$

Thus, it suffices to show that

$$\frac{1}{2a^2+b^2+c^2} \ge \frac{1}{2(b^2+c^2)},$$

which reduces to $b^2+c^2 \ge 2a^2$.
Case 2: $2a^2 \ge b^2+c^2$. By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2} \ge \frac{[c(b-a)+b(a-c)]^2}{c^2(a^2+b^2)+b^2(a^2+c^2)} = \frac{a^2(b-c)^2}{a^2(b^2+c^2)+2b^2c^2}.$$

Therefore, it suffices to prove that

$$\frac{a^2}{a^2(b^2+c^2)+2b^2c^2} \ge \frac{1}{2(b^2+c^2)},$$

which reduces to $a^2(b^2+c^2) \ge 2b^2c^2$. This is true since

$$2a^2(b^2+c^2)-4b^2c^2 \ge (b^2+c^2)^2-4b^2c^2 = (b^2-c^2)^2.$$

The equality holds for $a=b=c$.

(b) The inequality follows from the inequality in (a) by replacing a with $-a$. The equality holds for $-a=b=c$.

□

P 2.19. *Let a, b, c be real numbers, no two of which are zero. If $bc \ge 0$, then*

(a) $$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{(b+c)^2};$$

(b) $$\frac{(a+b)^2}{a^2+b^2}+\frac{(a+c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{(b+c)^2}.$$

(Vasile Cîrtoaje, 2011)

Solution. (a) Consider two cases: $a^2 \le bc$ and $a^2 \ge bc$.
Case 1: $a^2 \le bc$. By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2)}{a^2+c^2} \ge \frac{[(b-a)+(a-c)]^2}{(a^2+b^2)+(a^2+c^2)} = \frac{(b-c)^2}{2a^2+b^2+c^2}.$$

Thus, it suffices to show that

$$\frac{1}{2a^2+b^2+c^2} \ge \frac{1}{(b+c)^2},$$

which is equivalent to $a^2 \le bc$.
Case 2: $a^2 \ge bc$. By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2} \ge \frac{[c(b-a)+b(a-c)]^2}{c^2(a^2+b^2)+b^2(a^2+c^2)} = \frac{a^2(b-c)^2}{a^2(b^2+c^2)+2b^2c^2}.$$

Therefore, it suffices to prove that

$$\frac{a^2}{a^2(b^2+c^2)+2b^2c^2} \ge \frac{1}{(b+c)^2},$$

which reduces to $bc(a^2-bc) \ge 0$. The equality holds for $a=b=c$, for $b=0$ and $a=c$, and for $c=0$ and $a=b$.

(b) The inequality follows from the inequality in (a) by replacing a with $-a$. The equality holds for $-a=b=c$, for $b=0$ and $a+c=0$, and for $c=0$ and $a+b=0$.

□

P 2.20. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{|a-b|^3}{a^3+b^3}+\frac{|a-c|^3}{a^3+c^3} \ge \frac{|b-c|^3}{(b+c)^3}.$$

(Vasile Cîrtoaje, 2013)

Solution. Without loss of generality, assume that $b \ge c$. Thus, we have three cases to consider: $a \ge b \ge c$, $b \ge c \ge a$ and $b \ge a \ge c$.
Case 1: $a \ge b \ge c$. It suffices to show that

$$\frac{|a-c|^3}{(a+c)^3} \ge \frac{|b-c|)^3}{(b+c)^3},$$

which is equivalent to

$$\frac{a-c}{a+c} \ge \frac{b-c}{b+c}.$$

Indeed,

$$\frac{a-c}{a+c}-\frac{b-c}{b+c}=\frac{2c(a-b)}{(a+c)(b+c)}\ge 0.$$

Case 2: $b\ge c\ge a$. It suffices to show that

$$\frac{(b-a)^3}{a^3+b^3}\ge\frac{(b-c)^3}{(b+c)^3}.$$

Indeed,

$$\frac{(b-a)^3}{a^3+b^3}\ge\frac{(b-c)^3}{a^3+b^3}\ge\frac{(b-c)^3}{b^3+c^3}\ge\frac{b-c)^3}{(b+c)^3}.$$

Case 3: $b\ge a\ge c$. We need to prove that

$$\frac{(b-a)^3}{a^3+b^3}+\frac{(a-c)^3}{a^3+c^3}\ge\frac{(b-c)^3}{(b+c)^3}.$$

Using the substitution

$$x=\frac{b-a}{a+b},\quad y=\frac{a-c}{a+c},\quad 0\le x<1,\quad 0\le y\le 1,$$

we have

$$b=\frac{1+x}{1-x}a,\qquad c=\frac{1-y}{1+y}a,$$

$$(b-a)^3=\frac{8x^3}{(1-x)^3}a^3,\qquad (a-c)^3=\frac{8y^3}{(1+y)^3}a^3,$$

$$a^3+b^3=\frac{2(1+3x^3)}{(1-x)^3},\qquad a^3+c^3=\frac{2(1+3y^2)}{(1+y)^3},$$

$$\frac{b-c}{b+c}=\frac{x+y}{1+xy}.$$

Thus, the desired inequality becomes

$$\frac{4x^3}{1+3x^2}+\frac{4y^3}{1+3y^2}\ge\frac{(x+y)^3}{(1+xy)^3},$$

$$\frac{x^2+y^2-xy+3x^2y^2}{(1+3x^2)(1+3y^2)}\ge\frac{(x+y)^2}{4(1+xy)^3},$$

$$\frac{s-p+3p^2}{1+3s+9p^2}\ge\frac{s+2p}{4(1+p)^3},$$

where

$$s=x^2+y^2,\qquad p=xy,\quad 0\le p<1,\quad 2p\le s\le 1+p^2.$$

Therefore, we need to show that $f(s)\ge 0$, where

$$f(s)=4(1+p)^3(s-p+3p^2)-(s+2p)(3s+1+9p^2).$$

Since f is a concave function, it suffices to show that $f(2p) \geq 0$ and $f(1+p^2) \geq 0$. Indeed, we have

$$f(2p) = 4p^3(3p+1)(p+3) \geq 0,$$

$$f(1+p^2) = 16p^3(p+1)^2 \geq 0.$$

Thus, the proof is completed. The equality holds for $a = b = c$, for $b = 0$ and $a = c$, and for $c = 0$ and $a = b$.

□

P 2.21. *Let a, b, c be positive real numbers, $b \neq c$. Prove that*

$$\frac{ab}{(a+b)^2} + \frac{ac}{(a+c)^2} \leq \frac{(b+c)^2}{4(b-c)^2}.$$

(Vasile Cîrtoaje, 2010)

Solution. Write the inequality in the form

$$\frac{(a-b)^2}{(a+b)^2} + \frac{(a-c)^2}{(a+c)^2} + \frac{(b+c)^2}{(b-c)^2} \geq 2.$$

Replacing a be $-a$, the inequality becomes

$$\frac{(a+b)^2}{(a-b)^2} + \frac{(a+c)^2}{(a-c)^2} + \frac{(b+c)^2}{(b-c)^2} \geq 2. \qquad (*)$$

Making the substitution

$$x = \frac{a+b}{a-b}, \quad y = \frac{b+c}{b-c}, \quad z = \frac{c+a}{c-a},$$

we can write the inequality as

$$x^2 + y^2 + z^2 \geq 2.$$

From

$$x+1 = \frac{2a}{a-b}, \quad y+1 = \frac{2b}{b-c}, \quad z+1 = \frac{2c}{c-a}$$

and

$$x-1 = \frac{2b}{a-b}, \quad y-1 = \frac{2c}{b-c}, \quad z-1 = \frac{2a}{c-a},$$

we get

$$(x+1)(y+1)(z+1) = (x-1)(y-1)(z-1),$$

$$xy + yz + zx + 1 = 0.$$

Therefore, we have

$$x^2 + y^2 + z^2 - 2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) = (x+y+z)^2 \geq 0.$$

The inequality (*) is an equality for $x+y+z=0$; that is,

$$(a+b+c)(ab+bc+ca)-9abc=0.$$

Therefore, the original inequality is an equality for

$$(b+c-a)(bc-ab-ac)+9abc=0.$$

□

P 2.22. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{3bc+a^2}{b^2+c^2}\ge\frac{3ab-c^2}{a^2+b^2}+\frac{3ac-b^2}{a^2+c^2}.$$

(Vasile Cîrtoaje, 2014)

Solution *(by Nguyen Van Quy).* Write the inequality as

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{a^2+c^2}+\frac{c^2}{a^2+b^2}+\frac{3bc}{b^2+c^2}\ge\frac{3ab}{a^2+b^2}+\frac{3ac}{a^2+c^2}.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{b^2}{a^2+c^2}+\frac{c^2}{a^2+b^2}\ge\frac{(b^2+c^2)^2}{b^2(a^2+c^2)+c^2(a^2+b^2)}=\frac{(b^2+c^2)^2}{a^2(b^2+c^2)+2b^2c^2}$$
$$\ge 2-\frac{a^2(b^2+c^2)+2b^2c^2}{(b^2+c^2)^2}=2-\frac{a^2}{b^2+c^2}-\frac{2b^2c^2}{(b^2+c^2)^2},$$

hence

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{a^2+c^2}+\frac{c^2}{a^2+b^2}\ge 2-\frac{2b^2c^2}{(b^2+c^2)^2}.$$

Therefore, it suffices to show that

$$2-\frac{2b^2c^2}{(b^2+c^2)^2}+\frac{3bc}{b^2+c^2}\ge\frac{3ab}{a^2+b^2}+\frac{3ac}{a^2+c^2}.$$

This inequality is equivalent to

$$\left[\frac{1}{2}-\frac{2b^2c^2}{(b^2+c^2)^2}\right]+\left(\frac{3}{2}-\frac{3ab}{a^2+b^2}\right)+\left(\frac{3}{2}-\frac{3ac}{a^2+c^2}\right)\ge\left(\frac{3}{2}-\frac{3bc}{b^2+c^2}\right),$$

$$\frac{(b^2-c^2)^2}{3(b^2+c^2)^2}+\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2}\ge\frac{(b-c)^2}{b^2+c^2}.$$

Using the inequality in P 2.19-(a), namely

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2}\ge\frac{(b-c)^2}{(b+c)^2},$$

it is enough to prove that

$$\frac{(b+c)^2}{3(b^2+c^2)^2}+\frac{1}{(b+c)^2}\ge\frac{1}{b^2+c^2},$$

which is equivalent to

$$\frac{1}{(b+c)^2}\ge\frac{2(b^2-bc+c^2)}{3(b^2+c^2)^2}.$$

We have

$$\begin{aligned}3(b^2+c^2)^2-2(b+c)^2(b^2-bc+c^2)&=3(b^2+c^2)^2-2(b+c)(b^3+c^3)\\&=b^4+c^4+6b^2c^2-2bc(b^2+c^2)\\&\ge(b^2+c^2)^2-2bc(b^2+c^2)\\&=(b^2+c^2)(b-c)^2\ge 0.\end{aligned}$$

The equality holds for $a=b=c$.

□

P 2.23. *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that*

$$ab^2+bc^2+2ca^2\le 8.$$

Solution. Since the equality holds for $a=2$, $b=0$, $c=1$, we apply the AM-GM inequality to get

$$\frac{ca^2}{4}=c\cdot\frac{a}{2}\cdot\frac{a}{2}\le\frac{1}{27}\left(c+\frac{a}{2}+\frac{a}{2}\right)^3=\frac{1}{27}(c+a)^3\le\frac{1}{27}(a+b+c)^3=1.$$

Therefore, it suffices to show that

$$ab^2+bc^2+ca^2\le 4,$$

which is the inequality in P 1.1.

□

P 2.24. *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that*

$$ab^2+bc^2+\frac{3}{2}abc\le 4.$$

(Vasile Cîrtoaje and Vo Quoc Ba Can, 2007)

Solution. Consider two cases.

Case 1: $c \ge 2b$. We have

$$ab^2 + bc^2 + \frac{3}{2}abc = b(a+c)^2 - ab\left(a - b + \frac{c}{2}\right) \le b(a+c)^2$$

$$= 4b\left(\frac{a+c}{2}\right)\left(\frac{a+c}{2}\right) \le 4\left(\frac{b + \frac{a+c}{2} + \frac{a+c}{2}}{3}\right)^3 = 4.$$

Case 2: $2b > c$. Write the desired inequality as $f(a) \ge 0$, where

$$f(a) = 4\left(\frac{a+b+c}{3}\right)^3 - ab^2 - bc^2 - \frac{3}{2}abc,$$

with the derivative

$$f'(a) = 4\left(\frac{a+b+c}{3}\right)^2 - b^2 - \frac{3}{2}bc.$$

The equation $f'(a) = 0$ has the positive root

$$a_1 = \frac{3}{2}\sqrt{\frac{b(2b+3c)}{2}} - b - c = \frac{(2b-c)(5b+8c)}{6\sqrt{2b(2b+c)} + 8(b+c)}.$$

Since $f'(a) < 0$ for $0 \le a < a_1$ and $f'(a) > 0$ for $a > a_1$, $f(a)$ is decreasing on $[0, a_1]$ and increasing on $[a_1, \infty)$; consequently, $f(a) \ge f(a_1)$. To complete the proof, it suffices to show that $f(a_1) \ge 0$. Indeed, since

$$4\left(\frac{a_1+b+c}{3}\right)^2 = b^2 + \frac{3}{2}bc,$$

we have

$$\begin{aligned} f(a_1) &= 4\left(\frac{a_1+b+c}{3}\right)^3 - a_1\left(b^2 + \frac{3}{2}bc\right) - bc^2 \\ &= \frac{a_1+b+c}{3}\left(b^2 + \frac{3}{2}bc\right) - a_1\left(b^2 + \frac{3}{2}bc\right) - bc^2 \\ &= \frac{b+c-2a_1}{3}\left(b^2 + \frac{3}{2}bc\right) - bc^2 \\ &= \left(b + c - \sqrt{\frac{2b^2+3bc}{2}}\right)\left(b^2 + \frac{3}{2}bc\right) - bc^2 \\ &= \frac{b}{4}\left[4b^2 + 10bc + 2c^2 - (2b+3c)\sqrt{2b(2b+3c)}\right] \\ &= \frac{bc(2b-c)^2(b+2c)}{2[4b^2 + 10bc + 2c^2 + (2b+3c)\sqrt{2b(2b+3c)}]} \ge 0. \end{aligned}$$

Thus, the proof is completed. The equality holds for $a = 0$, $b = 1$, $c = 2$, and for $a = 1$, $b = 2$, $c = 0$.

□

P 2.25. *Let a, b, c be nonnegative real numbers such that $a + b + c = 5$. Prove that*

$$ab^2 + bc^2 + 2abc \le 20.$$

(Vo Quoc Ba Can, 2011)

Solution. Write the inequality as

$$b(ab + c^2 + 2ac) \le 20.$$

We see that the equality holds for $a = 1$ and $b = c = 2$. From $(a - b/2)^2 \ge 0$, it follows that

$$ab \le a^2 + \frac{b^2}{4}.$$

Therefore, for $b \le 4$, we have

$$\begin{aligned} b(ab + c^2 + 2ac) - 20 &\le b\left(a^2 + \frac{b^2}{4} + c^2 + 2ac\right) - 20 = b\left[(a + c)^2 + \frac{b^2}{4}\right] - 20 \\ &= b\left[(5 - b)^2 + \frac{b^2}{4}\right] - 20 = \frac{5}{4}(b - 4)(b - 2)^2 \le 0. \end{aligned}$$

Consider now that $b > 4$. Since

$$a = 5 - b - c \le 5 - b,$$

We have

$$\begin{aligned} ab^2 + bc^2 + 2abc - 20 &= ab^2 + b(5 - a - b)^2 + 2ab(5 - a - b) - 20 \\ &= b^3 + ab^2 - 10b^2 - a^2b + 25b - 20 \\ &\le b^3 + ab^2 - 10b^2 + 25b - 20 \\ &\le b^3 + (5 - b)b^2 - 10b^2 + 25b - 20 \\ &= -5(b - 4)(b - 1) < 0. \end{aligned}$$

□

P 2.26. *If a, b, c are nonnegative real numbers, then*

$$a^3 + b^3 + c^3 - a^2b - b^2c - c^2a \ge \frac{8}{9}(a - b)(b - c)^2.$$

Solution. Since

$$3(a^3+b^3+c^3-a^2b-b^2c-c^2a)=\sum(2a^3-3a^2b+b^3)=\sum(2a+b)(a-b)^2,$$

we can write the inequality as

$$(2a+b)(a-b)^2+(2b+c)(b-c)^2+(2c+a)(c-a)^2\ge\frac{8}{3}(a-b)(b-c)^2.$$

If $a\le b$, then

$$(2a+b)(a-b)^2+(2b+c)(b-c)^2+(2c+a)(c-a)^2\ge 0\ge\frac{8}{3}(a-b)(b-c)^2.$$

If $a\ge b$, then there are two cases to consider: $b\ge c$ and $b\le c$.
Case 1: $a\ge b\ge c$. It suffices to show that

$$(2c+a)(a-c)^2\ge\frac{8}{3}(a-b)(b-c)^2.$$

By the AM-GM inequality, we have

$$\begin{aligned}(a-b)(b-c)^2&=4(a-b)\left(\frac{b-c}{2}\right)\left(\frac{b-c}{2}\right)\\&\le 4\left[\frac{(a-b)+(b-c)/2+(b-c)/2}{3}\right]^3\\&=\frac{4}{27}(a-c)^3.\end{aligned}$$

Therefore, it suffices to show that

$$(2c+a)(a-c)^2\ge\frac{32}{81}(a-c)^3,$$

which is obvious.
Case 2: $a\ge b$, $c\ge b$. Making the substitution

$$a=b+p,\qquad c=b+q,\qquad p,q\ge 0,$$

the inequality becomes

$$(3b+2p)p^2+(3b+q)q^2+(3b+p+2q)(p-q)^2\ge\frac{8}{3}pq^2,$$

$$3[p^2+q^2+(p-q)^2]b+2p^3+q^3+(p+2q)(p-q)^2\ge\frac{8}{3}pq^2.$$

It suffices to show that

$$2p^3+q^3+(p+2q)(p-q)^2\ge\frac{8}{3}pq^2,$$

which is equivalent to

$$2p^3 + 2q^3 \geq \frac{34}{9}pq^2.$$

By the AM-GM inequality, we have

$$2p^3 + 2q^3 = 2p^3 + q^3 + q^3 \geq 3\sqrt[3]{2p^3q^6} \geq \frac{34}{9}pq^2,$$

because

$$3\sqrt[3]{2} > \frac{34}{9}.$$

The equality holds for $a = b = c$.

□

P 2.27. *If a, b, c are positive real numbers, then*

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3 + \frac{(a-c)^2}{ab+bc+ca}.$$

(Vasile Cîrtoaje and Vo Quoc Ba Can, 2008)

First Solution. By expanding, the inequality can be written as

$$b^2 + \frac{bc^2}{a} + \frac{ca^2}{b} + \frac{ab^2}{c} \geq 2ab + 2bc.$$

We can get this inequality by summing the AM-GM inequalities

$$ab + \frac{bc^2}{a} \geq 2bc,$$

$$b^2 + \frac{ca^2}{b} + \frac{ab^2}{c} \geq 3ab.$$

The equality holds for $a = b = c$.

Second Solution. From

$$\begin{aligned}
(a+b+c)\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a} - 3\right) &= \sum \frac{a^2}{b} + \sum \frac{bc}{a} - 2\sum a \\
&= \sum \left(\frac{a^2}{b} - 2a + b\right) + \sum \left(\frac{bc}{a} - b\right) \\
&= \sum \left(\frac{a^2}{b} - 2a + b\right) + \frac{1}{2}\sum \left(\frac{ab}{c} + \frac{ac}{b} - 2a\right) \\
&= \sum \frac{(a-b)^2}{b} + \frac{1}{2}\sum \frac{a(b-c)^2}{bc},
\end{aligned}$$

we get

$$(a+b+c)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right) \ge \frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(c-a)^2}{a}.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c} \ge \frac{(a-c)^2}{b+c}.$$

Therefore,

$$(a+b+c)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right) \ge \frac{(a-c)^2}{b+c}+\frac{(c-a)^2}{a},$$

which is equivalent to

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3 \ge \frac{(a-c)^2}{a(b+c)}.$$

From this result, the desired inequality follows immediately.

□

P 2.28. *If a, b, c are positive real numbers, then*

(a)
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3+\frac{4(a-c)^2}{(a+b+c)^2};$$

(b)
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3+\frac{5(a-c)^2}{(a+b+c)^2}.$$

(Vo Quoc Ba Can and Vasile Cîrtoaje, 2009)

Solution. As we have shown at the second solution of the previous problem P 2.27:

$$(a+b+c)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right) = \sum \frac{(a-b)^2}{b}+\frac{1}{2}\sum \frac{a(b-c)^2}{bc},$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3 \ge \frac{(a-c)^2}{a(b+c)}.$$

(a) According to the upper inequality, it suffices to show that

$$\frac{1}{a(b+c)} \ge \frac{4}{(a+b+c)^2}.$$

Indeed,

$$\frac{1}{a(b+c)}-\frac{4}{(a+b+c)^2} = \frac{(a-b-c)^2}{a(b+c)(a+b+c)^2} \ge 0.$$

The equality holds for $a = b = c$.

(b) According to the upper identity, write the inequality as

$$(a+b+c)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right) \ge \frac{5(a-c)^2}{a+b+c},$$

$$\sum \frac{(a-b)^2}{b} + \frac{1}{2}\sum \frac{a(b-c)^2}{bc} \ge \frac{5(a-c)^2}{a+b+c},$$

$$\frac{(a-b)^2}{b} + \frac{(b-c)^2}{c} + \frac{c(a-b)^2}{2ab} + \frac{a(b-c)^2}{2bc} \ge \left(\frac{5}{a+b+c} - \frac{1}{a} - \frac{b}{2ac}\right)(a-c)^2.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{b} + \frac{(b-c)^2}{c} \ge \frac{[(a-b)+(b-c)]^2}{b+c},$$

$$\frac{c(a-b)^2}{2ab} + \frac{a(b-c)^2}{2bc} \ge \frac{[(a-b)+(b-c)]^2}{\frac{2ab}{c}+\frac{2bc}{a}} = \frac{ac(a-c)^2}{2b(a^2+c^2)}.$$

Thus, we only need to show that

$$\frac{1}{b+c} + \frac{ac}{2b(a^2+c^2)} \ge \frac{5}{a+b+c} - \frac{1}{a} - \frac{b}{2ac},$$

which is equivalent to

$$\left(\frac{1}{a}+\frac{1}{b+c}\right) + \frac{ac}{2b(a^2+c^2)} + \frac{b}{2ac} \ge \frac{5}{a+b+c}.$$

This inequality is true because, by the AM-HM inequality and the AM-GM inequality, we have

$$\frac{1}{a}+\frac{1}{b+c} \ge \frac{4}{a+(b+c)}$$

and

$$\frac{ac}{2b(a^2+c^2)} + \frac{b}{2ac} \ge \frac{1}{\sqrt{a^2+c^2}} > \frac{1}{a+c} > \frac{1}{a+b+c}.$$

The equality holds for $a=b=c$.

□

P 2.29. *If $a \ge b \ge c > 0$, then*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3 + \frac{3(b-c)^2}{ab+bc+ca}.$$

First Solution. Since

$$\frac{a}{b}+\frac{c}{a}-1-\frac{c}{b}=\frac{(a-b)(a-c)}{ab}\geq 0,$$

it suffices to show that

$$\frac{b}{c}+\frac{c}{b}-2\geq \frac{3(b-c)^2}{ab+bc+ca}.$$

Indeed, we have

$$\frac{b}{c}+\frac{c}{b}-2-\frac{3(b-c)^2}{ab+bc+ca}=\frac{(b-c)^2(ab+ac-2bc)}{bc(ab+bc+ca)}.$$

The equality holds for $a=b=c$.

Second Solution. Since

$$ab+bc+ca\geq 3bc,$$

it suffices to show that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3+\frac{(b-c)^2}{bc},$$

which is equivalent to

$$\frac{a}{b}+\frac{c}{a}\geq 1+\frac{c}{b},$$

$$\frac{(a-b)(a-c)}{ab}\geq 0.$$

□

P 2.30. *Let a,b,c be positive real numbers such that $abc=1$. Prove that*

(a) if $a\geq b\geq 1\geq c$, *then*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3+\frac{2(a-b)^2}{ab};$$

(b) if $a\geq 1\geq b\geq c$, *then*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3+\frac{2(b-c)^2}{bc}.$$

(Vasile Cîrtoaje, 2010)

Solution. (a) Write the inequality as

$$f(c) \geq \frac{a}{b} + 2\frac{b}{a} - 1,$$

where

$$f(c) = \frac{b}{c} + \frac{c}{a}.$$

From

$$b^3 \geq 1 = abc,$$

we find

$$b^2 \geq ac.$$

We will show that

$$f(c) \geq f\left(\frac{b^2}{a}\right) \geq \frac{a}{b} + 2\frac{b}{a} - 1.$$

The left inequality is equivalent to

$$\frac{b}{c} + \frac{c}{a} \geq \frac{a}{b} + \frac{b^2}{a^2},$$

$$\frac{b^2 - ac}{bc} \geq \frac{b^2 - ac}{a^2} \geq 0,$$

$$(a^2 - bc)(b^2 - ac) \geq 0.$$

The right inequality reduces to

$$\left(\frac{b}{a} - 1\right)^2 \geq 0.$$

The equality holds for $a = b = c = 1$.

(b) Write the inequality as

$$f(a) \geq \frac{b}{c} + 2\frac{c}{b} - 1,$$

where

$$f(a) = \frac{a}{b} + \frac{c}{a}.$$

From

$$b^3 \leq 1 = abc,$$

we find

$$b^2 \leq ac.$$

We will show that

$$f(a) \geq f\left(\frac{b^2}{c}\right) \geq \frac{b}{c} + 2\frac{c}{b} - 1$$

The left inequality is equivalent to

$$\frac{a}{b} + \frac{c}{a} \geq \frac{b}{c} + \frac{c^2}{b^2},$$

$$\frac{ac-b^2}{bc} \ge \frac{c(ac-b^2}{ab^2} \ge 0,$$

$$(ab-c^2)(ac-b^2) \ge 0.$$

The right inequality reduces to

$$\left(\frac{c}{b}-1\right)^2 \ge 0.$$

The equality holds for $a=b=c=1$.

□

P 2.31. *Let a,b,c be positive real numbers such that*

$$a \ge 1 \ge b \ge c, \qquad abc = 1.$$

prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3+\frac{9(b-c)^2}{ab+bc+ca}.$$

(Vasile Cîrtoaje, 2010)

Solution. From $b^3 \le 1 = abc$, we find $b^2 \le ac$. We will show that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge \frac{2b}{c}+\frac{c^2}{b^2} \ge 3+\frac{9(b-c)^2}{ab+bc+ca}.$$

The left inequality is equivalent to

$$\frac{a}{b}+\frac{c}{a} \ge \frac{b}{c}+\frac{c^2}{b^2},$$

$$\frac{a}{b}-\frac{b}{c}+\left(\frac{c}{a}-\frac{c^2}{b^2}\right) \ge 0,$$

$$\frac{ac-b^2}{bc}+\frac{c(b^2-ac)}{ab^2} \ge 0,$$

$$\frac{(ac-b^2)(ab-c^2)}{ab^2c} \ge 0.$$

The right inequality is equivalent to

$$\frac{2b}{c}+\frac{c^2}{b^2}-3 \ge \frac{9(b-c)^2}{ab+bc+ca}.$$

$$\frac{(b-c)^2(2b+c)}{b^2c} \ge \frac{9(b-c)^2}{ab+bc+ca}.$$

We need to show that

$$\frac{(2b+c)}{b^2c} \ge \frac{9}{a(b+c)+bc}.$$

This is true if

$$\frac{(2b+c)}{b^2c} \geq \frac{9}{b(b+c)+bc},$$

which is equivalent to

$$\frac{2(b-c)^2}{b^2c(b+2c)} \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 2.32. *Let a, b, c be positive real numbers such that*

$$a \geq 1 \geq b \geq c, \qquad a+b+c = 3.$$

prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3 + \frac{4(b-c)^2}{b^2+c^2}.$$

(Vasile Cîrtoaje, 2010)

Solution. From

$$3b \leq 3 = a+b+c,$$

we find

$$2b \leq a+c, \qquad a \geq 2b-c.$$

We will show that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{2b-c}{b} + \frac{b}{c} + \frac{c}{2b-c} \geq 3 + \frac{4(b-c)^2}{b^2+c^2}.$$

The left inequality is equivalent to

$$\frac{a}{b} + \frac{c}{a} \geq \frac{2b-c}{b} + \frac{c}{2b-c},$$

$$\frac{a+c-2b}{b} - \frac{c(a+c-2b)}{a(2b-c)} \geq 0,$$

$$\frac{(a+c-2b)[a(b-c)+b(a-c)]}{ab(2b-c)} \geq 0.$$

The right inequality is equivalent to

$$\frac{(b-c)^2(2b+c)}{bc(2b-c)} \geq \frac{4(b-c)^2}{b^2+c^2}.$$

We need to show that

$$\frac{(2b+c)}{bc(2b-c)} \geq \frac{4}{b^2+c^2},$$

which is equivalent to

$$2b^3 - 7b^2c + 6bc^2 + c^3 \geq 0,$$

$$2b(b-2c)^2 + (b-c)^2c \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 2.33. *Let a, b, c be positive real numbers such that*

$$a \geq b \geq 1 \geq c, \qquad a + b + c = 3.$$

Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3 + \frac{3(a-b)^2}{ab}.$$

(Vasile Cîrtoaje, 2008)

Solution. From

$$3b \geq 3 = a + b + c,$$

we get

$$2b \geq a + c, \qquad c \leq 2b - a.$$

We will show that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a}{b} + \frac{b}{2b-a} + \frac{2b-a}{a} \geq 3 + \frac{3(a-b)^2}{ab}.$$

The left inequality is equivalent to

$$\frac{b}{c} + \frac{c}{a} \geq \frac{b}{2b-a} + \frac{2b-a}{a},$$

$$(2b-a-c)[b(a-c) + c(a-b)] \geq 0.$$

The right inequality is equivalent to

$$\frac{a}{b} + \frac{b}{2b-a} + \frac{2b-a}{a} - 3 \geq \frac{3(a-b)^2}{ab},$$

$$\frac{(a-b)^2(4b-a)}{ab(2b-a)} \geq \frac{3(a-b)^2}{ab},$$

$$\frac{2(a-b)^3}{ab(2b-a)} \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 2.34. *If a, b, c are positive real numbers, then*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3+\frac{2(a-c)^2}{(a+c)^2}.$$

Solution. Since

$$\frac{a}{b}+\frac{b}{c}\geq 2\sqrt{\frac{a}{c}},$$

it suffices to show that

$$\frac{c}{a}+2\sqrt{\frac{a}{c}}\geq 3+\frac{2(a-c)^2}{(a+c)^2}.$$

Using the substitution $x=\sqrt{\dfrac{a}{c}}$, this inequality becomes as follows:

$$\frac{1}{x^2}+2x\geq 3+\frac{2(x^2-1)^2}{(x^2+1)^2},$$

$$\frac{(x-1)^2(2x+1)}{x^2}\geq\frac{2(x^2-1)^2}{(x^2+1)^2}.$$

We need to show that

$$\frac{2x+1}{x^2}\geq\frac{2(x+1)^2}{(x^2+1)^2},$$

which is equivalent to

$$2x^5-3x^4+2x+1\geq 0.$$

For $0<x\leq 1$, we have

$$2x^5-3x^4+2x+1>-3x^4+2x+1\geq -3x+2x+1\geq 0.$$

Also, for $x\geq 1$, we have

$$2x^5-3x^4+2x+1>2x^5-3x^4+2x-1=(x-1)^2(2x^3+x^2-1)\geq 0.$$

The equality holds for $a=b=c$.

□

P 2.35. *If a, b, c are positive real numbers, then*

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq a+b+c+\frac{4(a-c)^2}{a+b+c}.$$

(Balkan MO, 2005, 2008)

Solution. Write the inequality as follows:

$$\left(\frac{a^2}{b}+b-2a\right)+\left(\frac{b^2}{c}+c-2b\right)+\left(\frac{c^2}{a}+a-2c\right)\ge\frac{4(a-c)^2}{a+b+c},$$

$$\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(a-c)^2}{a}\ge\frac{4(a-c)^2}{a+b+c}.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(a-c)^2}{a}\ge\frac{[(a-b)+(b-c)+(a-c)]^2}{b+c+a}=\frac{4(a-c)^2}{a+b+c}.$$

The equality holds for $a=b=c$, and also for $a=b+c$ and $\dfrac{b}{c}=\dfrac{1+\sqrt{5}}{2}$.

□

P 2.36. *If $a\ge b\ge c>0$, then*

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\ge a+b+c+\frac{6(b-c)^2}{a+b+c}.$$

(Vasile Cîrtoaje, 2014)

Solution. Write the inequality as follows:

$$\left(\frac{a^2}{b}+b-2a\right)+\left(\frac{b^2}{c}+c-2b\right)+\left(\frac{c^2}{a}+a-2c\right)\ge\frac{6(b-c)^2}{a+b+c},$$

$$\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(a-c)^2}{a}\ge\frac{6(b-c)^2}{a+b+c},$$

$$\frac{(a-b)^2}{b}+\frac{(a-c)^2}{a}+\frac{(a+b-5c)(b-c)^2}{c(a+b+c)}\ge 0.$$

Since

$$(a-c)^2=[(a-b)+(b-c)]^2=(a-b)^2+2(a-b)(b-c)+(b-c)^2,$$

we have

$$\frac{(a-b)^2}{b}+\frac{(a-c)^2}{a}\ge\frac{(a-c)^2}{a}\ge\frac{2(a-b)(b-c)+(b-c)^2}{a}.$$

Therefore, it suffices to show that

$$\frac{2(a-b)(b-c)+(b-c)^2}{a}+\frac{(a+b-5c)(b-c)^2}{c(a+b+c)}\ge 0,$$

which can be written as

$$\frac{2(a-b)(b-c)}{a}+\frac{(a-c)^2+ab+bc-2ca}{ac(a+b+c)}(b-c)^2\ge 0.$$

Since

$$(a-c)^2+ab+bc-2ca=(a-c)^2+a(b-c)-c(a-b)\geq -c(a-b),$$

it is enough to prove that

$$\frac{2(a-b)(b-c)}{a}-\frac{a-b}{a(a+b+c)}(b-c)^2\geq 0.$$

Indeed,

$$\frac{2(a-b)(b-c)}{a}-\frac{a-b}{a(a+b+c)}(b-c)^2=\frac{(a-b)(b-c)}{a}\left(2-\frac{b-c}{a+b+c}\right)\geq 0.$$

The equality holds for $a=b=c$.

□

P 2.37. *If $a\geq b\geq c>0$, then*

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}>5(a-b).$$

(Vasile Cîrtoaje, 2014)

Solution. Consider two cases: $a\leq 2b$ and $a\geq 2b$.

Case 1: $a\leq 2b$. It suffices to show that

$$\frac{a^2}{b}+\frac{b^2}{b}\geq 5(a-b),$$

which is equivalent to the obvious inequality

$$(2b-a)(3b-a)\geq 0.$$

Case 2: $a\geq 2b$. Since

$$\frac{b^2}{c}+\frac{c^2}{a}-b-\frac{b^2}{a}=(b-c)\left(\frac{b}{c}-\frac{b+c}{a}\right)$$

$$\geq (b-c)\left(\frac{b}{c}-\frac{b+c}{2b}\right)=\frac{(b-c)^2(2b+c)}{2bc}\geq 0,$$

it suffices to show that

$$\frac{a^2}{b}+b+\frac{b^2}{a}\geq 5(a-b),$$

which is equivalent to

$$x(x-2)(3-x)<1,$$

where $x=a/b\geq 2$. For the non-trivial case $2\leq x\leq 3$, we have

$$x(x-2)(3-x)\geq x\left[\frac{(x-2)+(3-x)}{2}\right]^2=\frac{x}{4}<1.$$

□

P 2.38. *Let a, b, c be positive real numbers such that*

$$a \geq b \geq 1 \geq c, \qquad a+b+c=3.$$

Prove that

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \geq 3+\frac{11(a-c)^2}{4(a+c)}.$$

(Vasile Cîrtoaje, 2010)

Solution. We have

$$a+b+c=3 \leq b, \qquad 2b \geq a+c.$$

Thus, we need to prove the homogeneous inequality

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \geq a+b+c+\frac{11(a-c)^2}{4(a+c)}$$

for

$$a \geq b \geq \frac{a+c}{2}.$$

Denote

$$f(a,b,c)=\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}-a-b-c.$$

We will show that

$$f(a,b,c) \geq f\left(a, \frac{a+c}{2}, c\right) \geq \frac{11(a-c)^2}{4(a+c)}.$$

Write the left inequality as follows:

$$\left(\frac{a^2}{b}-\frac{2a^2}{a+c}\right)+\left[\frac{b^2}{c}-\frac{(a+c)^2}{4c}\right]-\left(b-\frac{a+c}{2}\right) \geq 0,$$

$$(2b-a-c)\left[-\frac{a^2}{b(a+c)}+\frac{2b+a+c}{4c}-\frac{1}{2}\right] \geq 0.$$

Since $2b-a-c \geq 0$, we only need to show that

$$\frac{2b+a+c}{4c} \geq \frac{a^2}{b(a+c)}+\frac{1}{2}.$$

It suffices to prove this inequality for $b=\dfrac{a+c}{2}$. Making this, the inequality becomes

$$\frac{a(a-c)^2}{2c(a+c)^2} \geq 0.$$

To prove the right inequality, we find

$$f\left(a, \frac{a+c}{2}, c\right)=\frac{(a-c)^2(a^2+7ac+4c^2)}{4ac(a+c)},$$

hence

$$f\left(a,\frac{a+c}{2},c\right)-\frac{11(a-c)^2}{4(a+c)}=\frac{(a-c)^2(a-2c)^2}{4ac(a+c)}\ge 0.$$

The equality holds for $a=b=c=1$, and also for $\frac{a}{4}=\frac{b}{3}=\frac{c}{2}$ (that is, for $a=\frac{4}{3}$, $b=1$, $c=\frac{2}{3}$).

□

P 2.39. *If a,b,c are positive real numbers, then*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{3}{2}+\frac{27(b-c)^2}{16(a+b+c)^2}.$$

(Vasile Cîrtoaje, 2014)

Solution. Write the inequality as follows:

$$\sum\left(\frac{a}{b+c}+1\right)\ge\frac{9}{2}+\frac{27(b-c)^2}{16(a+b+c)^2},$$

$$\left[\sum(b+c)\right]\left(\sum\frac{1}{b+c}\right)\ge 9+\frac{27(b-c)^2}{2\left[\sum(b+c)\right]^2}.$$

Replacing $b+c$, $c+a$, $a+b$ by a, b, c, respectively, we need to show that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge 9+\frac{27(b-c)^2}{2(a+b+c)^2},$$

where a,b,c are the side-lengths of a non-degenerate triangle. Write this inequality in the form

$$\frac{a+b+c}{a}+(a+b+c)\left(\frac{1}{b}+\frac{1}{c}\right)+\frac{54bc}{(a+b+c)^2}\ge 9+\frac{27(b+c)^2}{2(a+b+c)^2}.$$

Applying the AM-GM inequality gives

$$(a+b+c)\left(\frac{1}{b}+\frac{1}{c}\right)+\frac{54bc}{(a+b+c)^2}\ge 6\sqrt{\frac{6(b+c)}{a+b+c}}.$$

Therefore, it suffices to show that

$$\frac{a+b+c}{a}+6\sqrt{\frac{6(b+c)}{a+b+c}}\ge 9+\frac{27(b+c)^2}{2(a+b+c)^2},$$

which can be rewritten as

$$\frac{1}{1-\dfrac{b+c}{a+b+c}}+6\sqrt{\frac{6(b+c)}{a+b+c}}\ge 9+\frac{27(b+c)^2}{2(a+b+c)^2}.$$

Using the substitution

$$\frac{b+c}{a+b+c} = \frac{2}{3}t^2, \quad t^2 > \frac{3}{4},$$

this inequality becomes

$$\frac{1}{3-2t^2} + 4t \ge 3 + 2t^4,$$

$$2t^6 - 3t^4 - 4t^3 + 3t^2 + 6t - 4 \ge 0,$$

$$(t-1)^2(2t^4 + 4t^3 + 3t^2 - 2t - 4) \ge 0,$$

$$(t-1)^2\left[(4t^2-3)(t^2+2t+2) + t^2 + 2t - 2\right] \ge 0.$$

Clearly, the last inequality is true for $t^2 > 3/4$. The original inequality is an equality for $a = b = c$.

□

P 2.40. *Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$. Prove that*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2} + \frac{9(b-c)^2}{4(a+b+c)^2}.$$

(Vasile Cîrtoaje, 2014)

Solution. Write the inequality as

$$\sum \left(\frac{a}{b+c} + 1\right) \ge \frac{9}{2} + \frac{9(b-c)^2}{4(a+b+c)^2},$$

$$\left[\sum(b+c)\right]\left(\sum \frac{1}{b+c}\right) \ge 9 + \frac{18(b-c)^2}{[(b+c)+(c+a)+(a+b)]^2}.$$

Replacing $b+c$, $c+a$, $a+b$ by a, b, c, respectively, we need to show that

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \ge 9 + \frac{18(b-c)^2}{(a+b+c)^2},$$

where a, b, c are the side-lengths of a non-degenerate triangle, $a = \max\{a, b, c\}$. Since

$$(a+b+c)^2 \ge \frac{9}{4}(b+c)^2 \ge 9bc,$$

it suffices to show that

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \ge 9 + \frac{2(b-c)^2}{bc}.$$

Write the inequality as follows:

$$\frac{(a-b)^2}{ab} + \frac{(a-c)^2}{ac} + \frac{(b-c)^2}{bc} \ge \frac{2(b-c)^2}{bc},$$

$$c(a-b)^2 + b(a-c)^2 \geq a(b-c)^2,$$

$$(b+c)a^2 - (b+c)^2 a + bc(b+c) \geq 0,$$

$$(b+c)(a-b)(a-c) \geq 0.$$

Clearly, the last inequality is true. The original inequality is an equality for $a = b = c$.

□

P 2.41. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2} + \frac{(b-c)^2}{2(b+c)^2}.$$

(Vasile Cîrtoaje, 2014)

First Solution. Write the inequality as follows:

$$\frac{2bc}{(b+c)^2} + \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq 2,$$

$$\frac{a(b+c)+2bc}{(b+c)^2} + \frac{b}{c+a} + \frac{c}{a+b} \geq 2,$$

By the Cauchy-Schwarz inequality, we have

$$\frac{b}{c+a} + \frac{c}{a+b} \geq \frac{(b+c)^2}{b(c+a)+c(a+b)} = \frac{(b+c)^2}{a(b+c)+2bc}.$$

Therefore, it suffices to prove that

$$\frac{a(b+c)+2bc}{(b+c)^2} + \frac{(b+c)^2}{a(b+c)+2bc} \geq 2,$$

which is obvious. The original inequality is an equality for $a = b = c$, for $a = b$ and $c = 0$, and for $a = c$ and $b = 0$.

Second Solution. Write the inequality as follows:

$$\sum \left(\frac{a}{b+c} + 1 \right) \geq \frac{9}{2} + \frac{(b-c)^2}{2(b+c)^2},$$

$$\left[\sum (b+c) \right] \left(\sum \frac{1}{b+c} \right) \geq 9 + \frac{(b-c)^2}{(b+c)^2}.$$

Replacing $b+c$, $c+a$, $a+b$ by a, b, c, respectively, we need to show that

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 9 + \frac{(b-c)^2}{a^2},$$

where a, b, c are the lengths of the sides of a triangle. Write this inequality as

$$\frac{(a-b)^2}{ab} + \frac{(a-c)^2}{ac} + \frac{(b-c)^2}{bc} \ge \frac{(b-c)^2}{a^2},$$

$$a[c(a-b)^2 + b(a-c)^2] \ge (bc-a^2)(b-c)^2.$$

Without loss of generality, assume that $b \ge c$. Since $a \ge b-c$, it suffices to show that

$$c(a-b)^2 + b(a-c)^2 \ge (bc-a^2)(b-c).$$

Indeed, we have

$$c(a-b)^2 + b(a-c)^2 - (bc-a^2)(b-c) = 2b(a-c)^2 \ge 0.$$

□

P 2.42. *Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$. Prove that*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2} + \frac{(b-c)^2}{4bc}.$$

(Vasile Cîrtoaje, 2014)

First Solution *(by Nguyen Van Quy).* Notice that for $a = \min\{a, b, c\}$, we have

$$4bc = (2b)(2c) \ge (a+b)(a+c) \ge 2a(b+c),$$

hence

$$\frac{a}{b+c} \ge \frac{2a^2}{(a+b)(a+c)}, \quad \frac{(b-c)^2}{4bc} \le \frac{(b-c)^2}{(a+b)(a+c)}.$$

So, it suffices to show that

$$\frac{2a^2}{(a+b)(a+c)} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2} + \frac{(b-c)^2}{(a+b)(a+c)},$$

which is equivalent to the obvious inequality

$$(a-b)(a-c) \ge 0.$$

The proof is completed. The original inequality is an equality for $a = b = c$.

Second Solution. Let

$$E(a, b, c) = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}.$$

Without loss of generality, assume that $b \le c$, hence $a \le b \le c$. We will show that

$$E(a, b, c) \ge E(b, b, c) \ge \frac{3}{2} + \frac{(b-c)^2}{4bc}.$$

We have

$$E(a,b,c) - E(b,b,c) = \frac{a-b}{b+c} + \frac{b(b-a)}{(a+c)(b+c)} + \frac{c(b-a)}{2b(a+b)}$$
$$= (b-a)\left[\frac{(b-a)-c}{(a+c)(b+c)} + \frac{c}{2b(a+b)}\right]$$
$$= \frac{(b-a)[2b(b^2-a^2)+c(c-b)(a+2b+c)]}{2b(a+b)(a+c)(b+c)} \ge 0$$

and

$$E(b,b,c) - \frac{3}{2} - \frac{(b-c)^2}{4bc} = \left(\frac{2b}{b+c} + \frac{c}{2b} - \frac{3}{2}\right) - \frac{(b-c)^2}{4bc}$$
$$= \frac{(b-c)^2}{2b(b+c)} - \frac{(b-c)^2}{4bc}$$
$$= \frac{(c-b)^3}{4bc(b+c)} \ge 0.$$

□

P 2.43. *Let a, b, c be positive real numbers such that*

$$a \le 1 \le b \le c, \qquad a+b+c = 3,$$

then

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2} + \frac{3(b-c)^2}{4bc}.$$

(Vasile Cîrtoaje, 2014)

Solution. From

$$3b \ge 3 = a+b+c,$$

we get

$$a \le 2b-c, \qquad 2b > c.$$

Let

$$E(a,b,c) = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}.$$

We will show that

$$E(a,b,c) \ge E(2b-c,b,c) \ge \frac{3}{2} + \frac{3(b-c)^2}{4bc}.$$

We have

$$E(a,b,c) - E(2b-c,b,c) = (2b-a-c)F,$$

where

$$F = \frac{-1}{b+c} + \frac{1}{2(c+a)} + \frac{c}{(a+b)(3b-c)}.$$

Since $2b-a-c\ge 0$, we need to show that $F\ge 0$. This is true because

$$\begin{aligned}F &= \frac{1}{2}\left(-\frac{1}{b+c}+\frac{1}{c+a}\right)-\frac{1}{2(b+c)}+\frac{c}{(a+b)(3b-c)}\\ &\ge -\frac{1}{2(b+c)}+\frac{c}{(a+b)(3b-c)} \ge -\frac{1}{2(a+b)}+\frac{c}{(a+b)(3b-c)}\\ &= \frac{3(c-b)}{2(a+b)(3b-c)}\ge 0.\end{aligned}$$

In what concerns the right inequality, we have

$$\begin{aligned}E(2b-c,b,c)-\frac{3}{2}-\frac{3(b-c)^2}{4bc} &= 3(b-c)^2\left[\frac{1}{(b+c)(3b-c)}-\frac{1}{4bc}\right]\\ &= \frac{-3(b-c)^3(3b+c)}{4bc(b+c)(3b-c)}\ge 0.\end{aligned}$$

The proof is completed. The original inequality is an equality for $a=b=c=1$.

□

P 2.44. *Let a,b,c be nonnegative real numbers such that*

$$a\ge 1\ge b\ge c,\qquad a+b+c=3,$$

then

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{3}{2}+\frac{(b-c)^2}{(b+c)^2}.$$

(Vasile Cîrtoaje, 2014)

Solution. From

$$3b\le 3=a+b+c,$$

we get

$$a\ge 2b-c.$$

Let

$$E(a,b,c)=\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

We will show that

$$E(a,b,c)\ge E(2b-c,b,c)\ge\frac{3}{2}+\frac{(b-c)^2}{(b+c)^2}.$$

We have

$$E(a,b,c)-E(2b-c,b,c)=(a-2b+c)F,$$

where

$$F=\frac{1}{b+c}-\frac{1}{2(c+a)}-\frac{c}{(a+b)(3b-c)}.$$

Since $a-2b+c\ge 0$, we need to show that $F\ge 0$. This is true because

$$\begin{aligned} F &= \frac{1}{2}\left(\frac{1}{b+c}-\frac{1}{c+a}\right)+\frac{1}{2(b+c)}-\frac{c}{(a+b)(3b-c)} \\ &\ge \frac{1}{2(b+c)}-\frac{c}{(a+b)(3b-c)} \ge \frac{1}{2(a+b)}-\frac{c}{(a+b)(3b-c)} \\ &= \frac{3(b-c)}{2(a+b)(3b-c)} \ge 0. \end{aligned}$$

The right inequality is also true because

$$\begin{aligned} E(2b-c,b,c)-\frac{3}{2}-\frac{(b-c)^2}{(b+c)^2} &= \frac{(b-c)^2}{b+c}\left[\frac{3}{3b-c}-\frac{1}{b+c}\right] \\ &= \frac{4c(b-c)^2}{(b+c)^2(3b-c)} \ge 0. \end{aligned}$$

The proof is completed. The original inequality is an equality for $a=b=c=1$, and also for $a=2$, $b=1$, $c=0$.

□

P 2.45. *Let a,b,c be positive real numbers such that $a=\min\{a,b,c\}$. Prove that*

(a) $$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{2(b-c)^2}{3(b^2+c^2)}\le 1;$$

(b) $$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(b-c)^2}{b^2+bc+c^2}\le 1;$$

(c) $$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(a-b)^2}{2(a^2+b^2)}\le 1.$$

(Vasile Cîrtoaje, 2014)

Solution. (a) *First Solution.* Since

$$3(b^2+c^2)\ge 2(a^2+b^2+c^2),$$

it suffices to show that

$$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(b-c)^2}{a^2+b^2+c^2}\le 1.$$

This inequality is equivalent to

$$(a-b)(a-c)\ge 0,$$

which is clearly true. The equality holds for $a=b=c$.

Second Solution. Write the inequality as follows:

$$\frac{4(b-c)^2}{3(b^2+c^2)}\le\frac{(b-c)^2+(a-b)^2+(a-c)^2}{a^2+b^2+c^2},$$

$$3(b^2+c^2)[(a-b)^2+(a-c)^2] \ge (b-c)^2(4a^2+b^2+c^2),$$

$$3(b^2+c^2)[(b-c)^2+2(a-b)(a-c)] \ge (b-c)^2(4a^2+b^2+c^2),$$

$$6(b^2+c^2)(a-b)(a-c)+2(b-c)^2(b^2+c^2-2a^2) \ge 0.$$

The last inequality is true because $(a-b)(a-c) \ge 0$ and $b^2+c^2-2a^2 \ge 0$.

(b) Without loss of generality, assume that

$$a \le b \le c.$$

Write the inequality as

$$\frac{ab+bc+ca}{a^2+b^2+c^2} \le \frac{3bc}{b^2+bc+c^2};$$

that is,

$$E(a,b,c) \ge 0,$$

where

$$E(a,b,c) = 3bca^2-(b+c)(b^2+c^2+bc)a+bc(2b^2+2c^2-bc).$$

We will show that

$$E(a,b,c) \ge E(b,b,c) \ge 0.$$

We have

$$\begin{aligned} E(a,b,c)-E(b,b,c) &= 3bc(a^2-b^2)-(b+c)(b^2+c^2+bc)(a-b) \\ &= (b-a)[(b+c)(b^2+c^2+bc)-3bc(a+b)] \\ &\ge (b-a)[(b+c)(b^2+c^2+bc)-3bc(c+b)] \\ &= (b-a)(b+c)(b-c)^2 \ge 0. \end{aligned}$$

Also,

$$E(b,b,c) = b(c-b)^3 \ge 0.$$

The equality holds for $a=b=c$, and also for $a=b=0$ or $a=c=0$.

(c) Write the inequality as follows:

$$\frac{ab+(a+b)c}{a^2+b^2+c^2} \le \frac{(a+b)^2}{2(a^2+b^2)},$$

$$(a+b)^2c^2-2(a+b)(a^2+b^2)c+(a^2+b^2)^2 \ge 0,$$

$$[(a+b)c-(a^2+b^2)]^2 \ge 0.$$

The equality holds for $c = \dfrac{a^2+b^2}{a+b}$.

□

P 2.46. *Let a, b, c be positive real numbers such that*

$$a \le 1 \le b \le c, \qquad a+b+c=3,$$

then

$$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(b-c)^2}{bc} \le 1.$$

(Vasile Cîrtoaje, 2014)

Solution. From

$$3b \ge 3 = a+b+c,$$

we get

$$a \le 2b-c.$$

Write the inequality as follows:

$$\frac{2(b-c)^2}{bc} \le \frac{(b-c)^2+(a-b)^2+(a-c)^2}{a^2+b^2+c^2},$$

$$(b-a)^2+(c-a)^2 \ge \left(\frac{2a^2+2b^2+2c^2}{bc}-1\right)(c-b)^2,$$

$$(c-b)^2+2(b-a)(c-a) \ge \left(\frac{2a^2+2b^2+2c^2}{bc}-1\right)(c-b)^2,$$

$$(b-a)(c-a) \ge \left(\frac{a^2+b^2+c^2}{bc}-1\right)(c-b)^2.$$

Since

$$b-a \ge b-(2b-c)=c-b \ge 0, \qquad c-a \ge c-(2b-c)=2(c-b) \ge 0,$$

it suffices to show that

$$2 \ge \frac{a^2+b^2+c^2}{bc}-1,$$

which is equivalent to

$$3bc \ge a^2+b^2+c^2.$$

This is true if

$$3bc \ge (2b-c)^2+b^2+c^2,$$

which reduces to

$$7bc \ge 5b^2+2c^2,$$

$$(c-b)(5b-2c) \ge 0.$$

Thus, we only need to show that $5b-2c \ge 0$. Indeed, we have

$$5b-2c > 2(2b-c) \ge 2a > 0.$$

The equality holds for $a=b=c=1$.

□

P 2.47. *Let a, b, c be nonnegative real numbers such that $a = \max\{a, b, c\}$ and $b + c > 0$. Prove that*

(a) $$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{(b-c)^2}{2(ab+bc+ca)} \le 1;$$

(b) $$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{2(b-c)^2}{(a+b+c)^2} \le 1.$$

(Vasile Cîrtoaje, 2014)

Solution. Without loss of generality, assume that $a \ge b \ge c$.

(a) Write the inequality as follows:

$$\frac{(b-c)^2}{ab+bc+ca} \le \frac{(b-c)^2+(a-b)^2+(a-c)^2}{a^2+b^2+c^2},$$

$$(ab+bc+ca)[(a-b)^2+(a-c)^2] \ge (b-c)^2(a^2+b^2+c^2-ab-bc-ca).$$

Since

$$ab+bc+ca \ge ab \ge b^2 \ge (b-c)^2,$$

it suffices to show that

$$(a-b)^2+(a-c)^2 \ge a^2+b^2+c^2-ab-bc-ca.$$

Indeed,

$$(a-b)^2+(a-c)^2-(a^2+b^2+c^2-ab-bc-ca) = (a-b)(a-c) \ge 0.$$

The equality holds for $a = b = c$, for $a = b$ and $c = 0$, and for $a = c$ and $b = 0$.

(b) Write the inequality as follows:

$$\frac{4(b-c)^2}{(a+b+c)^2} \le \frac{(b-c)^2+(a-b)^2+(a-c)^2}{a^2+b^2+c^2},$$

$$(a+b+c)^2[(a-b)^2+(a-c)^2] \ge (b-c)^2[3(a^2+b^2+c^2)-2(ab+bc+ca)],$$

$$(a+b+c)^2[(b-c)^2+2(a-b)(a-c)] \ge (b-c)^2[3(a^2+b^2+c^2)-2(ab+bc+ca)],$$

$$(a+b+c)^2(a-b)(a-c) \ge (b-c)^2[a^2+b^2+c^2-2(ab+bc+ca)].$$

Since

$$a^2+b^2+c^2-2(ab+bc+ca) = (a-b)^2-c(2a+2b-c) \le (a-b)^2,$$

it suffices to show that

$$(a+b+c)^2(a-c) \ge (b-c)^2(a-b).$$

This inequality is true because

$$(a+b+c)^2 \ge (b-c)^2$$

and

$$a-c \ge a-b.$$

The equality holds for $a = b = c$, for $a = b$ and $c = 0$, and for $a = c$ and $b = 0$.

□

P 2.48. *Let a, b, c be positive real numbers. Prove that*

(a) if $a \ge b \ge c$, then

$$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{(a-c)^2}{a^2-ac+c^2} \ge 1;$$

(b) if $a \ge 1 \ge b \ge c$ and $abc = 1$, then

$$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{(b-c)^2}{b^2-bc+c^2} \le 1.$$

(Vasile Cîrtoaje, 2014)

Solution. (a) Write the inequality as follows:

$$\frac{ab+bc+ca}{a^2+b^2+c^2} \ge \frac{ac}{a^2-ac+c^2},$$

$$acb^2 - (a+c)(a^2-ac+c^2)b + a^2c^2 \le 0,$$

$$acb^2 - (a^3+c^3)b + a^2c^2 \le 0,$$

$$(ab-c^2)(bc-a^2) \le 0.$$

Because $ab-c^2 \ge 0$ and $bc-a^2 \le 0$, the conclusion follows. The equality holds for $a = b = c$.

(b) From

$$b^3 \le 1 = abc,$$

it follows that

$$b^2 \le ac.$$

Write the inequality as follows:

$$\frac{ab+bc+ca}{a^2+b^2+c^2} \le \frac{bc}{b^2-bc+c^2},$$

$$bca^2 - (b+c)(b^2-bc+c^2)a + b^2c^2 \ge 0,$$

$$bca^2 - (b^3+c^3)a + b^2c^2 \ge 0,$$

$$(ab-c^2)(ac-b^2) \ge 0.$$

The inequality is true because $ab-c^2 \ge 0$ and $ac-b^2 \ge 0$. The equality holds for $a = b = c = 1$.

□

P 2.49. *Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$. Prove that*

(a)
$$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1 + \frac{4(b-c)^2}{3(b+c)^2};$$

(b)
$$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1 + \frac{(a-b)^2}{(a+b)^2}.$$

(Vasile Cîrtoaje, 2014)

Solution. (a) *First Solution.* Since

$$3(b+c)^2 \ge 12bc \ge 4(ab+bc+ca),$$

it suffices to prove that

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1 + \frac{(b-c)^2}{ab+bc+ca},$$

which is equivalent to the obvious inequality

$$(a-b)(a-c) \ge 0.$$

The equality holds for $a = b = c$.

Second Solution. Since $(b+c)^2 \ge 4bc$, it suffices to prove that

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1 + \frac{(b-c)^2}{3bc}.$$

Write this inequality as follows:

$$\frac{(a-b)^2+(a-c)^2+(b-c)^2}{ab+bc+ca} \ge \frac{2(b-c)^2}{3bc},$$

$$3bc[(a-b)^2+(a-c)^2] \ge (b-c)^2(2ab+2ac-bc),$$

$$3bc[(b-c)^2+2(a-b)(a-c)] \ge (b-c)^2(2ab+2ac-bc),$$

$$6bc(a-b)(a-c)+2(b-c)^2(2bc-ab-ac) \ge 0.$$

The last inequality is true because $(a-b)(a-c) \ge 0$ and

$$2bc-ab-ac = b(c-a)+c(b-a) \ge 0.$$

(b) Write the inequality as follows:

$$\frac{a^2+b^2+c^2}{ab+(a+b)c} \ge \frac{2(a^2+b^2)}{(a+b)^2},$$

$$(a+b)^2c^2 - 2(a+b)(a^2+b^2)c + (a^2+b^2)^2 \ge 0,$$

$$[(a+b)c-(a^2+b^2)]^2 \ge 0.$$

The equality holds for $c = \dfrac{a^2+b^2}{a+b}$.

□

P 2.50. *If a, b, c are positive real numbers, then*

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1+\frac{9(a-c)^2}{4(a+b+c)^2}.$$

(Vasile Cîrtoaje, 2014)

Solution. Write the inequality as follows:

$$\frac{(b-c)^2+(a-b)^2+(a-c)^2}{ab+bc+ca} \ge \frac{9(a-c)^2}{2(a+b+c)^2},$$

$$2(a+b+c)^2[(b-c)^2+(a-b)^2] \ge (a-c)^2[5(ab+bc+ca)-2(a^2+b^2+c^2)],$$

$$2(a+b+c)^2[(a-c)^2-2(a-b)(b-c)] \ge (a-c)^2[5(ab+bc+ca)-2(a^2+b^2+c^2)],$$

$$(a-c)^2[4(a^2+b^2+c^2)-(ab+bc+ca)] \ge 4(a+b+c)^2(a-b)(a-c).$$

Consider further the nontrivial case $(a-b)(a-c) \ge 0$. Since

$$(a-c)^2 = [(a-b)+(b-c)]^2 \ge 4(a-b)(b-c),$$

it suffices to show that

$$4(a^2+b^2+c^2)-(ab+bc+ca) \ge (a+b+c)^2.$$

Indeed,

$$4(a^2+b^2+c^2)-(ab+bc+ca)-(a+b+c)^2 = 3(a^2+b^2+c^2-ab-bc-ca) \ge 0.$$

The equality holds for $a = b = c$.

□

P 2.51. *Let a, b, c be nonnegative real numbers, no two of which are zero. If $a = \min\{a, b, c\}$, then*

$$\frac{1}{\sqrt{a^2-ab+b^2}}+\frac{1}{\sqrt{b^2-bc+c^2}}+\frac{1}{\sqrt{c^2-ca+a^2}} \ge \frac{6}{b+c}.$$

Solution. Since

$$\frac{1}{\sqrt{a^2-ab+b^2}}+\frac{1}{\sqrt{b^2-bc+c^2}}+\frac{1}{\sqrt{c^2-ca+a^2}} \ge \frac{1}{b}+\frac{1}{\sqrt{b^2-bc+c^2}}+\frac{1}{c},$$

it suffices to show that

$$\frac{1}{b}+\frac{1}{\sqrt{b^2-bc+c^2}}+\frac{1}{c} \ge \frac{6}{b+c}.$$

Write this inequality as

$$\frac{b}{c}+\frac{c}{b}+\sqrt{\frac{b^2+c^2+2bc}{b^2+c^2-bc}} \ge 4,$$

which is equivalent to

$$\sqrt{\frac{x+2}{x-1}} \geq 4-x,$$

where $x = \dfrac{b}{c} + \dfrac{c}{b}$, $x \geq 2$. Consider the non-trivial case $2 \leq x \leq 4$. The inequality is true if

$$\frac{x+2}{x-1} \geq (4-x)^2,$$

which is equivalent to

$$(x-2)(x^2-7x+9) \leq 0.$$

This inequality is true because

$$x^2-7x+9 < x^2-7x+10 = (x-2)(x-5) \leq 0.$$

The equality holds for $a=b=c$, and also $a=0$ and $b=c$.

□

P 2.52. *If $a \geq 1 \geq b \geq c \geq 0$ such that*

$$ab+bc+ca = abc+2,$$

then

$$ac \leq 4-2\sqrt{2}.$$

(Vasile Cîrtoaje, 2012)

Solution. By hypothesis, we have

$$a = \frac{2-bc}{b+c-bc}.$$

Since

$$ac \leq \frac{1}{2}a(b+c) = \frac{(2-bc)(b+c)}{2(b+c-bc)} = \frac{2-bc}{2-\frac{2bc}{b+c}} \leq \frac{2-bc}{2-\sqrt{bc}},$$

it suffices to show that

$$\frac{2-bc}{2-\sqrt{bc}} \leq 4-2\sqrt{2},$$

which is equivalent to

$$(\sqrt{bc}-2+\sqrt{2})^2 \geq 0.$$

The equality holds for $a=2$ and $b=c=2-\sqrt{2}$.

□

P 2.53. *If a, b, c are nonnegative real numbers such that*

$$ab + bc + ca = 3, \qquad a \le 1 \le b \le c,$$

then

(a) $$a + b + c \le 4;$$

(b) $$2a + b + c \le 4.$$

Solution. From

$$(1 - b)(1 - c) \ge 0,$$

we get

$$bc \ge b + c - 1.$$

Therefore, we have

$$3 = a(b + c) + bc \ge a(b + c) + b + c - 1 = (a + 1)(b + c) - 1,$$

$$b + c \le \frac{4}{a + 1},$$

hence

$$a + b + c - 4 \le a + \frac{4}{a + 1} - 4 = \frac{a(a - 3)}{a + 1} \le 0,$$

$$2a + b + c - 4 \le 2a + \frac{4}{a + 1} - 4 = \frac{2a(a - 1)}{a + 1} \le 0.$$

The equality holds for $a = 0$, $b = 1$ and $c = 3$. In addition, the inequality (b) is also an equality for $a = b = c = 1$.

□

P 2.54. *Let a, b, c be nonnegative real numbers such that $a \le b \le c$. Prove that*

(a) *if* $a + b + c = 3$, *then*

$$a^4(b^4 + c^4) \le 2;$$

(b) *if* $a + b + c = 2$, *then*

$$c^4(a^4 + b^4) \le 1.$$

(Vasile Cîrtoaje, 2012)

Solution. (a) Let x, y be nonnegative real numbers. We claim that

$$x^4 - y^4 \ge 4y^3(x - y).$$

Indeed, this inequality follows from

$$\begin{aligned} x^4 - y^4 - 4y^3(x - y) &= (x - y)(x^3 + x^2y + xy^2 - 3y^3) \\ &= (x - y)[(x^3 - y^3) + y(x^2 - y^2) + y^2(x - y)]. \end{aligned}$$

Using this inequality, we can show that

$$b^4 + c^4 \le a^4 + (b + c - a)^4.$$

Indeed, we have

$$\begin{aligned} a^4 + (b + c - a)^4 - b^4 - c^4 &= (a^4 - b^4) + (b + c - a)^4 - c^4 \\ &\ge 4b^3(a - b) + 4c^3(b + c - a - c) \\ &= 4(a - b)(b^3 - c^3) \ge 0. \end{aligned}$$

Thus, it suffices to show that

$$a^4[a^4 + (b + c - a)^4] \le 2,$$

which is equivalent to $f(a) \le 2$, where

$$f(a) = a^8 + a^4(3 - 2a)^4, \quad 0 \le a \le 1.$$

If $f'(a) \ge 0$ for $0 \le a \le 1$, then $f(a)$ is increasing, hence $f(a) \le f(1) = 2$. From the derivative

$$f'(a) = 4a^3[2a^4 - (4a - 3)(3 - 2a)^3],$$

we need to show that

$$2a^4 \ge (4a - 3)(3 - 2a)^3.$$

This inequality is true for the trivial case $0 \le a \le 3/4$. Consider further that $3/4 < a \le 1$. We need to show that $h(a) \ge 0$, where

$$h(a) = \ln 2 + 4 \ln a - \ln(4a - 3) - 3 \ln(3 - 2a), \quad 3/4 < a \le 1.$$

From

$$h'(a) = \frac{4}{a} - \frac{4}{4a - 3} + \frac{6}{3 - 2a} = \frac{6(7a - 6)}{a(4a - 3)(3 - 2a)},$$

it follows that $h(a)$ is decreasing on $(3/4, 6/7]$ and increasing on $[6/7, 1]$. Thus,

$$h(a) \ge h\left(\frac{6}{7}\right) = \ln 2 + 4 \ln \frac{6}{7} - \ln \frac{3}{7} - 3 \ln \frac{9}{7} = \ln \frac{32}{27} > 0.$$

The equality holds for $a = b = c = 1$.

(b) Since $a^4 + b^4 \le (a + b)^4$, it suffices to show that

$$c^4(a + b)^4 \le 1,$$

which is true if

$$c(a + b) \le 1.$$

Indeed, we have

$$1 - c(a + b) = 1 - c(2 - c) = (c - 1)^2 \ge 0.$$

The equality holds for $a = 0$ and $b = c = 1$.

□

P 2.55. *Let a, b, c be nonnegative real numbers such that*

$$a \le b \le c, \qquad a+b+c=3.$$

Find the greatest real number k such that

$$\sqrt{(56b^2+25)(56c^2+25)} + k(b-c)^2 \le 14(b+c)^2+25.$$

(Vasile Cîrtoaje, 2014)

Solution. For $a=b=0$ and $c=3$, the inequality becomes

$$115+9k \le 126+25, \qquad k \le 4.$$

To show that 4 is the greatest possible value of k, we need to prove the inequality

$$\sqrt{(56b^2+25)(56c^2+25)} + 4(b-c)^2 \le 14(b+c)^2+25,$$

which is equivalent to

$$\sqrt{(56b^2+25)(56c^2+25)} \le 10(b^2+c^2)+36bc+25.$$

By squaring, the inequality becomes as follows:

$$(10b^2+10c^2+36bc)^2 - 56^2b^2c^2 \ge 50[28(b^2+c^2)-(10b^2+10c^2+36bc)],$$

$$20(b-c)^2(5b^2+5c^2+46bc) \ge 900(b-c)^2,$$

$$20(b-c)^2(5b^2+5c^2+46bc-45) \ge 0.$$

Therefore, we need to show that

$$5(b+c)^2+36bc-45 \ge 0.$$

From $(a-b)(a-c) \ge 0$, we get

$$bc \ge a(b+c)-a^2 = a(3-a)-a^2 = 3a-2a^2.$$

Thus,

$$5(b+c)^2+36bc-45 \ge 5(3-a)^2+36(3a-2a^2)-45 = a(78-67a) \ge 0.$$

The proof is completed. If $k=4$, then the equality holds for $a=b=c=1$ and also for $a=b=0$ and $c=3$.

□

P 2.56. *If $a \ge b \ge c > 0$ such that $abc=1$, then*

$$3(a+b+c) \le 8+\frac{a}{c}.$$

(Vasile Cîrtoaje, 2009)

Solution. Write the inequality in the homogeneous form

$$\frac{3(a+b+c)}{\sqrt[3]{abc}} \le 8 + \frac{a}{c},$$

which is equivalent to

$$\frac{3(x^3+y^3+z^3)}{xyz} \le 8 + \frac{x^3}{z^3}, \qquad x \ge y \ge z > 0.$$

We show that

$$\frac{x^3+y^3+z^3}{xyz} \le \frac{x^3+2z^3}{xz^2} \le \frac{1}{3}\left(8+\frac{x^3}{z^3}\right).$$

Write the left inequality as

$$(y-z)[x^3+z^3-yz(y+z)] \ge 0.$$

This is true since

$$x^3+z^3-yz(y+z) \ge y^3+z^3-yz(y+z) = (y+z)(y-z)^2 \ge 0.$$

Write the right inequality as

$$(x-z)(x^3-2x^2z-2xz^2+6z^3) \ge 0.$$

This is also true since

$$x^3-2x^2z-2xz^2+6z^3 = (x-z)^3 + z(x^2-5xz+7z^2) \ge 0.$$

The equality holds for $a=b=c=1$.

□

P 2.57. *If $a \ge b \ge c > 0$, then*

$$(a+b-c)(a^2b-b^2c+c^2a) \ge (ab-bc+ca)^2.$$

Solution. Making the substitution

$$a=(p+1)c, \quad b=(q+1)c, \quad p \ge q \ge 0,$$

we get

$$a+b-c=(p+q+1)c,$$

$$a^2b-b^2c+c^2a=(p^2q+p^2+2pq-q^2+3p-q+1)c^3,$$

$$ab-bc+ca=(pq+2p+1)c^2.$$

Thus, the inequality becomes

$$(p+q+1)(p^2q+p^2+2pq-q^2+3p-q+1) \geq (pq+2p+1)^2,$$

which is equivalent to the obvious inequality

$$p^3(q+1)+q^2(p-q)+2q(p-q) \geq 0.$$

The equality holds for $a=b=c$.

□

P 2.58. *If $a \geq b \geq c > 0$, then*

$$\frac{ab+bc}{a^2+b^2+c^2} \leq \frac{1+\sqrt{3}}{4}.$$

Solution. Denote

$$k=\frac{1+\sqrt{3}}{4} \approx 0.683,$$

and write the inequality as $E(a,b,c) \geq 0$, where

$$E(a,b,c)=k(a^2+b^2+c^2)-ab-bc.$$

We show that

$$E(a,b,c) \geq E(b,b,c) \geq 0.$$

We have

$$E(a,b,c)-E(b,b,c)=(a-b)[ka-(1-k)b] \geq (2k-1)(a-b)b \geq 0$$

and

$$E(b,b,c)=(2k-1)b^2+kc^2-bc \geq 2\sqrt{k(2k-1)}bc-bc=0.$$

The equality holds for $a=b=\frac{1+\sqrt{3}}{2}c$.

□

P 2.59. *If $a \geq b \geq c \geq d > 0$, then*

$$\frac{ab+bc+cd}{a^2+b^2+c^2+d^2} \leq \frac{2+\sqrt{7}}{6}.$$

Solution. Write the inequality as $E(a,b,c,d) \ge 0$, where

$$E(a,b,c,d) = k(a^2+b^2+c^2+d^2) - ab - bc - cd, \qquad k = \frac{2+\sqrt{7}}{6} \approx 0.774.$$

We show that

$$E(a,b,c,d) \ge E(b,b,c,d) \ge E(c,c,c,d) \ge 0.$$

We have

$$E(a,b,c,d) - E(b,b,c,d) = (a-b)[ka-(1-k)b] \ge (2k-1)(a-b)b \ge 0,$$

$$E(b,b,c,d) - E(c,c,c,d) = (b-c)[(2k-1)b-(2-2k)c] \ge (4k-3)(b-c)c \ge 0$$

and

$$E(c,c,c,d) = (3k-2)c^2 + kd^2 - cd \ge 2\sqrt{k(3k-2)}cd - cd = 0.$$

The equality holds for $a = b = c = \dfrac{2+\sqrt{7}}{3}d$.

□

P 2.60. *If*

$$a \ge 1 \ge b \ge c \ge d \ge 0, \qquad a+b+c+d = 4,$$

then

$$ab+bc+cd \le 3.$$

Solution. Write the inequality in the homogeneous form $E(a,b,c,d) \ge 0$, where

$$E(a,b,c,d) = 3(a+b+c+d)^2 - 16(ab+bc+cd).$$

From

$$a+b+c+d = 4 \ge 4b,$$

we get

$$a \ge 3b-c-d.$$

We will show that

$$E(a,b,c,d) \ge E(3b-c-d,b,c,d) \ge 0.$$

We have

$$\begin{aligned}E(a,b,c,d) - E(3b-c-d,b,c,d) &= 3[(a+b+c+d)^2 - (4b)^2] - 16b(a-3b+c+d)\\ &= (a-3b+c+d)(3a-b+3c+3d) \ge 0.\end{aligned}$$

Also,

$$E(3b-c-d,b,c,d) = 48b^2 - 16(3b^2-bd+cd) = 16d(b-c) \ge 0.$$

The equality holds for

$$a \in [2,3], \quad b = 1, \quad c = 3-a, \quad d = 0.$$

□

P 2.61. *Let k and a,b,c be positive real numbers, and let*

$$E=(ka+b+c)\left(\frac{k}{a}+\frac{1}{b}+\frac{1}{c}\right),\qquad F=(ka^2+b^2+c^2)\left(\frac{k}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right).$$

(a) If $k\ge 1$, then

$$\sqrt{\frac{F-(k-2)^2}{2k}}+2\ge\frac{E-(k-2)^2}{2k};$$

(b) If $0<k\le 1$, then

$$\sqrt{\frac{F-k^2}{k+1}}+2\ge\frac{E-k^2}{k+1}.$$

(Vasile Cîrtoaje, 2007)

Solution. Due to homogeneity, we may assume that $bc=1$. Under this assumption, if we denote

$$x=a+\frac{1}{a},\qquad y=b+\frac{1}{b}=c+\frac{1}{c}$$

($x\ge 2$, $y\ge 2$), then

$$\begin{aligned}E&=\left(ka+b+\frac{1}{b}\right)\left(\frac{k}{a}+b+\frac{1}{b}\right)\\&=(ka+y)\left(\frac{k}{a}+y\right)\\&=k^2+kxy+y^2\end{aligned}$$

and

$$\begin{aligned}F&=\left(ka^2+b^2+\frac{1}{b^2}\right)\left(\frac{k}{a^2}+b^2+\frac{1}{b^2}\right)\\&=(ka^2+y^2-2)\left(\frac{k}{a^2}+y^2-2\right)\\&=k^2+k(x^2-2)(y^2-2)+(y^2-2)^2.\end{aligned}$$

(a) Write the inequality as

$$2kF-2k(k-2)^2\ge(E-k^2-4)^2.$$

We have

$$E-k^2-4=kxy+y^2-4>0,$$

$$(E-k^2-4)^2=k^2x^2y^2+2kxy(y^2-4)+(y^2-4)^2,$$

and

$$F-(k-2)^2 = 4k + k(x^2-2)(y^2-2) + y^2(y^2-4),$$

$$2kF - 2k(k-2)^2 = 8k^2 + 2k^2(x^2-2)(y^2-2) + 2ky^2(y^2-4).$$

Therefore,

$$2kF - 2k(k-2)^2 - (E-k^2-4)^2 = (y^2-4)[k^2(x^2-4) - 2ky(x-y) - (y^2-4)].$$

Since $y^2-4 \geq 0$, we still need to show that

$$k^2(x^2-4) - 2ky(x-y) \geq y^2-4.$$

We will show that

$$k^2(x^2-4) - 2ky(x-y) \geq (x^2-4) - 2y(x-y) \geq y^2-4.$$

The right inequality reduces to $(x-y)^2 \geq 0$, and the left inequality is equivalent to

$$(k-1)[(k+1)(x^2-4) - 2y(x-y)] \geq 0.$$

This is true because

$$(k+1)(x^2-4) - 2y(x-y) \geq 2(x^2-4) - 2y(x-y) = 2(x-y)^2 + 2(xy-4) \geq 0.$$

The equality holds for $b=c$. If $k=1$, then the equality holds for $a=b$ or $b=c$ or $c=a$.

(b) Write the inequality as

$$(k+1)(F-k^2) \geq (E-k^2-2k-2)^2.$$

We have

$$E-k^2-2k-2 = k(xy-2) + y^2 - 2 > 0,$$

$$(E-k^2-2k-2)^2 = k^2(xy-2)^2 + 2k(xy-2)(y^2-2) + (y^2-2)^2,$$

and

$$(k+1)(F-k^2) = k^2(x^2-2)(y^2-2) + k(y^2-2)(x^2+y^2-4) + (y^2-2)^2.$$

Thus,

$$(k+1)(F-k^2) - (E-k^2-2k-2)^2 = k(x-y)^2(y^2-2k-2)$$
$$\geq k(x-y)^2(y^2-4) \geq 0.$$

If $0 < k < 1$, then the equality holds for $a=b$ or $a=c$.

□

P 2.62. *If a,b,c are positive real numbers, then*

$$\frac{a}{2b+6c} + \frac{b}{7c+a} + \frac{25c}{9a+8b} > 1.$$

Solution. By the Cauchy-Schwarz inequality, we have

$$\frac{a}{2b+6c}+\frac{b}{7c+a}+\frac{25c}{9a+8b}\geq\frac{(a+b+5c)^2}{a(2b+6c)+b(7c+a)+c(9a+8b)}.$$

Therefore, it suffices to show that

$$(a+b+5c)^2\geq 3ab+15bc+15ca,$$

which is equivalent to

$$a^2+b^2+25c^2-ab-5bc-5ca\geq 0.$$

Indeed, we have

$$\begin{aligned}2(a^2+b^2+25c^2-ab-5bc-5ca)&=(a-b)^2+a^2+b^2+50c^2-10bc-10ca\\&=(a-b)^2+(a-5c)^2+(b-5c)^2\geq 0.\end{aligned}$$

□

P 2.63. *If a, b, c are positive real numbers such that*

$$\frac{1}{a}\geq\frac{1}{b}+\frac{1}{c},$$

then

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\frac{55}{12(a+b+c)}.$$

(Vasile Cîrtoaje, 2014)

Solution. Denote

$$x=\frac{bc}{b+c},\quad a\leq x,$$

and write the desired inequality as

$$\sum\frac{a+b+c}{b+c}\geq\frac{55}{12},$$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{19}{12}.$$

Using the Cauchy-Schwarz inequality

$$\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{(b+c)^2}{b(c+a)+c(a+b)},$$

it suffices to show that

$$F(a,b,c)\geq\frac{19}{12},$$

where

$$F(a,b,c) = \frac{a}{b+c} + \frac{(b+c)^2}{a(b+c)+2bc}.$$

We will show that

$$F(a,b,c) \ge F(x,b,c) \ge \frac{19}{12}.$$

Since

$$F(a,b,c) - F(x,b,c) = (x-a)\left[-\frac{1}{b+c} + \frac{(b+c)^3}{(a(b+c)+2bc)(x(b+c)+2bc)}\right],$$

we need to prove that

$$(b+c)^4 \ge [a(b+c)+2bc][(x(b+c)+2bc].$$

Since

$$a(b+c)+2bc \le x(b+c)+2bc,$$

it is enough to show that

$$(b+c)^2 \ge x(b+c)+2bc,$$

which is equivalent to the obvious inequality

$$(b+c)^2 \ge 3bc.$$

Also, we have

$$F(x,b,c) - \frac{19}{12} = \frac{bc}{(b+c)^2} + \frac{(b+c)^2}{3bc} - \frac{19}{12} = \frac{(b-c)^2(4b^2+5bc+4c^2)}{12bc(b+c)^2} \ge 0.$$

The equality occurs for $2a = b = c$.

□

P 2.64. *If a,b,c are positive real numbers such that*

$$\frac{1}{a} \ge \frac{1}{b} + \frac{1}{c},$$

then

$$\frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} \ge \frac{189}{40(a^2+b^2+c^2)}.$$

(Vasile Cîrtoaje, 2014)

Solution. Denote
$$x = \frac{bc}{b+c}, \quad a \le x,$$
and write the desired inequality as
$$\sum \frac{a^2+b^2+c^2}{b^2+c^2} \ge \frac{189}{40},$$
$$\frac{a^2}{b^2+c^2} + \frac{b^2}{c^2+a^2} + \frac{c^2}{a^2+b^2} \ge \frac{69}{40}.$$
Using the Cauchy-Schwarz inequality
$$\frac{b^2}{c^2+a^2} + \frac{c^2}{a^2+b^2} \ge \frac{(b^2+c^2)^2}{b^2(c^2+a^2)+c^2(a^2+b^2)},$$
it suffices to show that
$$F(a,b,c) \ge \frac{69}{40},$$
where
$$F(a,b,c) = \frac{a^2}{b^2+c^2} + \frac{(b^2+c^2)^2}{a^2(b^2+c^2)+2b^2c^2}.$$
We will show that
$$F(a,b,c) \ge F(x,b,c) \ge \frac{69}{40}.$$
Since
$$F(a,b,c) - F(x,b,c) = (x^2-a^2)\left[-\frac{1}{b^2+c^2} + \frac{(b^2+c^2)^3}{(a^2(b^2+c^2)+2b^2c^2)\,(x^2(b^2+c^2)+2b^2c^2)}\right],$$
we need to prove that
$$(b^2+c^2)^4 \ge [a^2(b^2+c^2)+2b^2c^2][x^2(b^2+c^2)+2b^2c^2].$$
Since
$$a^2(b^2+c^2)+2b^2c^2 \le x^2(b^2+c^2)+2b^2c^2,$$
it is enough to show that
$$(b^2+c^2)^2 \ge x^2(b^2+c^2)+2b^2c^2,$$
which is equivalent to
$$(b^4+c^4)(b+c)^2 \ge b^2c^2(b^2+c^2).$$
This inequality follows from $b^4+c^4 > b^2c^2$ and $(b+c)^2 > b^2+c^2$. Also, we have
$$F(x,b,c) = \frac{x^2}{b^2+c^2} + \frac{(b^2+c^2)^2}{x^2(b^2+c^2)+2b^2c^2}.$$
Since
$$2b^2c^2 \le 4x^2(b^2+c^2),$$

we have

$$F(x,b,c) \geq \frac{x^2}{b^2+c^2} + \frac{(b^2+c^2)^2}{5x^2(b^2+c^2)} = \frac{1}{t} + \frac{t}{5},$$

where

$$t = \frac{b^2+c^2}{x^2} \geq 8.$$

Therefore,

$$F(x,b,c) - \frac{69}{40} \geq \frac{1}{t} + \frac{t}{5} - \frac{69}{40} = \frac{(t-8)(8t-5)}{40t} \geq 0.$$

The equality occurs for $2a = b = c$.

□

P 2.65. *Find the best real numbers k, m, n such that*

$$(\sqrt{a}+\sqrt{b}+\sqrt{c})\sqrt{a+b+c} \geq ka + mb + nc$$

for all $a \geq b \geq c \geq 0$.

Solution. For $a = 1$ and $b = c = 0$, for $a = b = 1$ and $c = 0$, and for $a = b = c = 1$, we get respectively

$$k \leq 1, \quad k+m \leq 2\sqrt{2}, \quad k+m+n \leq 3\sqrt{3},$$

which yield

$$\begin{aligned} ka+mb+nc &= k(a-b) + (k+m)(b-c) + (k+m+nz)c \\ &\leq a - b + 2\sqrt{2}\ (b-c) + 3\sqrt{3}\ c \\ &= a + (2\sqrt{2}-1)b + (3\sqrt{3}-2\sqrt{2})c. \end{aligned}$$

Therefore, if the following inequality holds

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\sqrt{a+b+c} \geq a + (2\sqrt{2}-1)b + (3\sqrt{3}-2\sqrt{2})c,$$

then

$$k = 1, \quad m = 2\sqrt{2}-1, \quad n = 3\sqrt{3}-2\sqrt{2}$$

are the best real k, m, n. Since

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 = a + \left(2\sqrt{ab}+b\right) + \left(2\sqrt{ac}+2\sqrt{bc}+c\right) \geq a + 3b + 5c,$$

it suffices to show that

$$(a+3b+5c)(a+b+c) \geq [a + (2\sqrt{2}-1)b + (3\sqrt{3}-2\sqrt{2})c]^2,$$

which is equivalent to the obvious inequality

$$(3-2\sqrt{2})b(a-b) + (3+2\sqrt{2}-3\sqrt{3})c(a-b) + 3(5-2\sqrt{6})c(b-c) \geq 0.$$

If $k = 1,\ m = 2\sqrt{2}-1,\ n = 3\sqrt{3}-2\sqrt{2}$, then the equality holds for $a = b = c$, for $a = b$ and $c = 0$, and for $b = c = 0$.

□

P 2.66. *Let $a, b \in (0,1]$, $a \le b$.*

(a) If $a \le \frac{1}{e}$, then

$$2a^a \ge a^b + b^a;$$

(b) If $b \ge \frac{1}{e}$, then

$$2b^b \ge a^b + b^a.$$

(Vasile Cîrtoaje, 2012)

Solution. (a) We need to show that $f(a) \ge f(b)$, where

$$f(x) = a^x + x^a, \quad x \in [a,b].$$

This is true if $f(x)$ is decreasing; that is, if $f'(x) \le 0$ on $[a,b]$. Since the derivative

$$f'(x) = a(x^{a-1} + a^{x-1}\ln a) \le a(x^{a-1} - a^{x-1}),$$

it suffices to show that

$$x^{a-1} \le a^{x-1}$$

for $0 < a \le x \le 1$. Consider the non-trivial case $0 < a \le x < 1$, and write the inequality as $g(x) \ge g(a)$, where

$$g(x) = \frac{\ln x}{1-x}.$$

It suffices to show that $g'(x) \ge 0$ for $0 < x < 1$. We have

$$g'(x) = \frac{h(x)}{(1-x)^2}, \quad h(x) = \frac{1}{x} - 1 + \ln x.$$

Since

$$h'(x) = \frac{x-1}{x^2} < 0,$$

$h(x)$ is strictly decreasing, $h(x) > h(1) = 0$, $g'(x) > 0$. This completes the proof. The equality holds for $a = b$.

(b) We need to show that $f(b) \ge f(a)$, where

$$f(x) = x^b + b^x, \quad x \in [a,b].$$

This is true if $f(x)$ is increasing; that is, if $f'(x) \ge 0$ on $[a,b]$. Since the derivative

$$f'(x) = b(x^{b-1} + b^{x-1}\ln b) \ge b(x^{b-1} - b^{x-1}),$$

it suffices to show that

$$x^{b-1} \ge b^{x-1}$$

for $0 < x \le b \le 1$. As we shown at (a), this inequality is true. The equality holds for $a = b$.

□

P 2.67. *If* $0 \le a \le b$ *and* $b \ge \frac{1}{2}$, *then*

$$2b^{2b} \ge a^{2b} + b^{2a}.$$

(Vasile Cîrtoaje, 2012)

Solution. We need to show that $f(a) \le f(b)$, where

$$f(x) = x^{2b} + b^{2x}, \quad x \in [0, b].$$

From the derivative

$$f''(x) = 2b\left[2b^{2x-1}\ln^2 b + (2b-1)x^{2b-2}\right] > 0, \quad x \in (0, b],$$

it follows that $f(x)$ is convex on $[0, b]$. Therefore, we have

$$f(a) \le \max\{f(0), f(b)\}.$$

From this, it follows that $f(a) \le f(b)$ if $f(0) \le f(b)$. To prove that $f(0) \le f(b)$, we apply Bernoulli's inequality as follows:

$$\begin{aligned} f(b) - f(0) &= 2b^{2b} - 1 = 2[1 + (b-1)]^{2b} - 1 \\ &\ge 2[1 + 2b(b-1)] - 1 = (2b-1)^2 \ge 0. \end{aligned}$$

The equality holds for $a = b \ge \frac{1}{2}$, and also for $a = 0$ and $b = \frac{1}{2}$.

□

P 2.68. *If* $a \ge b \ge 0$, *then*

(a) $$a^{b-a} \le 1 + \frac{a-b}{\sqrt{a}};$$

(b) $$a^{a-b} \ge 1 - \frac{3(a-b)}{4\sqrt{a}}.$$

(Vasile Cîrtoaje, 2010)

Solution. (a) Write the inequality as

$$(a-b)\ln a + \ln\left(1 + \frac{a-b}{\sqrt{a}}\right) \ge 0,$$

which follows by adding the inequalities

$$\ln\left(1 + \frac{a-b}{\sqrt{a}}\right) - \frac{a-b}{\sqrt{a}} + \frac{(a-b)^2}{2a} \ge 0,$$

$$(a-b)\ln a + \frac{a-b}{\sqrt{a}} - \frac{(a-b)^2}{2a} \ge 0.$$

Denoting

$$x = \frac{a-b}{\sqrt{a}},$$

we can write the first inequality as $f(x) \ge 0$ for $x \ge 0$, where

$$f(x) = \ln(1+x) - x + \frac{x^2}{2}.$$

From the derivative

$$f'(x) = \frac{x^2}{1+x} \ge 0,$$

it follows that f is increasing, hence $f(x) \ge f(0) = 0$.

The second inequality is true if

$$\ln a + \frac{1}{\sqrt{a}} - \frac{a-b}{2a} \ge 0.$$

It suffices to prove that $g(a) \ge 0$, where

$$g(a) = \ln a + \frac{1}{\sqrt{a}} - \frac{1}{2}.$$

From

$$g'(a) = \frac{2\sqrt{a}-1}{2a\sqrt{a}},$$

it follows that g is decreasing on $(0, 1/4]$ and increasing on $[1/4, \infty)$; therefore,

$$g(a) \ge g\left(\frac{1}{4}\right) = \frac{3}{2} - \ln 4 > 0.$$

The equality holds for $a = b$.

(b) Consider the non-trivial case $1 - \dfrac{3(a-b)}{4\sqrt{a}} > 0$, write the inequality as

$$(a-b)\ln a \ge \ln\left(1 - \frac{3a-3b}{4\sqrt{a}}\right),$$

and prove it by adding the inequalities

$$0 \ge \ln\left(1 - \frac{3a-3b}{4\sqrt{a}}\right) + \frac{3(a-b)}{4\sqrt{a}},$$

$$(a-b)\ln a + \frac{3(a-b)}{4\sqrt{a}} \ge 0.$$

Denoting

$$x = \frac{3(a-b)}{4\sqrt{a}}, \qquad 0 \le x < 1,$$

we can write the first inequality as $f(x) \le 0$, where

$$f(x) = \ln(1-x) + x.$$

From the derivative

$$f'(x) = \frac{-x}{1-x} \le 0,$$

it follows that f is decreasing, hence $f(x) \le f(0) = 0$.

The second inequality is true if $g(a) \ge 0$, where

$$g(a) = \ln a + \frac{3}{4\sqrt{a}}.$$

From the derivative

$$g'(a) = \frac{8\sqrt{a}-3}{8a\sqrt{a}},$$

it follows that

$$g(a) \ge g\left(\frac{9}{64}\right) = 2\ln\frac{3e}{8} > 0.$$

The equality holds for $a = b$.

□

P 2.69. *If a, b, c are positive real numbers such that*

$$a \ge b \ge c, \qquad ab^2c^3 \ge 1,$$

then

$$a + 2b + 3c \ge \frac{1}{a} + \frac{2}{b} + \frac{3}{c}.$$

(Vasile Cîrtoaje, 2018)

Solution. It suffices to prove the homogeneous inequality

$$a + 2b + 3c \ge \sqrt[3]{ab^2c^3}\left(\frac{1}{a} + \frac{2}{b} + \frac{3}{c}\right).$$

Replacing a, b, c with a^3, b^3, c^3, the inequality becomes as follows:

$$a^3 + 2b^3 + 3c^3 \ge \frac{b^2c^3}{a^2} + \frac{2ac^3}{b} + 3ab^2,$$

$$a^3 + 2b^3 - 3ab^2 \ge \frac{c^3}{a^2b}(2a^3 - 3a^2b + b^3),$$

$$(a-b)^2(a+2b) \geq \frac{c^3}{a^2b}(a-b)^2(2a+b).$$

Thus, we need to show that

$$a^2b(a+2b) \geq c^3(2a+b)$$

for $a \geq b \geq c$. Since $c^3 \leq ab^2$, we have

$$a^2b(a+2b) - c^3(2a+b) \geq a^2b(a+2b) - ab^2(2a+b) = ab(a^2-b^2) \geq 0.$$

The equality occurs for $a = b = 1/c \geq 1$.

□

P 2.70. *If a,b,c are positive real numbers such that*

$$a+b+c = 3, \qquad a \leq b \leq c,$$

then

$$\frac{1}{a} + \frac{2}{b} \geq a^2 + b^2 + c^2.$$

(Vasile Cîrtoaje, 2020)

Solution. Let

$$f(a,b,c) = \frac{1}{a} + \frac{2}{b} - a^2 - b^2 - c^2.$$

We will show that

$$f(a,b,c) \geq f(a,x,x) \geq 0,$$

where

$$x = \frac{b+c}{2} = \frac{3-a}{2}.$$

Since

$$f(a,b,c) - f(a,x,x) = \frac{2}{b} - \frac{2}{x} - (b^2+c^2-2x^2)$$
$$= \frac{2(c-b)}{b(b+c)} - \frac{(c-b)^2}{2} = \frac{(c-b)(b^3-bc^2+4)}{2b(b+c)},$$

we need to show that

$$b^3 - bc^2 + 4 \geq 0.$$

Since $b + c < 3$, we have

$$b^3 - bc^2 + 4 > b^3 - b(3-b)^2 + 4 = 6b^2 + 4 - 9b \geq (4\sqrt{6} - 9)b > 0.$$

Also, since $a \leq 1$, we have

$$f(a,x,x) = \frac{1}{a} + \frac{2}{x} - a^2 - 2x^2 = \frac{1}{a} + \frac{4}{3-a} - a^2 - \frac{1}{2}(3-a)^2$$
$$= \frac{a^4 - 5a^3 + 9a^2 - 7a + 2}{a(3-a)} = \frac{(1-a)^3(2-a)}{a(3-a)} \geq 0.$$

The equality occurs for $a = b = c = 1$.

□

P 2.71. *If a, b, c are positive real numbers such that*

$$a+b+c=3, \qquad a \le b \le c,$$

then

$$\frac{2}{a}+\frac{3}{b}+\frac{1}{c} \ge 2(a^2+b^2+c^2).$$

(Vasile Cîrtoaje, 2020)

Solution. From

$$a \le b = 3-a-c,$$

we get

$$a \le \frac{3-c}{2}.$$

For fixed b, write the inequality as $f(a) \ge 0$, where

$$f(a) = \frac{2}{a}+\frac{3}{b}+\frac{1}{c}-2(a^2+b^2+c^2), \qquad c = 3-a-b.$$

We have

$$f'(a) = -\frac{2}{a^2}+\frac{1}{c^2}-4(a-c) = \frac{1}{c^2}+4c-2g(a), \qquad g(a) = 2a+\frac{1}{a^2}.$$

Since

$$g'(a) = 2-\frac{2}{a^3} \le 0,$$

$g(a)$ is decreasing, hence

$$g(a) \ge g\left(\frac{3-c}{2}\right)$$

and

$$f'(a) \le \frac{1}{c^2}+4c-2g\left(\frac{3-c}{2}\right) = 6(c-1)-\frac{7c^2+6c-9}{c^2(3-c)^2}$$

$$\le 6(c-1)-\frac{16}{81}(7c^2+6c-9) = \frac{-2}{81}(56c^2+171-195c)$$

$$\le \frac{-2}{27}(4\sqrt{266}-65)c < 0.$$

Therefore, $f(a)$ is decreasing. On the other hand, from $a \le b$ and $b \le c = 3-a-b$, we get

$$a \le b, \qquad a \le 3-2b.$$

There are two cases to consider: $b \in (0,1]$ and $b \in [1, 3/2)$.

Case 1: $b \in (0,1]$. Since $a \le b$, we have

$$f(a) \ge f(b) = \frac{5}{b}+\frac{1}{c}-2(2b^2+c^2), \qquad c = 3-2b,$$

hence

$$f(a) \geq \frac{5}{b} + \frac{1}{3-2b} - 4b^2 - 2(3-2b)^2$$
$$= \frac{3(5-3b)}{b(3-2b)} - 3(4b^2 - 8b + 6)$$
$$= \frac{3(8b^4 - 28b^3 + 36b^2 - 21b + 5)}{b(3-2b)}$$
$$\geq \frac{3(8b^4 - 27b^3 + 35b^2 - 21b + 5)}{b(3-2b)}$$
$$= \frac{3(b-1)^2(8b^2 - 11b + 5)}{b(3-2b)} \geq 0.$$

Case 2: $b \in [1, 3/2)$. Since $a \leq 3 - b$, we have

$$f(a) \geq f(3-b) = \frac{2}{3-2b} + \frac{3}{b} + \frac{1}{c} - 2(3-2b)^2 - 2(b^2 + c^2), \qquad c = b,$$

hence

$$f(a) \geq f(3-b) = \frac{2}{3-2b} + \frac{4}{b} - 2(3-2b)^2 - 4b^2$$
$$= \frac{6(2-b)}{b(3-2b)} - 6(2b^2 - 4b + 3)$$
$$= \frac{12(2b^4 - 7b^3 + 9b^2 - 5b + 1)}{b(3-2b)}$$
$$= \frac{12(b-1)^3(2b-1)}{b(3-2b)} \geq 0.$$

The equality occurs for $a = b = c = 1$.

Remark. Since

$$\frac{2}{a} + \frac{3}{b} + \frac{1}{c} \leq 2\left(\frac{1}{a} + \frac{2}{b}\right),$$

the inequality is stronger than the one of P 2.70.

□

P 2.72. *If a, b, c are positive real numbers such that*

$$a + b + c = 3, \qquad a \leq b \leq c,$$

then

$$\frac{31}{a} + \frac{25}{b} + \frac{25}{c} \geq 27(a^2 + b^2 + c^2).$$

(Vasile Cîrtoaje, 2020)

Solution. From

$$a \le b = 3 - a - c,$$

we get

$$a \le \frac{3-c}{2}.$$

For fixed $c \in [1,3)$,write the inequality as $f(a) \ge 0$, where $a \le \dfrac{3-c}{2}$ and

$$f(a) = \frac{31}{a} + \frac{25}{b} + \frac{25}{c} - 27(a^2+b^2+c^2), \qquad b = 3-a-c.$$

We will show that

$$f(a) \ge f\left(\frac{3-c}{2}\right) \ge 0.$$

Since $a+b \le 2$, we have

$$\frac{a+b}{a^2b^2} \ge \frac{16}{(a+b)^3} \ge 2,$$

therefore

$$f'a) = -\frac{31}{a^2} + \frac{25}{b^2} - 27(2a-2b) < -\frac{27}{a^2} + \frac{27}{b^2} - 54(a-b)$$

$$= 27(a-b)\left(\frac{a+b}{a^2b^2} - 2\right) \le 0,$$

$f(a)$ is decreasing, hence $f(a)$ is minimum for $a = \dfrac{3-c}{2}$, when

$$b = 3-a-c = \frac{3-c}{2} = a.$$

So, we have

$$f\left(\frac{3-c}{2}\right) = \frac{56}{a} + \frac{25}{c} - 27(2a^2+c^2)$$

$$= \frac{112}{3-c} + \frac{25}{c} - \frac{27(3-c)^2}{2} - 27c^2$$

$$= \frac{3(27c^4 - 135c^3 + 243c^2 - 185c + 50)}{2c(3-c)}$$

$$= \frac{3(c-1)(3c-2)(3c-5)^2}{2c(3-c)} \ge 0.$$

The equality occurs for $a = b = c = 1$, and also for $a = b = \dfrac{2}{3}$ and $c = \dfrac{5}{3}$.

Remark. Actually, the following stronger inequalities are true:

$$\frac{29}{a} + \frac{27}{b} + \frac{25}{c} \ge 27(a^2+b^2+c^2),$$

$$\frac{28}{a}+\frac{28}{b}+\frac{25}{c}\ge 27(a^2+b^2+c^2). \tag{*}$$

For (*), we have

$$f(a)=\frac{28}{a}+\frac{28}{b}+\frac{25}{c}-27(a^2+b^2+c^2), \qquad b=3-a-c,$$

$$f'a)=-\frac{28}{a^2}+\frac{28}{b^2}-27(2a-2b)\le-\frac{27}{a^2}+\frac{27}{b^2}-54(a-b)$$

$$=27(a-b)\left(\frac{a+b}{a^2b^2}-2\right)\le 0$$

and

$$f\left(\frac{3-c}{2}\right)=\frac{56}{a}+\frac{25}{c}-27(2a^2+c^2)$$

$$=\frac{3(c-1)(3c-2)(3c-5)^2}{2c(3-c)}\ge 0.$$

On the other hand, we can prove the inequality (*) by showing that

$$f(a,b,c)\ge f(x,x,c)\ge 0,$$

where

$$f(a,b,c)=\frac{28}{a}+\frac{28}{b}+\frac{25}{c}-27(a^2+b^2+c^2), \qquad x=\frac{a+b}{2}=\frac{3-c}{2}.$$

We have

$$f(a,b,c)-f(x,x,c)=28\left(\frac{1}{a}+\frac{1}{b}-\frac{2}{x}\right)-27(a^2+b^2-2x^2)$$

$$=\frac{1}{2}(a-b)^2\left[\frac{56}{ab(a+b)}-27\right]\ge\frac{27}{2}(a-b)^2\left[\frac{2}{ab(a+b)}-1\right]\ge 0$$

and

$$f(x,x,c)=\frac{56}{x}+\frac{25}{c}-27(2x^2+c^2)=\frac{3(c-1)(3c-2)(3c-5)^2}{2c(3-c)}\ge 0.$$

□

P 2.73. *If a,b,c are the lengths of the sides of a triangle, then*

$$a^3(b+c)+bc(b^2+c^2)\ge a(b^3+c^3).$$

(Vasile Cîrtoaje, 2010)

First Solution. Because the inequality is symmetric in b and c, we may assume that $b \ge c$. Consider the following two cases.

Case 1: $a \ge b$. It suffices to show that

$$a^3(b+c) \ge a(b^3+c^3).$$

We have

$$a^3(b+c) - a(b^3+c^3) \ge ab^2(b+c) - a(b^3+c^3) = ac(b^2-c^2) \ge 0.$$

Case 2: $a \le b$. Write the inequality as

$$c(a^3+b^3) - c^3(a-b) + ab(a^2-b^2) \ge 0.$$

It suffices to show that

$$c(a^3+b^3) + ab(a^2-b^2) \ge 0.$$

We have

$$c(a^3+b^3) + ab(a^2-b^2) \ge c(a^3+b^3) - abc(a+b) = c(a+b)(a-b)^2 \ge 0.$$

The equality holds for a degenerate triangle with $a = b$ and $c = 0$, or $a = c$ and $b = 0$.

Second Solution. Consider two cases.

Case 1: $b^2+c^2 \ge a(b+c)$. Write the inequality as

$$bc(b^2+c^2) \ge a(b+c)(b^2+c^2-bc-a^2).$$

It suffices to show that

$$bc \ge b^2+c^2-bc-a^2,$$

which is equivalent to the obvious inequality

$$a^2 \ge (b-c)^2.$$

Case 2: $a(b+c) \ge b^2+c^2$. Write the inequality as

$$a(b+c)(a^2+bc) \ge (b^2+c^2)(ab+ac-bc).$$

It suffices to show that

$$a^2+bc \ge ab+ac-bc,$$

which is equivalent to the obvious inequality

$$bc \ge (a-c)(b-a).$$

□

P 2.74. *If a, b, c are the lengths of the sides of a triangle, then*

$$\frac{(a+b)^2}{2ab+c^2} + \frac{(a+c)^2}{2ac+b^2} \geq \frac{(b+c)^2}{2bc+a^2}.$$

(Vasile Cîrtoaje, 2010)

Solution. By the Cauchy-Schwarz inequality, we have

$$\frac{(a+b)^2}{2ab+c^2} + \frac{(a+c)^2}{2ac+b^2} \geq \frac{(2a+b+c)^2}{2a(b+c)+b^2+c^2}.$$

Therefore, it suffices to show that

$$\frac{(2a+b+c)^2}{2a(b+c)+b^2+c^2} \geq \frac{(b+c)^2}{2bc+a^2}.$$

We will show that

$$\frac{(2a+b+c)^2}{2a(b+c)+b^2+c^2} \geq 2 \geq \frac{(b+c)^2}{2bc+a^2}.$$

The left inequality reduces to $4a^2 \geq (b-c)^2$, and the right inequality reduces to $2a^2 \geq (b-c)^2$. These are true because $a^2 \geq (b-c)^2$. The equality holds for a degenerate triangle with $a = 0$ and $b = c$.

□

P 2.75. *If a, b, c are the lengths of the sides of a triangle, then*

$$\frac{a+b}{ab+c^2} + \frac{a+c}{ac+b^2} \geq \frac{b+c}{bc+a^2}.$$

(Vasile Cîrtoaje, 2010)

Solution. Without loss of generality, assume that $b \geq c$. Since $a+b \geq a+c$ and

$$ab+c^2-(ac+b^2) = (b-c)(a-b-c) \leq 0,$$

by Chebyshev's inequality, we have

$$\frac{a+b}{ab+c^2} + \frac{a+c}{ac+b^2} \geq \frac{1}{2}[(a+b)+(a+c)]\left(\frac{1}{ab+c^2} + \frac{1}{ac+b^2}\right)$$

$$\geq \frac{2(2a+b+c)^2}{a(b+c)+b^2+c^2}.$$

On the other hand,

$$\frac{b+c}{bc+a^2} \leq \frac{b+c}{\frac{1}{2}(b-c)^2+bc} = \frac{2(b+c)}{b^2+c^2}.$$

Therefore, it suffices to show that

$$\frac{2(2a+b+c)}{a(b+c)+b^2+c^2} \ge \frac{2(b+c)}{b^2+c^2},$$

which is equivalent to $a(b-c)^2 \ge 0$. The equality holds for a degenerate triangle with $a = 0$ and $b = c$.

□

P 2.76. *If a, b, c are the lengths of the sides of a triangle, then*

$$\frac{b(a+c)}{ac+b^2} + \frac{c(a+b)}{ab+c^2} \ge \frac{a(b+c)}{bc+a^2}.$$

(Vo Quoc Ba Can, 2010)

Solution. Without loss of generality, assume that $b \ge c$. Since

$$ab+c^2-(ac+b^2) = (b-c)(a-b-c) \le 0,$$

it suffices to prove that

$$\frac{b(a+c)}{ac+b^2} + \frac{c(a+b)}{ac+b^2} \ge \frac{a(b+c)}{bc+a^2},$$

which is equivalent to

$$\frac{2bc+a(b+c)}{ac+b^2} \ge \frac{a(b+c)}{bc+a^2},$$

$$\frac{2bc}{ac+b^2} \ge a(b+c)\left(\frac{1}{bc+a^2} - \frac{1}{ac+b^2}\right),$$

$$2bc(bc+a^2) \ge a(b+c)(b-a)(a+b-c).$$

Consider the nontrivial case $b \ge a$. Since $c \ge b-a$, it suffices to show that

$$2b(bc+a^2) \ge a(b+c)(a+b-c).$$

We have

$$2b(bc+a^2) - a(b+c)(a+b-c) = ab(a-b) + c(2b^2-a^2+ac)$$

$$\ge -abc + c(2b^2-a^2+ac) = ac(b+c-a) + 2bc(b-a) \ge 0.$$

The equality holds for a degenerate triangle with $a = b$ and $c = 0$, or $a = c$ and $b = 0$.

□

P 2.77. *If a, b, c, d are positive real numbers such that*

$$a \geq b \geq c \geq d, \qquad ab^2c^3d^6 \geq 1,$$

then

$$a + 2b + 3c + 6d \geq \frac{1}{a} + \frac{2}{b} + \frac{3}{c} + \frac{6}{d}.$$

(Vasile Cîrtoaje, 2018)

Solution. It suffices to prove the homogeneous inequality

$$a + 2b + 3c + 6d \geq \sqrt[6]{ab^2c^3d^6}\left(\frac{1}{a} + \frac{2}{b} + \frac{3}{c} + \frac{6}{d}\right).$$

Replacing a, b, c, d with a^6, b^6, c^6, d^6, we need to show that

$$a^6 + 2b^6 + 3c^6 \geq \left(\frac{b^2c^3}{a^5} + \frac{2ac^3}{b^4} + \frac{3ab^2}{c^3} - 6\right)d^6 + 6ab^2c^3$$

for $a \geq b \geq c \geq d$. By the AM-GM inequality, we have

$$\frac{b^2c^3}{a^5} + \frac{2ac^3}{b^4} + \frac{3ab^2}{c^3} - 6 \geq 6\sqrt[6]{\frac{b^2c^3}{a^5} \cdot \left(\frac{ac^3}{b^4} \cdot\right)^2 \left(\frac{ab^2}{c^3}\right)^3} - 6 = 0.$$

Since $d^6 \leq ab^2c^3$, it suffices to show that

$$a^6 + 2b^6 + 3c^6 \geq \left(\frac{b^2c^3}{a^5} + \frac{2ac^3}{b^4} + \frac{3ab^2}{c^3} - 6\right)ab^2c^3 + 6ab^2c^3,$$

which is equivalent to

$$a^6 + 2b^6 + 3c^6 \geq \frac{b^4c^6}{a^4} + \frac{2a^2c^6}{b^2} + 3a^2b^4,$$

$$a^6 + 2b^6 - 3a^2b^4 \geq \left(\frac{b^4}{a^4} + \frac{2a^2}{b^2} - 3\right)c^6,$$

$$(a^2 - b^2)^2(a^2 + 2b^2) \geq \frac{(a^2 - b^2)^2(2a^2 + b^2)c^6}{a^4b^2}.$$

We need to show that

$$a^4b^2(a^2 + 2b^2) \geq (2a^2 + b^2)c^6.$$

Since $c^6 \leq a^2b^4$, we have

$$a^4b^2(a^2 + 2b^2) - (2a^2 + b^2)c^6 \geq a^4b^2(a^2 + 2b^2) - (2a^2 + b^2)a^2b^4 = a^2b^2(a^4 - b^4) \geq 0.$$

The equality occurs for $a = b = c = d = 1$.

Remark. By induction method, we can prove the following generalization.

- *If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are positive real numbers such that*

$$a_1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1 a_2^2 a_3^3 a_4^6 \cdots a_n^{k_n} \ge 1, \qquad k_n = 3 \cdot 2^{n-3},$$

then

$$a_1 + 2a_2 + 3a_3 + 6a_4 + \cdots + k_n a_n \ge \frac{1}{a_1} + \frac{2}{a_2} + \frac{3}{a_3} + \frac{6}{a_4} + \cdots + \frac{k_n}{a_n},$$

with equality for $a_1 = a_2 = \cdots = a_n$.

For $n = 3$, we get the inequalities in P 2.69

□

P 2.78. *If a, b, c, d are positive real numbers such that*

$$a \ge b \ge c \ge d, \qquad abc^2d^4 \ge 1,$$

then

$$a + b + 2c + 4d \ge \frac{1}{a} + \frac{1}{b} + \frac{2}{c} + \frac{4}{d}.$$

(Vasile Cîrtoaje, 2018)

Solution. It suffices to prove the homogeneous inequality

$$a + b + 2c + 4d \ge \sqrt[4]{abc^2d^4}\left(\frac{1}{a} + \frac{1}{b} + \frac{2}{c} + \frac{4}{d}\right).$$

Replacing a, b, c, d with a^4, b^4, c^4, d^4, we need to show that

$$a^4 + b^4 + 2c^4 \ge \left(\frac{bc^2}{a^3} + \frac{ac^2}{b^3} + \frac{2ab}{c^2} - 4\right)d^4 + 4abc^2$$

for $a \ge b \ge c \ge d$. By the AM-GM inequality, we have

$$\frac{bc^2}{a^3} + \frac{ac^2}{b^3} + \frac{2ab}{c^2} - 4 \ge 4\sqrt[4]{\frac{bc^2}{a^3} \cdot \frac{ac^2}{b^3} \cdot \left(\frac{ab}{c^2}\right)^2} - 4 = 0.$$

Since $d^4 \le abc^2$, it suffices to show that

$$a^4 + b^4 + 2c^4 \ge \left(\frac{bc^2}{a^3} + \frac{ac^2}{b^3} + \frac{2ab}{c^2} - 4\right)abc^2 + 4abc^2$$

which is equivalent to

$$a^4 + b^4 + 2c^4 \ge \frac{b^2c^4}{a^2} + \frac{a^2c^4}{b^2} + 2a^2b^2,$$

$$(a^2 - b^2)^2 \ge \frac{(a^2 - b^2)^2 c^4}{a^2b^2},$$

$$(a^2-b^2)^2\left(1-\frac{c^4}{a^2b^2}\right)\geq 0.$$

The equality occurs for $a=b=c=d=1$.

Remark. By induction method, we can prove the following generalization.

- *If $a_1,a_2,\dots,a_n$ $(n\geq 3)$ are positive real numbers such that*

$$a_1\geq a_2\geq\cdots\geq a_n,\qquad a_1a_2a_3^2a_4^4\cdots a_n^{2^{n-2}}\geq 1,$$

then

$$a_1+a_2+2a_3+4a_4+\cdots+2^{n-2}a_n\geq\frac{1}{a_1}+\frac{1}{a_2}+\frac{2}{a_3}+\frac{4}{a_4}+\cdots+\frac{2^{n-2}}{a_n},$$

with equality for $a_1=a_2=\cdots=a_n$.

For $n=4$, we get the inequalities in P 2.78.

□

P 2.79. *If a,b,c,d,e,f are positive real numbers such that*

$$abcdef\geq 1,\qquad a\geq b\geq c\geq d\geq e\geq f,\qquad af\geq be\geq cd,$$

then

$$a+b+c+d+e+f\geq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\frac{1}{f}.$$

(Vasile Cîrtoaje, 2018)

Solution. Write the inequality as

$$(a+f)\left(1-\frac{1}{af}\right)+(b+e)\left(1-\frac{1}{be}\right)+(c+d)\left(1-\frac{1}{cd}\right)\geq 0.$$

For

$$af=k=constant,$$

we claim that the sum $a+f$ is minimum for $a=\frac{k}{e}\geq b$ and $f=e$. Indeed, we have

$$a+f-\frac{k}{e}-e=a+f-\frac{af}{e}-e=a-e-\left(\frac{a}{e}-1\right)f=\frac{(a-e)(e-f)}{e}\geq 0.$$

In addition, for

$$cd=k=constant,$$

we claim that the sum $c+d$ is maximum for $c=\frac{k}{e}\leq b$ and $d=e$. Indeed, we have

$$c+d-\frac{k}{e}-e=c+d-\frac{cd}{e}-e=c-e-\left(\frac{c}{e}-1\right)d=\frac{-(c-e)(d-e)}{e}\leq 0.$$

Thus, it suffices to prove the inequality for $f = e$ and $d = e$, that is for $d = e = f$. So, we need to show that

$$a + b + c + 3d \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{3}{d}$$

for

$$a \geq b \geq c \geq d, \qquad abcd^3 \geq 1.$$

It suffices to prove the homogeneous inequality

$$a + b + c + 3d \geq \sqrt[3]{abcd^3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{3}{d}\right).$$

Replacing a, b, c, d with a^3, b^3, c^3, d^3, we need to show that

$$a^3 + b^3 + c^3 \geq \left(\frac{bc}{a^2} + \frac{ca}{b^2} + \frac{ab}{c^2} - 3\right)d^3 + 3abc$$

for $a \geq b \geq c \geq d$. By the AM-GM inequality, we have

$$\frac{bc}{a^2} + \frac{ca}{b^2} + \frac{ab}{c^2} - 3 \geq 0.$$

Since $d^3 \leq c^3$, it suffices to show that

$$a^3 + b^3 + c^3 \geq \left(\frac{bc}{a^2} + \frac{ca}{b^2} + \frac{ab}{c^2} - 3\right)c^3 + 3abc,$$

which can be written as follows:

$$a^3 + b^3 + 4c^3 \geq \frac{bc^4}{a^2} + \frac{ac^4}{c^2} + 4abc,$$

$$(a^3 + b^3)\left(1 - \frac{c^4}{a^2b^2}\right) - 4c(ab - c^2) \geq 0,$$

$$(ab - c^2)[(a^3 + b^3)(ab + c^2) - 4a^2b^2c] \geq 0.$$

It is true since

$$(a^3 + b^3)(ab + c^2) - 4a^2b^2c \geq 2ab\sqrt{ab}\left(ab + c^2\right) - 4a^2b^2c = 2ab\sqrt{ab}\left(\sqrt{ab} - c\right)^2 \geq 0.$$

The equality occurs for $af = be = cd = 1$.

□

P 2.80. *Let a, b, c, d be nonnegative real numbers such that*

$$a^2 - ab + b^2 = c^2 - cd + d^2.$$

Prove that

$$(a + b)(c + d) \geq 2(ab + cd).$$

(Vasile Cîrtoaje, 2000)

Solution. Let
$$x = a^2 - ab + b^2 = c^2 - cd + d^2.$$
Without loss of generality, assume that $ab \ge cd$. Then,
$$x \ge ab \ge cd, \quad (a+b)^2 = x + 3ab, \quad (c+d)^2 = x + 3cd.$$
By squaring, the desired inequality can be restated as
$$(x+3ab)(x+3cd) \ge 4(ab+cd)^2.$$
It is true since
$$(x+3ab)(x+3cd) - 4(ab+cd)^2 \ge (ab+3ab)(ab+3cd) - 4(ab+cd)^2$$
$$= 4cd(ab-cd) \ge 0.$$
The equality occurs for $a = b = c = d$, and also for $a = b = c$ and $d = 0$ (or any cyclic permutation).

□

P 2.81. *Let a, b, c, d be nonnegative real numbers such that*
$$a^2 - ab + b^2 = c^2 - cd + d^2.$$
Prove that
$$\frac{1}{a^2+b^2} + \frac{1}{c^2+d^2} \le \frac{8}{(a+b)^2 + (c+d)^2}.$$
(Vasile Cîrtoaje and Relic-93, 2021)

Solution. Let
$$x = a^2 - ab + b^2 = c^2 - cd + d^2.$$
Without loss of generality, assume that $ab \ge cd$. Then, $x \ge ab \ge cd$ and
$$a^2 + b^2 = x + ab, \quad c^2 + d^2 = x + cd, \quad (a+b)^2 = x + 3ab, \quad (c+d)^2 = x + 3cd.$$
The required inequality can be rewritten as
$$\frac{1}{x+ab} + \frac{1}{x+cd} \le \frac{8}{2x + 3(ab+cd)},$$
$$3(a^2b^2 + c^2d^2) \le 4x^2 + 2abcd.$$
It is true if
$$3(a^2b^2 + c^2d^2) \le 4a^2b^2 + 2abcd,$$
which is equivalent to
$$(ab - cd)(ab + 3cd) \ge 0.$$
The equality occurs for $a = b = c = d$.

□

P 2.82. *Let a, b, c, d be nonnegative real numbers such that*

$$a^2 - ab + b^2 = c^2 - cd + d^2.$$

Prove that

$$\frac{1}{a^2+ab+b^2} + \frac{1}{c^2+cd+d^2} \le \frac{8}{3(a+b)(c+d)}.$$

(Anhduy98, 2021)

Solution. Without loss of generality, assume that $ab \ge cd$. Let

$$x = a^2 - ab + b^2 = c^2 - cd + d^2, \quad y = ab, \quad z = cd.$$

Then, $x \ge y \ge z$ and

$$a^2 + ab + b^2 = x + 2y, \quad c^2 + cd + d^2 = x + 2z, \quad (a+b)^2 = x + 3y, \quad (c+d)^2 = x + 3z.$$

The required inequality can be rewritten as $F(x, y, z) \le 0$, where

$$F(x, y, z) = \frac{1}{x+2y} + \frac{1}{x+2z} - \frac{8}{3\sqrt{(x+3y)(x+3z)}}.$$

We will show that

$$F(x, y, z) \le F(x, x, z) \le 0.$$

The left inequality is equivalent to

$$\frac{4}{\sqrt{x+3z}}\left(\frac{1}{\sqrt{x+3y}} - \frac{1}{2\sqrt{x}}\right) \ge \frac{x-y}{x(x+2y)}\ ,$$

$$\frac{6(x-y)}{\sqrt{x(x+3y)(x+3z)}\left(2\sqrt{x}+\sqrt{x+3z}\right)} \ge \frac{x-y}{x(x+2y)}\ .$$

It is true if

$$\frac{6}{\sqrt{(x+3y)(x+3z)}\left(2\sqrt{x}+\sqrt{x+3z}\right)} \ge \frac{1}{(x+2y)\sqrt{x}}\ .$$

Since $x \ge y \ge z$, we only need to show that

$$\frac{6}{(x+3y)\left(2\sqrt{x}+\sqrt{4x}\right)} \ge \frac{1}{(x+2y)\sqrt{x}}\ ,$$

which is clearly true.

The right inequality $F(x, x, z) \le 0$ is equivalent to

$$\frac{1}{3x} + \frac{1}{x+2z} \le \frac{4}{3\sqrt{x(x+3z)}}\ ,$$

$$(2x+z)^2(x+3z) \le 4x(x+2z)^2.$$

It is true because

$$4x(x+2z)^2 - (2x+z)^2(x+3z) = 3(x-z)z^2 \geq 0.$$

The equality occurs for $a = b = c = d$, and also for $a = b = c$ and $d = 0$ (or any cyclic permutation).

□

P 2.83. *Let a, b, c, d be nonnegative real numbers such that*

$$a^2 - ab + b^2 = c^2 - cd + d^2.$$

Prove that

$$\frac{1}{(ac+bd)^4} + \frac{1}{(ad+bc)^4} \leq \frac{2}{(ab+cd)^4}.$$

(Vasile Cîrtoaje, 2021)

Solution. Due to homogeneity, we may set

$$a^2 - ab + b^2 = c^2 - cd + d^2 = 1.$$

Let

$$x = ab, \quad y = cd, \quad s = x+y, \quad p = xy.$$

From $1 = a^2 - ab + b^2 \geq ab$, we get $x \leq 1$. Similarly, $y \leq 1$, hence $p \leq 1$. In addition, from

$$(1-x)1-y) \geq 0,$$

we get

$$s \leq 1 + p.$$

Since

$$(ac+bd)(ad+bc) = ab(c^2+d^2) + cd(a^2+b^2) = x(1+y) + y(1+x) = s + 2p,$$

$$(ac+bd)^2 + (ad+bc)^2 = (a^2+b^2)(c^2+d^2) + 4abcd = (1+x)(1+y) + 4xy = 1 + s + 5p,$$

$$(ac+bd)^4 + (ad+bc)^4 = \left[(ac+bd)^2 + (ad+bc)^2\right]^2 - 2(ac+bd)^2(ad+bc)^2$$
$$= (1+s+5p)^2 - 2(s+2p)^2 \ ,$$

we need to show that

$$\frac{(1+s+5p)^2 - 2(s+2p)^2}{(s+2p)^4} \leq \frac{2}{s^4} \ ,$$

that is equivalent to $f(s,p) \geq g(s,p)$, where

$$f(s,p) = 2\left(1 + \frac{2p}{s}\right)^4, \qquad g(s,p) = (1+s+5p)^2 - 2(s+2p)^2 \ .$$

Since

$$f(s,p) \ge f(1+p,p)$$

and

$$g(s,p) - g(1+p,p) = (s-1-p)(3+s+11p) - 2(s-1-p)(1+s+5p) = -(s-1-p)^2 \le 0,$$

it is enough to show that

$$f(1+p,p) \ge g(1+p,p),$$

that is

$$\frac{2(1+3p)^4}{(1+p)^4} \ge 2(1+3p)^2,$$

$$p(1-p)(1+3p)^2(2+5p+p^2) \ge 0.$$

The equality occurs for $a = b = c = d$, and also for $a = b = c$ and $d = 0$ (or any cyclic permutation).

□

P 2.84. *Let a, b, c, d be nonnegative real numbers such that $a \ge b \ge c \ge d$ and*

$$a+b+c+d = 13, \qquad a^2+b^2+c^2+d^2 = 43.$$

Prove that

$$ab \ge cd + 3.$$

(PMO, 2021)

Solution *(by Doxuantrong).* From

$$43 - a^2 = b^2 + c^2 + d^2 \ge \frac{1}{3}(b+c+d)^2 = \frac{1}{3}(13-a)^2,$$

we get

$$(a-4)(2a-5) \le 0,$$

hence $\frac{5}{2} \le a, b, c, d \le 4$. On the other hand, we write the required inequality as follows:

$$2ab \ge 2cd + 6,$$

$$(a+b)^2 - (a^2+b^2) \ge (c+d)^2 - (c^2+d^2) + 6,$$

$$(13-c-d)^2 - (43-c^2-d^2) \ge (c+d)^2 - (c^2+d^2) + 6,$$

$$c^2 + d^2 + 60 \ge 13(c+d),$$

$$(c-d)^2 + (c+d)^2 + 120 \ge 26(c+d),$$

$$(c-d)^2 \ge (c+d-6)(20-c-d).$$

Thus, it suffices to show that $c+d \le 6$, that is equivalent to $a+b \ge 7$. If $a=4$, then

$$a+b \ge a+\frac{b+c+d}{3} = a+\frac{13-a}{3} = 7.$$

Consider further that $a<4$. From

$$(b-c)(b-d) \ge 0,$$

we get

$$b^2-(c+d)b+cd \ge 0,$$

that is equivalent to

$$2b^2-2(c+d)b+(c+d)^2-(c^2+d^2) \ge 0,$$

$$b^2+(b-c-d)^2-(c^2+d^2) \ge 0,$$

$$b^2+(a+2b-13)^2-(43-a^2-b^2) \ge 0,$$

$$3b^2-2(13-a)b+a^2-13a+63 \ge 0,$$

$$3b \ge 13-a+\sqrt{(4-a)(2a-5)}.$$

Note that we cannot have $3b \le 13-a-\sqrt{(4-a)(2a-5)}$ because this involves a contradiction:

$$13-a = b+c+d \le 3b \le 13-a-\sqrt{(4-a)(2a-5)} < 13-a.$$

From

$$3a \ge 3b \ge 13-a+\sqrt{(4-a)(2a-5)},$$

we get

$$4a-13 \ge \sqrt{(4-a)(2a-5)},$$

$$(2a-7)(a-3) \ge 0,$$

hence $a \ge 7/2$. As a consequence, we have

$$3(a+b-7) = 3(a-7)+3b \ge 3(a-7)+13-a+\sqrt{(4-a)(2a-5)}$$

$$= \sqrt{4-a}\left(\sqrt{2a-5}-2\sqrt{4-a}\right) = \frac{3\sqrt{4-a}\ (2a-7)}{\sqrt{2a-5}+2\sqrt{4-a}} \ge 0.$$

The equality occurs for $a=4$ and $b=c=d=3$.

Second solution (by *KaiRain*) To show that $a+b \ge 7$, the key is

$$a^2+b^2+c^2+d^2+6(ab+cd) = (a+b+c+d)^2+2(a-c)(b-d)+2(a-d)(b-c)$$

$$\ge (a+b+c+d)^2,$$

which gives

$$ab+cd \ge 21,$$

$$(a+b)^2+(c+d)^2 \ge a^2+b^2+c^2+d^2+42,$$

$$(a+b)^2+(13-a-b)^2 \ge 85,$$
$$(a+b-6)(a+b-7) \ge 0,$$
$$a+b \ge 7.$$

Hence,

$$ab-cd \ge ab-\frac{c^2+d^2}{2} = ab+\frac{a^2+b^2-43}{2} = \frac{(a+b)^2-43}{2} \ge 3.$$

□

P 2.85. *Let a,b,c,d be nonnegative real numbers such that $a \ge b \ge c \ge d$ and*

$$a+b+c+d=13, \qquad a^2+b^2+c^2+d^2=43.$$

Prove that

$$\frac{83}{4} \le ac+bd \le \frac{169}{8}.$$

(Vasile Cîrtoaje, 2021)

Solution. As shown at P 2.84, we have

$$\frac{5}{2} \le a,b,c,d \le 4.$$

Since

$$2(ac+bd) = (a+c)^2+(b+d)^2-(a^2+b^2+c^2+d^2) = (a+c)^2+(13-a-c)^2-43$$
$$= 2(a+c)^2-26(a+c)+126,$$

the left required inequality is equivalent to

$$\left(a+c-\frac{13}{2}\right)^2 \ge 0\ ,$$

and the right required inequality is equivalent to

$$8(a+c)^2-104(a+c)+335 \ge 0.$$

Since

$$a+c \ge \frac{a+b+c+d}{2} = \frac{13}{2}\ ,$$

we only need yo show that

$$a+c \le \frac{26+\sqrt{6}}{4}\ .$$

From

$$(c-b)(c-d) \le 0,$$

we get

$$c^2-(b+d)b+bd\le 0,$$

that is equivalent to

$$c^2+(b+d-c)^2-b^2-d^2\le 0\ ,$$

$$c^2+(13-a-2c)^2+a^2+c^2-43\le 0\ ,$$

$$3c^2-2(13-a)c+a^2-13a+63\le 0\ ,$$

$$c\le C,\qquad C=\frac{13-a+\sqrt{(4-a)(2a-5)}}{3}\ .$$

So, it suffices to show that

$$a+C\le\frac{26+\sqrt{6}}{4},$$

which is equivalent to

$$26+3\sqrt{6}-8a\ge 4\sqrt{(4-a)(2a-5)},$$

$$(\sqrt{6}+2)(4-a)+\frac{\sqrt{6}-2}{2}(2a-5)\ge 4\sqrt{(4-a)(2a-5)}\ .$$

Clearly, the last inequality is true (by the AM-GM inequality).

The left inequality is an equality for $a+c=b+d=\frac{13}{2}$ and $ac+bc=\frac{83}{4}$, while the right inequality is an equality for $a=\frac{13+\sqrt{6}}{4}$, $b=c=\frac{13}{4}$ and $d=\frac{13-\sqrt{6}}{4}$.

□

P 2.86. *If a,b,c,d are positive real numbers such that*

$$a^2+b^2+c^2+d^2=4,\qquad a\le b\le c\le d,$$

then

$$\frac{1}{a}+a+b+c+d\ge 5.$$

(Vasile Cîrtoaje, 2021)

Solution. Write the inequality in the homogeneous form

$$\frac{a^2+b^2+c^2+d^2}{4}+a(a+b+c+d)\ge 5a\sqrt{\frac{a^2+b^2+c^2+d^2}{4}}.$$

For fixed a,b,d, we need to prove that $f(c)\ge 0$, where

$$f(c)=5a^2+b^2+c^2+d^2+4a(b+c+d)-10a\sqrt{a^2+b^2+c^2+d^2},\qquad c\in[b,d].$$

From

$$f'(c)=2c+4a-\frac{10ac}{\sqrt{a^2+b^2+c^2+d^2}}\ge 4a+2c-\frac{10ac}{\sqrt{2(a^2+c^2)}}$$

$$\geq 4\sqrt{2ac} - 5\sqrt{ac} = (4\sqrt{2} - 5)\sqrt{ac} > 0,$$

it follows that $f(c)$ is increasing, hence $f(c) \geq f(b)$. The inequality $f(b) \geq 0$ is equivalent to

$$5a^2 + 2b^2 + d^2 + 4a(2b + d) - 10a\sqrt{a^2 + 2b^2 + d^2} \geq 0.$$

For fixed a and d, we need to show that $g(b) \geq 0$, where

$$g(b) = 5a^2 + 2b^2 + d^2 + 4a(2b + d) - 10a\sqrt{a^2 + 2b^2 + d^2}, \qquad b \in [a, d].$$

From

$$g'(b) = 4b + 8a - \frac{20ab}{\sqrt{a^2 + 2b^2 + d^2}} \geq 4b + 8a - \frac{20ab}{\sqrt{a^2 + 3b^2}}$$

$$\geq 8\sqrt{2ab} - \frac{20\sqrt{ab}}{\sqrt{2\sqrt{3}}} = 4\left(2\sqrt{2} - \frac{5}{\sqrt{2\sqrt{3}}}\right)\sqrt{ab} > 0,$$

it follows that $g(b)$ is increasing, hence $g(b) \geq g(a)$, that is

$$g(b) \geq 15a^2 + 4ad + d^2 - 10a\sqrt{3a^2 + d^2}.$$

Thus, we only need to show that

$$15a^2 + 4ad + d^2 \geq 10a\sqrt{3a^2 + d^2}.$$

Due to homogeneity, we may set $a = 1$, hence $d \geq 1$. We need to show that

$$(15 + 4d + d^2)^2 \geq 100(3 + d^2),$$

which is equivalent to

$$d^4 + 8d^3 - 54d^2 + 120d - 75 \geq 0,$$

$$(d - 1)(d^3 + 9d^2 - 45d + 75) \geq 0.$$

This is true because

$$d^3 + 9d^2 - 45d + 75 > 9d^2 - 45d + 63 = 9(d^2 - 5d + 7) > 0.$$

The equality holds for $a = b = c = d = 1$.

Remark. Similarly, we can prove the following stronger inequality

$$\frac{3}{4a} + a + b + c + d \geq \frac{19}{4}.$$

□

P 2.87. *If a, b, c, d are real numbers, then*

$$6(a^2 + b^2 + c^2 + d^2) + (a + b + c + d)^2 \geq 12(ab + bc + cd).$$

(Vasile Cîrtoaje, 2005)

Solution. Let

$$E(a,b,c,d) = 6(a^2+b^2+c^2+d^2) + (a+b+c+d)^2 - 12(ab+bc+cd).$$

First Solution. We have

$$E(x+a, x+b, x+c, x+d) =$$
$$= 4x^2 + 4(2a-b-c+2d)x + 7(a^2+b^2+c^2+d^2) + 2(ac+ad+bd) - 10(ab+bc+cd)$$
$$= (2x+2a-b-c+2d)^2 + 3(a^2+2b^2+2c^2+d^2-2ab+2ac-2ad-4bc+2bd-2cd)$$
$$= (2x+2a-b-c+2d)^2 + 3(b-c)^2 + 3(a-b+c-d)^2.$$

For $x = 0$, we get

$$E(a,b,c,d) = (2a-b-c+2d)^2 + 3(b-c)^2 + 3(a-b+c-d)^2 \geq 0.$$

The equality holds for $2a = b = c = 2d$.

Second Solution. Let

$$x = a-b, \qquad y = c-d.$$

We have

$$\begin{aligned}
E &= 6[(a-b)^2 + (c-d)^2] + (a+b+c+d)^2 - 12bc \\
&= 6(x^2+y^2) + [x+y+2(b+c)]^2 - 12bc \\
&= 3(x-y)^2 + 3(x+y)^2 + [x+y+2(b+c)]^2 - 12bc \\
&= 3(x-y)^2 + 4(x+y)^2 + 4(x+y)(b+c) + (b+c)^2 + 3(b-c)^2 \\
&= 3(x-y)^2 + (2x+2y+b+c)^2 + 3(b-c)^2 \geq 0.
\end{aligned}$$

□

P 2.88. *If a,b,c,d are positive real numbers, then*

$$\frac{1}{a^2+ab} + \frac{1}{b^2+bc} + \frac{1}{c^2+cd} + \frac{1}{d^2+da} \geq \frac{4}{ac+bd}.$$

Solution. Write the inequality as follows:

$$\sum \left(\frac{ac+bd}{a^2+ab} + 1 \right) \geq 8,$$

$$\sum \frac{a(c+a)+b(d+a)}{a(a+b)} \geq 8,$$

$$\sum \frac{c+a}{a+b} + \sum \frac{b(d+a)}{a(a+b)} \geq 8.$$

By the AM-GM inequality, we have

$$\sum \frac{b(d+a)}{a(a+b)} \ge 4\sqrt[4]{\prod \frac{b(d+a)}{a(a+b)}} = 4.$$

Therefore, it suffices to prove the inequality

$$\sum \frac{c+a}{a+b} \ge 4,$$

which is equivalent to

$$(a+c)\left(\frac{1}{a+b}+\frac{1}{c+d}\right)+(b+d)\left(\frac{1}{b+c}+\frac{1}{d+a}\right) \ge 4.$$

This inequality follows immediately from

$$\frac{1}{a+b}+\frac{1}{c+d} \ge \frac{4}{(a+b)+(c+d)}$$

and

$$\frac{1}{b+c}+\frac{1}{d+a} \ge \frac{4}{(b+c)+(d+a)}.$$

The equality occurs for $a=b=c=d$.

□

P 2.89. *If a,b,c,d are positive real numbers, then*

$$\frac{1}{a(1+b)}+\frac{1}{b(1+a)}+\frac{1}{c(1+d)}+\frac{1}{d(1+c)} \ge \frac{16}{1+8\sqrt{abcd}}.$$

(Vasile Cîrtoaje, 2007)

Solution. Let

$$x=\sqrt{ab}, \quad y=\sqrt{cd}.$$

Write the inequality as

$$\frac{a+b+2ab}{ab(1+a)(1+b)}+\frac{c+d+2cd}{cd(1+c)(1+d)} \ge \frac{16}{1+8\sqrt{abcd}}.$$

We claim that

$$x \ge 1 \quad \Longrightarrow \quad \frac{a+b+2ab}{ab(1+a)(1+b)} \ge \frac{1}{ab},$$

and

$$x \le 1 \quad \Longrightarrow \quad \frac{a+b+2ab}{ab(1+a)(1+b)} \ge \frac{2}{\sqrt{ab}+ab}.$$

The first inequality is equivalent to $ab \ge 1$, while the second inequality is equivalent to

$$\left(1-\sqrt{ab}\right)\left(\sqrt{a}-\sqrt{b}\right)^2 \ge 0.$$

Similarly, we have

$$y \ge 1 \quad \Longrightarrow \quad \frac{c+d+2cd}{cd(1+d)(1+d)} \ge \frac{1}{cd}$$

and

$$y \le 1 \quad \Longrightarrow \quad \frac{c+d+2cd}{cd(1+d)(1+d)} \ge \frac{2}{\sqrt{cd}+cd}.$$

There are four cases to consider.

Case 1: $x \ge 1$, $y \ge 1$. It suffices to show that

$$\frac{1}{x^2}+\frac{1}{y^2} \ge \frac{16}{1+8xy}.$$

Indeed, we have

$$\frac{1}{x^2}+\frac{1}{y^2} \ge \frac{2}{xy} > \frac{16}{1+8xy}.$$

Case 2: $x \le 1$, $y \le 1$. It suffices to show that

$$\frac{2}{x+x^2}+\frac{2}{y+y^2} \ge \frac{16}{1+8xy}.$$

Putting $s = x + y$ and $p = \sqrt{xy}$, this inequality becomes

$$\frac{s^2+s-2p^2}{p^2(s+p^2+1)} \ge \frac{8}{1+8p^2},$$

$$(1+8p^2)s^2+s-24p^4-10p^2 \ge 0.$$

Since $s \ge 2p$, we get

$$\begin{aligned}(1+8p^2)s^2+s-24p^4-10p^2 &\ge 4(1+8p^2)p^2+2p-24p^4-10p^2\\ &= 2p(p+1)(2p-1)^2 \ge 0.\end{aligned}$$

Case 3: $x \ge 1$, $y \le 1$. It suffices to show that

$$\frac{1}{x^2}+\frac{2}{y+y^2} \ge \frac{16}{1+8xy}.$$

This inequality is equivalent in succession to

$$(1+8xy)(2x^2+y^2+y) \ge 16x^2y(1+y),$$

$$(1+8xy)(x-y)^2+8x^3y+8xy^2-16x^2y+2xy+x^2+y \ge 0,$$

$$(1+8xy)(x-y)^2+8xy(x-1)^2+8xy^2+x^2+y \ge 6xy.$$

The last inequality is true since the AM-GM inequality yields

$$8xy^2 + x^2 + y \geq 3\sqrt[3]{8xy^2 \cdot x^2 \cdot y} = 3\sqrt[3]{8x^3y^3} = 6xy.$$

Case 4: $x \leq 1$, $y \geq 1$. It suffices to show that

$$\frac{2}{x + x^2} + \frac{1}{y^2} \geq \frac{16}{1 + 8xy},$$

which is equivalent to

$$(1 + 8xy)(x - y)^2 + 8xy(y - 1)^2 + 8x^2y + y^2 + x \geq 6xy.$$

As in the case 3, we have

$$8x^2y + y^2 + x \geq 3\sqrt[3]{8x^2y \cdot y^2 \cdot x} = 3\sqrt[3]{8x^3y^3} = 6xy.$$

The proof is completed. The equality holds for $a = b = c = d = \dfrac{1}{2}$.

□

P 2.90. *If a, b, c, d are positive real numbers such that $a \geq b \geq c \geq d$ and*

$$a + b + c + d = 4,$$

then

$$ac + bd \leq 2.$$

(Vasile Cîrtoaje, 2019)

Solution. Write the inequality in the homogeneous form

$$(a + b + c + d)^2 \geq 8(ac + bd).$$

We have

$$(a + b + c + d)^2 - 8(ac + bd) = a^2 + 2(b + d - 3c)a + (b + c + d)^2 - 8bd$$

$$= (a + b + d - 3c)^2 - (b + d - 3c)^2 + (b + d + c)^2 - 8bd$$

$$= (a + b + d - 3c)^2 + 8(b - c)(c - d) \geq 0.$$

The equality holds for $b = c = 1$ and $a + d = 2$.

□

P 2.91. *If a, b, c, d are positive real numbers such that $a \geq b \geq c \geq d$ and*

$$a + b + c + d = 4,$$

then

$$2\left(\frac{1}{b} + \frac{1}{d}\right) \geq a^2 + b^2 + c^2 + d^2.$$

(Vasile Cîrtoaje, 2019)

Solution. Write the inequality in the homogeneous form

$$(a + b + c + d)^3\left(\frac{1}{b} + \frac{1}{d}\right) - 32(a^2 + b^2 + c^2 + d^2) \geq 0.$$

For fixed b, c, d, the inequality becomes $f(a) \geq 0$, with

$$f'(a) = 3(a + b + c + d)^2\left(\frac{1}{b} + \frac{1}{d}\right) - 64a.$$

For $a + b + c + d = 4$, when $a = 4 - b - c - d \leq 4 - b - 2d$, we have

$$\frac{1}{16}f'(a) \geq 3\left(\frac{1}{b} + \frac{1}{d}\right) - 4(4 - b - 2d)$$

$$= \left(\frac{3}{b} + 4b\right) + \left(\frac{3}{d} + 8d\right) - 16 \geq 4(\sqrt{3} + \sqrt{(}6) - 4) > 0.$$

Therefore, $f(a)$ is increasing, hence $f(a) \geq f(b)$. Similarly, for fixed a, b, d, the inequality becomes $g(c) \geq 0$, with

$$g'(c) = 3(a + b + c + d)^2\left(\frac{1}{b} + \frac{1}{d}\right) - 64c \geq f'(a) > 0.$$

Therefore, $g(c)$ is increasing, hence $g(c) \geq g(d)$. As a consequence, it suffices to prove the original inequality for $a = b$ and $c = d$. So, we only need to show that $b + d = 2$ involves

$$\frac{1}{b} + \frac{1}{d} \geq b^2 + d^2,$$

which is equivalent to

$$(bd - 1)^2 \geq 0.$$

The equality holds for $a = b = c = d = 1$.

□

P 2.92. *Let a, b, c, d be positive real numbers such that $a \ge b \ge c \ge d$ and*

$$ab + bc + cd + da = 3.$$

Prove that

$$a^3bcd < 4.$$

(Vasile Cîrtoaje, 2012)

Solution. Write the desired inequality as

$$4(ab + bc + cd + da)^3 > 27a^3bcd,$$

$$4\left(b + d + \frac{bc + cd}{a}\right)^3 > 27bcd.$$

It suffices to show that

$$4(b + d)^3 \ge 27bcd.$$

Indeed, by the AM-GM inequality, we have

$$(b + d)^3 = \left(\frac{b}{2} + \frac{b}{2} + d\right)^3 \ge 27\left(\frac{b}{2}\right)^2 d \ge \frac{27bcd}{4}.$$

□

P 2.93. *Let a, b, c, d be positive real numbers such that $a \ge b \ge c \ge d$ and*

$$ab + bc + cd + da = 6.$$

Prove that

$$acd \le 2.$$

(Vasile Cîrtoaje, 2012)

Solution. Write the desired inequality in the homogeneous form

$$(a + c)^3(b + d)^3 \ge 54a^2c^2d^2.$$

Since $b \ge c$, we only need to show that

$$(a + c)^3(c + d)^3 \ge 54a^2c^2d^2.$$

By the AM-GM inequality, we have

$$(a + c)^3 = \left(\frac{a}{2} + \frac{a}{2} + c\right)^3 \ge 27\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)c = \frac{27}{4}a^2c.$$

Thus, it suffices to show that

$$(c + d)^3 \ge 8cd^2.$$

Indeed,

$$(c + d)^3 - 8cd^2 = (c - d)(c^2 + 4cd - d^2) \ge 0.$$

The equality holds for $a = 2$ and $b = c = d = 1$.

□

P 2.94. *Let a, b, c, d be positive real numbers such that $a \ge b \ge c \ge d$ and*

$$ab + bc + cd + da = 9.$$

Prove that

$$abd \le 4.$$

(Vasile Cîrtoaje, 2012)

Solution. Write the desired inequality in the homogeneous form

$$(a+c)^3(b+d)^3 \ge \frac{729}{16}a^2b^2d^2.$$

Since $c \ge d$, we only need to show that

$$(a+d)^3(b+d)^3 \ge \frac{729}{16}a^2b^2d^2.$$

By the AM-GM inequality, we have

$$(a+d)^3 = \left(\frac{a}{2} + \frac{a}{2} + d\right)^3 \ge 27\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)d = \frac{27}{4}a^2d$$

and, similarly,

$$(b+d)^3 \ge \frac{27}{4}b^2d$$

Multiplying these inequalities, the desired inequality holds. The equality occurs for $a = b = 2$ and $c = d = 1$.

□

P 2.95. *Let a, b, c, d be positive real numbers such that $a \ge b \ge c \ge d$ and*

$$a^2 + b^2 + c^2 + d^2 = 10.$$

Prove that

$$2b + 4d \le 3c + 5.$$

(Vasile Cîrtoaje, 2012)

Solution. Write the desired inequality in the homogeneous form

$$2b - 3c + 4d \le \sqrt{\frac{5}{2}(a^2 + b^2 + c^2 + d^2)}.$$

It is true if

$$5(a^2 + b^2 + c^2 + d^2) \ge 2(2b - 3c + 4d)^2.$$

Since $a \ge b$, it remains to show that

$$5(2b^2 + c^2 + d^2) \ge 2(2b - 3c + 4d)^2,$$

which is equivalent to

$$2b^2 + 24bc + 48cd \geq 13c^2 + 27d^2 + 32bd.$$

Since $d^2 \leq cd$, it suffices to prove that

$$2b^2 + 24bc + 48cd \geq 13c^2 + 27cd + 32bd,$$

which is equivalent to

$$2b^2 + 24bc \geq 13c^2 + (32b - 21c)d.$$

Since $32b - 21c > 0$ and $c \geq d$, it is enough to show that

$$2b^2 + 24bc \geq 13c^2 + (32b - 21c)c.$$

This reduces to the obvious inequality

$$2(b - 2c)^2 \geq 0.$$

The equality holds for $a = b = 2$ and $c = d = 1$.

□

P 2.96. *Let a, b, c, d be positive real numbers such that $a \leq b \leq c \leq d$ and*

$$abcd = 1.$$

Prove that

$$4 + \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 2(a + b)(c + d).$$

Solution. Since

$$\frac{b}{c} + \frac{d}{a} - \frac{b}{a} - \frac{d}{c} = \frac{(d - b)(c - a)}{ca} \geq 0,$$

we only need to prove that

$$4 + \frac{a}{b} + \frac{b}{a} + \frac{c}{d} + \frac{d}{c} \geq 2(a + b)(c + d),$$

which is equivalent to

$$\frac{(a + b)^2}{ab} + \frac{(c + d)^2}{cd} \geq 2(a + b)(c + d),$$

$$\left(\frac{a + b}{\sqrt{ab}} - \frac{c + d}{\sqrt{cd}}\right)^2 \geq 0.$$

The proof is completed. The equality holds for $a = b = c = d = 1$.

□

P 2.97. *Let a, b, c, d be positive real numbers such that $a \ge b \ge c \ge d$ and*

$$3(a^2+b^2+c^2+d^2) = (a+b+c+d)^2.$$

Prove that

(a)
$$\frac{a+d}{b+c} \le 2;$$

(b)
$$\frac{a+c}{b+d} \le \frac{7+2\sqrt{6}}{5};$$

(c)
$$\frac{a+c}{c+d} \le \frac{3+\sqrt{5}}{2}.$$

(Vasile Cîrtoaje, 2010)

Solution. (a) Since

$$(a+d)(b+c) - 2(ad+bc) = (a-b)(c-d) + (a-c)(b-d) \ge 0,$$

we have

$$\begin{aligned} a^2+b^2+c^2+d^2 &= (a+d)^2 + (b+c)^2 - 2(ad+bc) \\ &\ge (a+d)^2 + (b+c)^2 - (a+d)(b+c), \end{aligned}$$

hence

$$\frac{1}{3}(a+b+c+d)^2 \ge (a+d)^2 + (b+c)^2 - (a+d)(b+c),$$

$$\left(\frac{a+d}{b+c} - 2\right)\left(\frac{a+d}{b+c} - \frac{1}{2}\right) \le 0,$$

from where the desired result follows. The equality holds for $a/3 = b = c = d$.

(b) From $(a-d)(b-c) \ge 0$ and the AM-GM inequality, we have

$$2(ac+bc) \le (a+d)(b+c) \le \frac{(a+b+c+d)^2}{4},$$

hence

$$\begin{aligned} a^2+b^2+c^2+d^2 &= (a+c)^2 + (b+d)^2 - 2(ac+bd) \\ &\ge (a+c)^2 + (b+d)^2 - \frac{(a+b+c+d)^2}{4}, \end{aligned}$$

$$\frac{1}{3}(a+b+c+d)^2 \ge (a+c)^2 + (b+d)^2 - \frac{(a+b+c+d)^2}{4},$$

$$\left(\frac{a+c}{b+d} - \frac{7+2\sqrt{6}}{2}\right)\left(\frac{a+c}{b+d} - \frac{7-2\sqrt{6}}{2}\right) \le 0,$$

from where the desired result follows. The equality holds for

$$(3-\sqrt{6})a = b = c = (3+\sqrt{6})d.$$

(c) Writing the hypothesis $3(a^2+b^2+c^2+d^2) = (a+b+c+d)^2$ as

$$b^2-(a+c+d)b+a^2+c^2+d^2-ac-cd-da=0,$$

$$(2b-a-c-d)^2 = 3(2ac+2cd+2da-a^2-c^2-d^2),$$

it follows that

$$2ac+2cd+2da \ge a^2+c^2+d^2,$$

$$a^2-2(c+d)a+(c-d)^2 \le 0,$$

$$a \le c+d+2\sqrt{cd}.$$

Thus, it suffices to prove that

$$\frac{2c+d+2\sqrt{cd}}{c+d} \le \frac{3+\sqrt{5}}{2},$$

which is equivalent to

$$(\sqrt{5}-1)c+(\sqrt{5}+1)d \ge 4\sqrt{cd}.$$

This inequality follows immediately from the AM-GM inequality. The equality holds for

$$\frac{a}{3+\sqrt{5}} = \frac{b}{4} = \frac{c}{2} = \frac{d}{3-\sqrt{5}}.$$

□

P 2.98. *Let a,b,c,d be nonnegative real numbers such that $a \ge b \ge c \ge d$ and*

$$2(a^2+b^2+c^2+d^2) = (a+b+c+d)^2.$$

Prove that

$$a \ge b+3c+(2\sqrt{3}-1)d.$$

(Vasile Cîrtoaje, 2010)

First Solution. For $c=d=0$, the desired inequality is an equality. Assume further that $c>0$. From the hypothesis $2(a^2+b^2+c^2+d^2) = (a+b+c+d)^2$, we get

$$a = b+c+d \pm 2\sqrt{bc+cd+db}.$$

It is not possible to have

$$a = b+c+d-2\sqrt{bc+cd+db},$$

because this equality and $a \ge b$ involve

$$c + d \ge 2\sqrt{bc + cd + db},$$

$$(c - d)^2 \ge 4b(c + d),$$

$$(c - d)^2 \ge 4c(c + d),$$

$$d^2 \ge 3c(c + 2d),$$

which is not true. Thus, we have

$$a = b + c + d + 2\sqrt{bc + cd + db}.$$

Using this equality, we can rewrite the desired inequality as

$$b + c + d - 2\sqrt{bc + cd + db} \ge b + 3c + (2\sqrt{3} - 1)d,$$

$$\sqrt{b(c + d) + cd} \ge c + (\sqrt{3} - 1)d.$$

Since $b \ge c$, it suffices to show that

$$\sqrt{c(c + d) + cd} \ge c + (\sqrt{3} - 1)d.$$

By squaring, we get the obvious inequality $d(c - d) \ge 0$. The equality holds for $a = b$ and $c = d = 0$, for $\frac{a}{4} = b = c$ and $d = 0$, and for $\frac{a}{3 + 2\sqrt{3}} = b = c = d$.

Second Solution (by *Vo Quoc Ba Can*). Write the hypothesis $2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2$ as

$$(a - b)^2 + (c - d)^2 \ge 2(a + b)(c + d).$$

Since

$$a + b \ge (a - b) + 2c,$$

we get

$$(a - b)^2 + (c - d)^2 \ge 2[(a - b) + 2c](c + d),$$

which is equivalent to

$$(a - b)^2 - 2(c + d)(a - b) - 3c^2 - 6cd + d^2 \ge 0.$$

From this, we get

$$a - b \ge c + d + 2\sqrt{c^2 + 2cd}.$$

Thus, the desired inequality

$$a - b \ge 3c + (2\sqrt{3} - 1)d$$

is true if

$$c + d + 2\sqrt{c^2 + 2cd} \ge 3c + (2\sqrt{3} - 1)d,$$

that is,

$$\sqrt{c^2 + 2cd} \ge c + (\sqrt{3} - 1)d.$$

By squaring, we get the obvious inequality $d(c - d) \ge 0$.

□

P 2.99. *If a, b, c, d, e are real numbers, then*

$$\frac{ab+bc+cd+de}{a^2+b^2+c^2+d^2+e^2} \le \frac{\sqrt{3}}{2}.$$

Solution. Using the AM-GM inequality, we have

$$a^2+b^2+c^2+d^2+e^2 = \left(a^2+\frac{1}{3}b^2\right)+\left(\frac{2}{3}b^2+\frac{1}{2}c^2\right)+\left(\frac{1}{2}c^2+\frac{2}{3}d^2\right)+\left(\frac{1}{3}d^2+e^2\right)$$

$$\ge 2\sqrt{a^2\cdot\frac{1}{3}b^2}+2\sqrt{\frac{2}{3}b^2\cdot\frac{1}{2}c^2}+2\sqrt{\frac{1}{2}c^2\cdot\frac{2}{3}d^2}+2\sqrt{\frac{1}{3}d^2\cdot e^2}$$

$$\ge \frac{2}{\sqrt{3}}(ab+bc+cd+da).$$

The equality holds for

$$a = \frac{b}{\sqrt{3}} = \frac{c}{2} = \frac{d}{\sqrt{3}} = e.$$

Remark. The following more general inequality holds

$$\frac{a_1a_2+a_2a_3+\cdots+a_{n-1}a_n}{a_1^2+a_2^2+\cdots+a_n^2} \le \cos\frac{\pi}{n+1},$$

with equality for

$$\frac{a_1}{\sin\frac{\pi}{n+1}} = \frac{a_2}{\sin\frac{2\pi}{n+1}} = \cdots = \frac{a_n}{\sin\frac{n\pi}{n+1}}.$$

Denoting

$$c_i = \frac{\sin\frac{(i+1)\pi}{n+1}}{2\sin\frac{i\pi}{n+1}}, \qquad i = 1, 2, \cdots, n-1,$$

we have

$$c_1 = \cos\frac{\pi}{n+1}, \qquad 4c_{n-1} = \frac{1}{\cos\frac{\pi}{n+1}},$$

$$\frac{1}{4c_i}+c_{i+1} = \cos\frac{\pi}{n+1}, \qquad i = 1, 2, \cdots, n-2,$$

hence

$$(a_1^2+a_2^2+\cdots+a_n^2)\cos\frac{\pi}{n+1} =$$

$$= c_1a_1^2+\left(\frac{1}{4c_1}+c_2\right)a_2^2+\cdots+\left(\frac{1}{4c_{n-2}}+c_{n-1}\right)a_{n-1}^2+\frac{1}{4c_{n-1}}a_n^2$$

$$= \left(c_1a_1^2+\frac{1}{4c_1}a_2^2\right)+\left(c_2a_2^2+\frac{1}{4c_2}a_3^2\right)+\cdots+\left(c_{n-1}a_{n-1}^2+\frac{1}{4c_{n-1}}a_n^2\right)$$

$$\geq 2\sqrt{c_1a_1^2 \cdot \frac{1}{4c_1}a_2^2} + 2\sqrt{c_2a_2^2 \cdot \frac{1}{4c_2}a_3^2} + \cdots + 2\sqrt{c_{n-1}a_{n-1}^2 \cdot \frac{1}{4c_{n-1}}a_n^2}$$

$$\geq a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n.$$

□

P 2.100. *If a, b, c, d, e are positive real numbers, then*

$$\frac{a^2b^2}{bd+ce} + \frac{b^2c^2}{cd+ae} + \frac{c^2a^2}{ad+be} \geq \frac{3abc}{d+e}.$$

Solution. Using the Cauchy-Schwarz inequality

$$\frac{a^2b^2}{bd+ce} + \frac{b^2c^2}{cd+ae} + \frac{c^2a^2}{ad+be} \geq \frac{(ab+bc+ca)^2}{(bd+ce)+(cd+ae)+(ad+be)},$$

it suffices to show that

$$\frac{(ab+bc+ca)^2}{(bd+ce)+(cd+ae)+(ad+be)} \geq \frac{3abc}{d+e},$$

which is equivalent to

$$\frac{(ab+bc+ca)^2}{a+b+c} \geq 3abc,$$

$$a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2 \geq 0.$$

The equality holds for $a = b = c$.

□

P 2.101. *Let a, b, c and x, y, z be positive real numbers such that*

$$x + y + z = a + b + c.$$

Prove that

$$ax^2 + by^2 + cz^2 + xyz \geq 4abc.$$

(Vasile Cîrtoaje, 1989)

First Solution. Write the inequality as $E \geq 0$, where

$$E = ax^2 + by^2 + cz^2 + xyz - 4abc.$$

Among the numbers

$$a - \frac{y+z}{2}, \quad b - \frac{z+x}{2}, \quad c - \frac{x+y}{2},$$

there are two of them with the same sign; let

$$pq \geq 0,$$

where

$$p = b - \frac{z+x}{2}, \quad q = c - \frac{x+y}{2}.$$

We have

$$b = p + \frac{x+z}{2}, \quad c = q + \frac{x+y}{2}, \quad a = x+y+z-b-c = \frac{y+z}{2} - p - q.$$

Then,

$$\begin{aligned} E = & \left(\frac{y+z}{2} - p - q\right)x^2 + \left(p + \frac{x+z}{2}\right)y^2 + \left(q + \frac{x+y}{2}\right)z^2 \\ & + xyz - 4\left(\frac{y+z}{2} - p - q\right)\left(p + \frac{x+z}{2}\right)\left(q + \frac{x+y}{2}\right) \\ = & 4pq(p+q) + 2p^2(x+y) + 2q^2(x+z) + 4pqx \\ = & 4q^2\left(p + \frac{x+z}{2}\right) + 4p^2\left(q + \frac{x+y}{2}\right) + 4pqx \\ = & 4(q^2b + p^2c + pqx) \geq 0. \end{aligned}$$

The equality holds for $a = \dfrac{y+z}{2}$, $b = \dfrac{z+x}{2}$, $c = \dfrac{x+y}{2}$.

Second Solution. Consider the following two cases.

Case 1: $x^2 \geq 4bc$. We have

$$ax^2 + by^2 + cz^2 + xyz - 4abc > ax^2 - 4abc \geq 0.$$

Case 2: $x^2 \leq 4bc$. Let

$$u = x + y + z = a + b + c.$$

Substituting

$$z = u - x - y, \quad a = u - b - c,$$

the inequality can be restated as

$$Au^2 + Bu + C \geq 0,$$

where

$$A = c,$$

$$B = (x^2 - 4bc) - 2c(x+y) + xy,$$

$$C = -(b+c)(x^2 - 4bc) + by^2 + c(x+y)^2 - xy(x+y).$$

Since the quadratic function $Au^2 + Bu + C$ has the discriminant

$$D = (x^2 - 4bc)(2c - x - y)^2 \leq 0,$$

the conclusion follows.

□

P 2.102. *Let a, b, c and x, y, z be positive real numbers such that*

$$x+y+z=a+b+c.$$

Prove that

$$\frac{x(3x+a)}{bc}+\frac{y(3y+b)}{ca}+\frac{z(3z+c)}{ab}\ge 12.$$

(Vasile Cîrtoaje, 1990)

Solution. Write the inequality as

$$ax^2+by^2+cz^2+\frac{1}{3}(a^2x+b^2y+c^2z)\ge 4abc.$$

Applying the Cauchy-Schwarz inequality, we have

$$a^2x+b^2y+c^2z\ge\frac{(a+b+c)^2}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}=\frac{xyz(x+y+z)^2}{xy+yz+zx}\ge 3xyz.$$

Therefore, it suffices to show that

$$ax^2+by^2+cz^2+xyz\ge 4abc,$$

which is just the inequality in the previous P 2.101. The equality holds for

$$x=y=z=a=b=c.$$

□

P 2.103. *Let a, b, c be given positive numbers. Find the minimum value $F(a, b, c)$ of*

$$E(x,y,z)=\frac{ax}{y+z}+\frac{by}{z+x}+\frac{cz}{x+y},$$

where x, y, z are nonnegative real numbers, no two of which are zero.

(Vasile Cîrtoaje, 2006)

Solution. Assume that

$$a=\max\{a,b,c\}.$$

There are two cases to consider.

Case 1: $\sqrt{a}<\sqrt{b}+\sqrt{c}$. Using the Cauchy-Schwarz inequality, we get

$$E=\sum\frac{a(x+y+z)-a(y+z)}{y+z}=(x+y+z)\sum\frac{a}{y+z}-\sum a$$

$$\ge(x+y+z)\frac{(\sum\sqrt{a})^2}{\sum(y+z)}-\sum a=\sqrt{ab}+\sqrt{bc}+\sqrt{ca}-\frac{a+b+c}{2}.$$

The equality holds for

$$\frac{y+z}{\sqrt{a}}=\frac{z+x}{\sqrt{b}}=\frac{x+y}{\sqrt{c}};$$

that is, for

$$\frac{x}{\sqrt{b}+\sqrt{c}-\sqrt{a}}=\frac{y}{\sqrt{c}+\sqrt{a}-\sqrt{b}}=\frac{z}{\sqrt{a}+\sqrt{b}-\sqrt{c}}.$$

Case 2: $\sqrt{a}\ge\sqrt{b}+\sqrt{c}$. Let us denote

$$A=(\sqrt{b}+\sqrt{c})^2,$$

$$X=\frac{y+z}{2},\qquad Y=\frac{z+x}{2},\qquad Z=\frac{x+y}{2},$$

hence

$$x=Y+Z-X,\qquad y=Z+X-Y,\qquad z=X+Y-Z.$$

We have

$$\begin{aligned}
E&\ge\frac{Ax}{y+z}+\frac{by}{z+x}+\frac{cz}{x+y}\\
&=\frac{A(Y+Z-X)}{2X}+\frac{b(Z+X-Y)}{2Y}+\frac{c(X+Y-Z)}{2Z}\\
&=\frac{1}{2}\left(A\frac{Y}{X}+b\frac{X}{Y}\right)+\frac{1}{2}\left(b\frac{Z}{Y}+c\frac{Y}{Z}\right)+\frac{1}{2}\left(c\frac{X}{Z}+A\frac{Z}{X}\right)-b-c-\sqrt{bc}\\
&\ge\sqrt{Ab}+\sqrt{bc}+\sqrt{cA}-b-c-\sqrt{bc}=2\sqrt{bc}.
\end{aligned}$$

The equality holds for $x=0$ and $\dfrac{y}{z}=\sqrt{\dfrac{c}{b}}$. Therefore, for $a=\max\{a,b,c\}$, we have

$$F(a,b,c)=\begin{cases}\sqrt{ab}+\sqrt{bc}+\sqrt{ca}-\dfrac{a+b+c}{2}, & \sqrt{a}<\sqrt{b}+\sqrt{c}\\ 2\sqrt{bc}, & \sqrt{a}\ge\sqrt{b}+\sqrt{c}\end{cases}.$$

□

P 2.104. *Let a,b,c and x,y,z be positive real numbers such that*

$$\frac{a}{yz}+\frac{b}{zx}+\frac{c}{xy}=1.$$

Prove that

(a) $$x+y+z\ge\sqrt{4(a+b+c+\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\ )+3\sqrt[3]{abc}};$$

(b) $$x+y+z>\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}.$$

Solution. (a) Write the desired inequality in the form

$$\left(\frac{a}{yz}+\frac{b}{zx}+\frac{c}{xy}\right)(x+y+z)^2 \ge 4\left(a+b+c+\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)+3\sqrt[3]{abc}.$$

We have

$$\left(\frac{a}{yz}+\frac{b}{zx}+\frac{c}{xy}\right)(x^2+y^2+z^2)=\sum\frac{ax^2}{yz}+\sum\frac{a(y^2+z^2)}{yz}.$$

In addition, by the AM-GM inequality, we get

$$\sum\frac{ax^2}{yz}\ge 3\sqrt[3]{abc},$$

$$\sum\frac{a(y^2+z^2)}{yz}\ge 2(a+b+c).$$

Therefore,

$$\left(\frac{a}{yz}+\frac{b}{zx}+\frac{c}{xy}\right)(x^2+y^2+z^2)\ge 3\sqrt[3]{abc}+2(a+b+c).$$

Adding this inequality to the Cauchy-Schwarz inequality

$$2\left(\frac{a}{yz}+\frac{b}{zx}+\frac{c}{xy}\right)(yz+zx+xy)\ge 2\left(\sqrt{a}+\sqrt{a}+\sqrt{c}\right)^2$$

yields the desired inequality. The equality holds for

$$x=y=z=\sqrt{3a}=\sqrt{3b}=\sqrt{3c}.$$

(b) According to the inequality in (a), it suffices to show that

$$4\left(a+b+c+\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)\ge\left(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\right)^2.$$

This inequality is equivalent to

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2\ge\sqrt{(a+b)(b+c)}+\sqrt{(b+c)(c+a)}+\sqrt{(c+a)(a+b)}\ ,$$

which follows immediately from the inequality P 2.24 in Volume 2.

□

P 2.105. *If a,b,c and x,y,z are nonnegative real numbers, then*

$$\frac{2}{(b+c)(y+z)}+\frac{2}{(c+a)(z+x)}+\frac{2}{(a+b)(x+y)}\ge\frac{9}{(b+c)x+(c+a)y+(a+b)z}.$$

(Ji Chen and Vasile Cîrtoaje, 2010)

Solution. Since

$$(b+c)x+(c+a)y+(a+b)z = a(y+z)+(b+c)x+bz+cy,$$

we can write the inequality as

$$\sum \frac{2a(y+z)+2(b+c)x+2(bz+cy)}{(b+c)(y+z)} \geq 9,$$

$$\sum \frac{2a}{b+c} + \sum \frac{2x}{y+z} \geq 9 - \sum \frac{2(bz+cy)}{(b+c)(y+z)},$$

$$\sum \frac{2a}{b+c} + \sum \frac{2x}{y+z} \geq 6 + \sum \left[1 - \frac{2(bz+cy)}{(b+c)(y+z)}\right],$$

$$\sum \frac{2a}{b+c} + \sum \frac{2x}{y+z} \geq 6 + \sum \frac{(b-c)(y-z)}{(b+c)(y+z)}.$$

Since

$$\sum \frac{(b-c)(y-z)}{(b+c)(y+z)} \leq \frac{1}{2}\sum \left(\frac{b-c}{b+c}\right)^2 + \frac{1}{2}\sum \left(\frac{y-z}{y+z}\right)^2,$$

it suffices to show that

$$\sum \frac{2a}{b+c} + \sum \frac{2x}{y+z} \geq 6 + \frac{1}{2}\sum \left(\frac{b-c}{b+c}\right)^2 + \frac{1}{2}\sum \left(\frac{y-z}{y+z}\right)^2,$$

which is equivalent to

$$\sum \frac{2a}{b+c} + \sum \frac{2x}{y+z} \geq 9 - \sum \frac{2bc}{(b+c)^2} - \sum \frac{2yz}{(y+z)^2},$$

$$\sum \left[\frac{2a}{b+c} + \frac{2bc}{(b+c)^2}\right] + \sum \left[\frac{2x}{y+z} + \frac{2yz}{(y+z)^2}\right] \geq 9,$$

$$2(ab+bc+ca)\sum \frac{1}{(b+c)^2} + 2(xy+yz+zx)\sum \frac{1}{(y+z)^2} \geq 9.$$

This inequality can be obtained by summing the known inequalities (see P 1.72 in Volume 2, case $k=2$)

$$4(ab+bc+ca)\sum \frac{1}{(b+c)^2} \geq 9,$$

$$4(xy+yz+zx)\sum \frac{1}{(y+z)^2} \geq 9.$$

The equality holds for $a=b=c$ and $x=y=z$, and also for $a=x=0$, $b=c$ and $y=z$ (or any cyclic permutation).

Remark. For $x=a$, $y=b$ and $z=c$, we get the known inequality (Iran 1996):

$$\frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} + \frac{1}{(b+c)^2} \geq \frac{9}{4(ab+bc+ca)}.$$

□

P 2.106. *Let a, b, c be the lengths of the sides of a triangle. If x, y, z are real numbers, then*

$$(ya^2 + zb^2 + xc^2)(za^2 + xb^2 + yc^2) \geq (xy + yz + zx)(a^2b^2 + b^2c^2 + c^2a^2).$$

(Vasile Cîrtoaje, 2001)

First Solution. Write the inequality as follows:

$$x^2b^2c^2 + y^2c^2a^2 + z^2a^2b^2 \geq \sum yza^2(b^2 + c^2 - a^2),$$

$$x^2b^2c^2 + y^2c^2a^2 + z^2a^2b^2 \geq 2abc \sum yza \cos A,$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \geq \frac{2yz \cos A}{bc} + \frac{2zx \cos B}{ca} + \frac{2xy \cos C}{ab},$$

$$\left(\frac{x}{a} - \frac{y}{b} \cos C - \frac{z}{c} \cos B\right)^2 + \left(\frac{y}{b} \sin C - \frac{z}{c} \sin B\right)^2 \geq 0.$$

The equality holds for

$$\frac{x}{a^2} = \frac{y}{b^2} = \frac{z}{c^2}.$$

Second Solution. Write the inequality as

$$b^2c^2x^2 - Bx + C \geq 0,$$

where

$$B = c^2(a^2 + b^2 - c^2)y + b^2(a^2 - b^2 + c^2)z,$$

$$C = a^2[c^2y^2 - (b^2 + c^2 - a^2)yz + b^2z^2].$$

It suffices to show that

$$B^2 - 4b^2c^2C \leq 0,$$

which is equivalent to

$$A(c^2y - b^2z)^2 \geq 0,$$

where

$$A = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4.$$

This inequality is true since

$$A = (a + b + c)(a + b - c)(b + c - a)(c + a - b) \geq 0.$$

Remark 1. For $x = 1/b$, $y = 1/c$ and $z = 1/a$, we get the well-known inequality from P 1.168-(a):

$$a^3b + b^3c + c^3a \geq a^2b^2 + b^2c^2 + c^2a^2.$$

Remark 2. For $x = 1/c^2$, $y = 1/a^2$ and $z = 1/b^2$, we get the elegant cyclic inequality of *Walker*:

$$3\left(\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}\right) \geq (a^2 + b^2 + c^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right).$$

□

P 2.107. *If a, b, c are nonnegative real numbers such that*

$$2(a+b+c)+ab+bc+ca=9,$$

then

$$(a+1)bc+3(b+c) \le \frac{16}{a+1}.$$

(Vasile Cîrtoaje, 2021)

Solution. Assume that a is fixed, and denote

$$x=\frac{b+c}{2}, \qquad y=bc, \qquad x^2 \ge y.$$

Thus, we need to show that

$$(a+1)^2y+6(a+1)x \le 16$$

for

$$2(a+2)x+y=9-2a, \qquad 0 \le a \le \frac{9}{2}, \qquad x^2 \ge y \ge 0.$$

From

$$2(a+2)x+x^2 \ge 9-2a,$$

we get

$$x \ge x_m, \qquad x_m=-a-2+\sqrt{a^2+2a+13},$$

with equality for $x^2=y$, and from

$$2(a+2)x \le 9-2a,$$

we get

$$x \le x_M, \qquad x_M=\frac{9-2a}{2(a+2)},$$

with equality for $y=0$. Write now the required inequality in the form

$$2(a+1)(a^2+3a-1)x+16 \ge (a+1)^2(9-2a).$$

Case 1: $a^2+3a-1 \ge 0$. It suffices to prove the required inequality for $x=x_m$, that is for $x^2=y$. So, we need to show that $(a+1)^2y+6(a+1)x \le 16$ for $x^2=y$, when

$$2(a+2)x+x^2=9-2a, \qquad a=\frac{9-4x-x^2}{2(1+x)}.$$

The inequality $(a+1)^2x^2+6(a+1)x \le 16$ is true if $(a+1)x \le 2$, which is equivalent to

$$\frac{x(11-2x-x^2)}{2(1+x)} \le 2,$$

$$x^3+2x^2-7x+4 \ge 0,$$

$$(x-1)^2(x+4) \geq 0.$$

Case 2: $a^2+3a-1 \leq 0$. It suffices to prove the required inequality for $x = x_M$, that is for $y = 0$. So, we need to show that $(a+1)^2y+6(a+1)x \leq 16$ for $y = 0$, when

$$2(a+2)x = 9-2a.$$

We have

$$16-(a+1)^2y-6(a+1)x = 16-6(a+1)x$$
$$= 16-\frac{3(a+1)(9-2a)}{a+2} = \frac{6a^2-5a+5}{a+2} > 0.$$

The equality holds for $a = b = c = 1$.

□

P 2.108. *If a, b, c are nonnegative real numbers such that*

$$2(a+b+c)+ab+bc+ca = 9,$$

then

$$\frac{1}{ab+4}+\frac{1}{ac+4}+\frac{1}{b+4}+\frac{1}{c+4} \geq \frac{4}{5}.$$

(Vasile Cîrtoaje, 2021)

Solution. By the AM-HM inequality, we have

$$\frac{1}{ab+4}+\frac{1}{b+4} \geq \frac{4}{(ab+4)+(b+4)} = \frac{4}{b(a+1)+8}$$

and

$$\frac{1}{ac+4}+\frac{1}{c+4} \geq \frac{4}{c(a+1)+8}.$$

Thus, it suffices to show that

$$\frac{1}{b(a+1)+8}+\frac{1}{c(a+1)+8} \geq \frac{1}{5},$$

which is equivalent to

$$(a+1)(b+c)+16 \geq \frac{[b(a+1)+8][c(a+1)+8]}{5},$$

$$(a+1)(b+c)+16 \geq \frac{(a+1)^2bc+8(a+1)(b+c)+64}{5},$$

$$16 \geq (a+1)^2bc+3(a+1)(b+c),$$

$$\frac{16}{a+1} \geq (a+1)bc+3(b+c).$$

The last inequality was proved at the previous P 2.107.

The equality holds for $a = b = c = 1$.

□

P 2.109. *If a, b, c are nonnegative real numbers such that*

$$6a^2 + 4a(b + c) + bc = 15,$$

then

$$\frac{4}{a^2 + 1} + \frac{1}{b^2 + 1} + \frac{1}{c^2 + 1} \ge 3.$$

(Vasile Cîrtoaje, 2021)

Solution. For $a = 0$, the inequality is clearly true. Assume that $a > 0$ and $b \ge c \ge 0$. For fixed a, from

$$6a^2 + 8a\sqrt{bc} + bc \le 15,$$

it follows that the product $p = bc$ has the maximum value when $b = c$ (for $b + c = 2\sqrt{bc}$), and the minimum value when $c = 0$. There are two cases to consider: $p \ge 1$ and $p \le 1$.

Case 1: $p \ge 1$. Since

$$\frac{1}{b^2 + 1} + \frac{1}{c^2 + 1} - \frac{2}{bc + 1} = \frac{(b - c)^2(bc - 1)}{(b^2 + 1)(c^2 + 1)(bc + 1)} \ge 0,$$

it suffices to show that

$$\frac{4}{a^2 + 1} + \frac{2}{p + 1} \ge 3.$$

For fixed a, p has the maximum value when $b = c$. Thus, we only need to consider the case $b = c$, that is to show that $6a^2 + 8ab + b^2 = 15$ implies

$$\frac{4}{a^2 + 1} + \frac{2}{b^2 + 1} \ge 3.$$

Write the inequality in the homogeneous form

$$\frac{4}{21a^2 + 8ab + b^2} + \frac{1}{3a^2 + 4ab + 8b^2} \ge \frac{3}{6a^2 + 8ab + b^2}.$$

It suffices to prove this inequality for $b = 0$ and $b = 1$. For $b = 0$, the inequality is clearly true, while for $b = 1$, it is equivalent to

$$(11a^2 + 8a + 11)(6a^2 + 8a + 1) \ge (21a^2 + 8a + 1)(3a^2 + 4a + 8),$$

$$3a^4 + 28a^3 - 62a^2 + 28a + 3 \ge 0,$$

$$(a - 1)^2(3a^2 + 34a + 3) \ge 0.$$

Case 2: $p \le 1$. Since

$$\frac{1}{b^2 + 1} + \frac{1}{c^2 + 1} = \frac{b^2 + c^2 + 2}{b^2c^2 + 1 + b^2 + c^2} = 1 + \frac{1 - p^2}{(1 - p)^2 + (b + c)^2},$$

we may write the inequality in the form

$$\frac{1-p^2}{(1-p)^2+(b+c)^2} \geq \frac{a^2-1}{2(a^2+1)}.$$

For $a \leq 1$, the inequality is clearly true. For fixed $a > 1$, since

$$b+c = \frac{15-6a^2-p}{4a},$$

the inequality is equivalent to

$$\frac{32a^2(1-p^2)}{16a^2(p-1)^2+(15-6a^2-p)^2} \geq \frac{a^2-1}{a^2+1},$$

or

$$Ap^2+Bp+C \geq 0,$$

where

$$A = -32a^2(a^2+1)-(a^2-1)(16a^2+1) = -(48a^4+17a^2-1).$$

Since $A < 0$ for $a > 1$, the polynomial Ap^2+Bp+C has the minimum value when p is minimum (when $c = 0$) or maximum (when $b = c$). Thus, we only need to consider these cases.

Sub-case 1: $c = 0$. We need to show that

$$\frac{4}{a^2+1}+\frac{1}{b^2+1} \geq 2$$

for

$$b = \frac{3(5-2a^2)}{4a}, \qquad 1 < a^2 \leq \frac{5}{2}.$$

The inequality is equivalent to

$$\frac{2}{a^2+1}+\frac{8a^2}{36a^4-164a^2+225} \geq 1,$$

$$225-381a^2+208a^4-36a^6 \geq 0,$$

$$(3-2a^2)^2(25-9a^2) \geq 0.$$

It is true since

$$25-9a^2 \geq 25-\frac{45}{2} > 0.$$

Sub-case 2: $b = c$. As shown previously, for the nontrivial case $b = c > 0$, the inequality reduces to the obvious form

$$(a-1)^2(3a^2+34a+3) \geq 0.$$

The equality holds for $a=b=c=1$, for $a=b=\sqrt{\dfrac{3}{2}}$ and $c=0$, and for $a=c=\sqrt{\dfrac{3}{2}}$ and $b=0$.

Remark. Similarly, we can prove the following generalization:

- *Let a,b,c be nonnegative real numbers such that*

$$k(k-1)a^2+2ka(b+c)+2bc=(k+1)(k+2),$$

where $k\ge 1$. Then

$$\frac{k}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}\ge\frac{k+2}{2},$$

with equality for $a=b=c=1$, for $a=b=\sqrt{\dfrac{k+2}{k}}$ and $c=0$, and for $a=c=\sqrt{\dfrac{k+2}{k}}$ and $b=0$.

□

P 2.110. *Let $a_1,\ a_2,\ \dots,\ a_n$ be positive real numbers such that $a_1\ge 2a_2$. Prove that*

$$(5n-1)(a_1^2+a_2^2+\cdots+a_n^2)\ge 5(a_1+a_2+\cdots+a_n)^2.$$

(Vasile Cîrtoaje, 2009)

Solution. Let

$$a_1=ka_2,\qquad k\ge 2.$$

By the Cauchy-Schwarz inequality, we have

$$a_1^2+a_2^2+\cdots+a_n^2=(k^2+1)a_2^2+a_3^2+\cdots+a_n^2$$

$$\ge\frac{[(k+1)a_2+a_3+\cdots+a_n]^2}{\dfrac{(k+1)^2}{k^2+1}+n-2}=\frac{(a_1+a_2+\cdots+a_n)^2}{\dfrac{2k}{k^2+1}+n-1}.$$

Therefore, it suffices to show that

$$\frac{5n-1}{5}\ge\frac{2k}{k^2+1}+n-1,$$

which is equivalent to the obvious inequality

$$(k-2)(2k-1)\ge 0.$$

The equality holds if and only if $k=2$ and

$$5a_2^2+a_3^2+\cdots+a_n^2=\frac{(3a_2+a_3+\cdots+a_n)^2}{\frac{9}{5}+n-2};$$

that is, if and only if

$$\frac{5a_1}{6}=\frac{5a_2}{3}=a_3=\cdots=a_n.$$

□

P 2.111. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 \geq 4a_2$, then*

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq \left(n + \frac{1}{2}\right)^2.$$

Solution. Setting

$$a_1 = ka_2, \qquad k \geq 4,$$

the inequality becomes

$$[(1+k)a_2 + a_3 + \cdots + a_n]\left(\frac{1+k}{ka_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}\right) \geq \left(n + \frac{1}{2}\right)^2.$$

By the Cauchy-Schwarz inequality, we have

$$[(1+k)a_2 + a_3 + \cdots + a_n]\left(\frac{1+k}{ka_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}\right) \geq \left(\frac{1+k}{\sqrt{k}} + n - 2\right)^2.$$

Thus, we only need to show that

$$\frac{1+k}{\sqrt{k}} + n - 2 \geq n + \frac{1}{2},$$

which reduces to

$$\left(\sqrt{k} - 2\right)\left(2\sqrt{k} - 1\right) \geq 0.$$

The equality holds if and only if $k = 4$ and

$$\frac{a_1}{2} = 2a_2 = a_3 = \cdots = a_n.$$

□

P 2.112. *Suppose $n \geq 3$ and $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1 \leq a_2 \leq \cdots \leq a_n$.*

(a) Prove that

$$\frac{a_1a_2 + a_2a_3 + \cdots + a_na_1}{n} \geq \left(\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}\right)^2;$$

(Vasile Cîrtoaje, GM-B, 2, 2023)

(b) If $k \geq k_1 = \dfrac{2}{1 + \sqrt{\dfrac{n}{n-2}}}$, prove that

$$\frac{a_1a_2 + a_2a_3 + \cdots + a_na_1}{n} \geq \left(\frac{ka_1 + a_2 + \cdots + a_{n-1}}{n-2+k}\right)^2.$$

(Vasile Cîrtoaje, Recreatii Matematice, 2, 2023)

(c) If $0 \le k \le k_2 = 1 + \dfrac{1}{1+\sqrt{\dfrac{n}{n-2}}}$ *, prove that*

$$\frac{a_1a_2 + a_2a_3 + \cdots + a_na_1}{n} \ge \left(\frac{a_1 + \cdots + a_{n-2} + ka_{n-1}}{n-2+k}\right)^2.$$

(Vasile Cîrtoaje, Math. Reflections, 2, 2025)

Solution. Denote

$$S = a_2 + \cdots + a_{n-1}, \qquad (n-2)a_1 \le S \le (n-2)a_n,$$

and write the inequality as follows:

$$a_1a_2 + a_2a_3 + \cdots + a_na_1 \ge \frac{n(ka_1 + S)^2}{(k+n-2)^2},$$

$$(n-1)(a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n) + (n-1)a_na_1 \ge \frac{n(n-1)(ka_1 + S)^2}{(k+n-2)^2}.$$

Since the sequences $(a_1, a_2, \ldots, a_{n-1})$ and $(a_2, a_3, \ldots, a_n)$ are increasing, by Chebyshev's inequality we have

$$(n-1)(a_1a_2+a_2a_3+\cdots+a_{n-1}a_n) \ge (a_1+a_2+\cdots+a_{n-1})(a_2+a_3+\cdots+a_n) = (a_1+S)(S+a_n).$$

Thus, it suffices to show that

$$(a_1 + S)(S + a_n) + (n-1)a_na_1 \ge \frac{n(n-1)(ka_1 + S)^2}{(k+n-2)^2}.$$

Since $a_n \ge \dfrac{S}{n-2}$, it is enough to prove that

$$(a_1 + S)\left(S + \frac{S}{n-2}\right) + \frac{(n-1)a_1S}{n-2} \ge \frac{n(n-1)(ka_1 + S)^2}{(k+n-2)^2},$$

which is equivalent to

$$(a_1 + S)S + a_1S \ge \frac{n(n-2)(ka_1 + S)^2}{(k+n-2)^2},$$

$$[S - (n-2)a_1](AS + nk^2a_1) \ge 0,$$

where

$$A = (k+n-2)^2 - n(n-2) = k^2 + 2(n-2)k - 2(n-2) \ge 0.$$

Clearly, the inequality holds if $A \ge 0$.

(a) For $k = 1$, we have $A = 1$. The equality occurs for $a_1 = a_2 = \cdots = a_n$.

(b) We have $A \ge 0$ for $k \ge k_1$. The equality occurs for $a_1 = a_2 = \cdots = a_n$. If $k = k_1$, then the equality also occurs when $a_1 = 0$ and $a_2 = \cdots = a_n$.

(c) Since $a_{n-1}a_n \ge a_{n-1}^2$ and $a_na_1 \ge a_{n-1}a_1$, it suffices to show that

$$(a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_1) + a_{n-1}^2 \ge \frac{n}{(n-2+k)^2} \cdot (a_1 + \cdots + a_{n-2} + ka_{n-1})^2 \qquad (*)$$

for $a_1, a_2, \ldots, a_{n-2} \in [0, a_{n-1}]$, which is a weaker condition than the original $0 \le a_1 \le a_2 \le \cdots \le a_{n-1}$. For fixed a_{n-1}, we write the inequality as $F(a_1, a_2, \ldots, a_{n-2}) \ge 0$, where

$$F(a_1, a_2, \ldots, a_{n-2}) = a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_1) + a_{n-1}^2 - \frac{n}{(n-2+k)^2} \cdot (a_1 + \cdots + a_{n-2} + ka_{n-1})^2.$$

Since $F(a_1, a_2, \ldots, a_{n-2}$ is a concave function in each variable, it has the minimum value for $a_1, a_2, \ldots, a_{n-2} \in \{0, a_n\}$. So, due to symmetry, it suffices to prove (*) for

$$a_1 = \cdots = a_j = 0, \qquad a_{j+1} = \cdots = a_{n-2} = a_{n-1},$$

where $j \in \{0, 1, \ldots, n-2\}$. For $j = 0$, the inequality (*) is an equality. For $j \in \{1, \ldots, n-2\}$, the inequality (*) is true if $f(j) \ge 0$, where

$$f(j) = (n-2+k)^2(n-j-1) - n(n-2+k-j)^2.$$

We will show that $f(j) \ge 0$ for all real $j \in [1, n-2]$. Since f is concave and $j \in [1, n-2]$, it suffices to show that $f(1) \ge 0$ and $f(n-2) \ge 0$. The inequality $f(1) \ge 0$ is equivalent to

$$(n-2+k)^2(n-2) \ge n(n-3+k)^2, \qquad (n-2+k)\sqrt{n-2} \ge (n-3+k)\sqrt{n},$$

$$k \le \frac{(n-2)\sqrt{n-2} - (n-3)\sqrt{n}}{\sqrt{n} - \sqrt{n-2}} = \frac{\sqrt{n(n-2)} - n + 4}{2}$$

$$= 1 + \frac{\sqrt{n-2}(\sqrt{n} - \sqrt{n-2})}{2} = 1 + \frac{\sqrt{n-2}}{\sqrt{n} + \sqrt{n-2}} = k_2.$$

The inequality $f(n-2) \ge 0$ is equivalent to

$$(n-2+k)^2 \ge nk^2, \qquad n-2+k \ge k\sqrt{n}, \qquad k \le \frac{n-2}{\sqrt{n}-1}.$$

For $n = 3$, the last inequality is equivalent to $k \le k_2$, while for $n \ge 4$ we have

$$k \le k_2 < 2 \le \frac{n-2}{\sqrt{n}-1}.$$

For $0 \le k \le k_2$, the equality occurs when $a_1 = a_2 = \cdots = a_n$. For $k = k_2$, the equality also occurs when $a_1 = 0$ and $a_2 = \cdots = a_n$.

Remark 1. Actually, k_1 is the least positive value of k such that such that

$$\frac{a_1a_2+a_2a_3+\cdots+a_na_1}{n} \ge \left(\frac{ka_1+a_2+\cdots+a_{n-1}}{k+n-2}\right)^2$$

for $n \ge 3$ and all $0 \le a_1 \le a_2 \le \cdots \le a_n$. Indeed, for $a_1 = 0$ and $a_2 = a_3 = \cdots = a_n = 1$, the inequality leads to the necessary condition $k \ge k_1$.

Remark 2. Actually, k_2 is the largest positive value of k such that such that

$$\frac{a_1a_2+a_2a_3+\cdots+a_na_1}{n} \ge \left(\frac{a_1+\cdots+a_{n-2}+ka_{n-1}}{n-2+k}\right)^2$$

for $n \ge 3$ and all $0 \le a_1 \le a_2 \le \cdots \le a_n$. Indeed, for $a_1 = 0$ and $a_2 = a_3 = \cdots = a_n = 1$, the inequality leads to the necessary condition $k \ge k_3$.

□

P 2.113. *If* $k \ge k_0 = 7-2\sqrt{6} \approx 2.101$ *and* $a \ge b \ge c \ge d \ge e \ge f \ge 0$, *then*

$$\left(\frac{ka+b+c+d+e+f}{k+5}\right)^2 \ge \frac{ab+bc+cd+de+ef+fa}{6}.$$

(Vasile Cîrtoaje, Math. Reflections, 3,2025)

Solution. Since $5a-b-c-d-e-f \ge 0$ and

$$\frac{ka+b+c+d+e+f}{k+5} = a - \frac{5a-b-c-d-e-f}{k+5},$$

it suffices to prove the desired inequality, for $k = k_0$, that is $(k+5)^2 = 24k$. Write the inequality in the homogeneous form $F(a,b,c,d,e) \ge 0$, where

$$F(a,b,c,d,e) = (ka+b+c+d+e+f)^2 - 4k(ab+bc+cd+de+ef+fa).$$

We will show that

$$F(a,b,c,d,e,f) \ge F(S,S,S,d,e,f) \ge F(S,S,S,d,s,s) \ge 0,$$

where

$$S = \frac{ka+b+c}{k+2}, \qquad s = \frac{e+f}{2}, \qquad S \ge d \ge s.$$

We have

$$\frac{F(a,b,c,d,e,f)-F(S,S,S,d,e,f)}{4k} = 2S^2 + d(S-c) - f(a-S) - ab - bc$$

$$\geq 2S^2 + d(S-c) - d(a-S) - ab - bc = 2S^2 + d(2S-a-c) - ab - bc.$$

Since

$$2S - a - c = \frac{(k-2)a + 2(b-c)}{k+2} > 0,$$

we get

$$\frac{F(a,b,c,d,e,f) - F(S,S,S,d,e,f)}{4k} \geq 2S^2 - ab - bc = \frac{E}{(k+2)^2},$$

where

$$E = 2(ka+b+c)^2 - (k+2)^2 b(a+c) = (2a^2 - ab - bc)k^2 + 4c(a-b)k + 2(b+c)^2 - 4b(a+c)$$

$$\geq 4(2a^2 - ab - bc) + 2(b+c)^2 - 4b(a+c) = 4a(a-b) + (b-c)^2 \geq 0.$$

We have

$$\frac{F(S,S,S,d,e,f) - F(S,S,S,d,s,s)}{4k} = s^2 - ef + d(s-e) + S(s-f)$$

$$= \frac{(e-f)^2}{4} + \frac{(e-f)(S-d)}{2} \geq 0.$$

We have

$$F(S,S,S,d,s,s) = [(k+2)S + d + 2s]^2 - 4k(2S^4 + Sd + ds + s^2 + sS)$$

$$= -As^2 + B(S,d)s + C(S,d),$$

where $A = 4(k-1)$. Since $A > 0$ and $0 \leq s \leq d$, to prove the inequality $F(S,S,S,d,s,s) \geq 0$, it suffices to consider the cases $s = 0$ and $s = d$. For $s = 0$, we need to show that

$$[(k+2)S + d]^2 - 4k(2S^2 + Sd) \geq 0,$$

which is equivalent to

$$[(k-2)S - d]^2 \geq 0.$$

For $s = d$, we need to show that

$$[(k+2)S + 3d]^2 - 8k(S^2 + Sd + d^2) \geq 0,$$

that is

$$(k-2)S^2 + 2(6-k)Sd - (8k-9)d^2 \geq 0.$$

Indeed, we have

$$(k-2)S^2 + 2(6-k)Sd - (8k-9)d^2 \geq (k-2)d^2 + 2(6-k)d^2 - (8k-9)d^2 = (k^2 - 14k + 25)d^2 = 0.$$

The equality occurs for $a = b = c = d = e = f$. If $k = 7 - 2\sqrt{6}$, then the equality also occurs for for $a = b = c = d/(k-2)$ and $e = f = 0$.

□

P 2.114. *If* $a_1 \ge a_2 \ge \cdots \ge a_9 \ge 0$, *then*

$$\left(\frac{4a_1 + a_2 + \cdots + a_9}{12}\right)^2 \ge \frac{a_1a_2 + a_2a_3 + \cdots + a_9a_1}{9}.$$

(Vasile Cîrtoaje, Math. Reflections, 6, 2023)

Solution. Write the inequality as $F(a_1, a_2, \dots, a_9) \ge 0$, where

$$F(a_1, a_2, \dots, a_9) = (4a_1 + a_2 + \cdots + a_9)^2 - 16(a_1a_2 + a_2a_3 + \cdots + a_9a_1).$$

We will show that

$$F(a_1, a_2, a_3, \dots, a_9) \ge F(a_2, a_2, a_3, \dots, a_9) \ge \cdots \ge F(a_8, a_8, \dots, a_8, a_9) \ge F(a_9, a_9, \dots, a_9, a_9) = 0,$$

that is

$$F(a_i, \dots, a_i, a_{i+1}, \dots, a_9) \ge F(a_{i+1}, \dots, a_{i+1}, a_{i+2}, \dots, a_9), \qquad i \in \{1, 2, \dots, 8\}.$$

Write this inequality as follows:

$$[(i+3)a_i + a_{i+1} + \cdots + a_9]^2 - 16[(i-1)a_i^2 + a_ia_{i+1} + \cdots + a_9a_i] \ge$$

$$\ge [(i+4)a_{i+1} + a_{i+2} + \cdots + a_9]^2 - 16[ia_{i+1}^2 + a_{i+1}a_{i+2} + \cdots + a_9a_{i+1}],$$

$$(i+3)(a_i - a_{i+1})[(i+3)a_i + (i+5)a_{i+1} + 2a_{i+2} + \cdots + 2a_9]$$
$$-16[(i-1)(a_i^2 - a_{i+1}^2) + a_{i+1}(a_i - a_{i+1}) + a_9(a_i - a_{i+1})] \ge 0,$$

$$(a_i - a_{i+1})E_i \ge 0,$$

where

$$E_i = (i-5)^2a_i + (i^2 - 8i + 15)a_{i+1} + 2(i+3)(a_{i+2} + \cdots + a_8) + 2(i-5)a_9.$$

Since

$$(i-5)^2a_i + (i^2 - 8i + 15)a_{i+1} \ge (i-5)^2a_{i+1} + (i^2 - 8i + 15)a_{i+1} = 2(i-4)(i-5)a_{i+1} \ge 0,$$

it suffices to show that

$$(i+3)(a_{i+2} + \cdots + a_8) + (i-5)a_9 \ge 0.$$

This is true for $i \ge 5$, while for $i \le 4$ we have

$$(i+3)(a_{i+2}+\cdots+a_8)+(i-5)a_9 \ge (i+3)(7-i)a_9+(i-5)a_9 \ge 3(i+3)a_9+(i-5)a_9 = 4(i+1)a_9 \ge 0.$$

The equality occurs for $a_1 = a_2 = \cdots = a_9 = 1$, and also for $a_1 = \cdots = a_5 = \frac{3}{2}$ and $a_6 = a_7 = a_8 = a_9 = 0$.

Remark. Similarly, we can prove the following generalization:

- *Let $n \ge 4$ be a perfect square. Then, $k_0 = (\sqrt{n}-1)^2$ is the least positive value of the constant k such that*

$$\left(\frac{ka_1 + a_2 + \cdots + a_n}{k+n-1}\right)^2 \ge \frac{a_1a_2 + a_2a_3 + \cdots + a_na_1}{n}$$

whenever $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$.

For $a_1 = \cdots = a_{k_0+1} = 1$ and $a_{k_0+2} = \cdots = a_n = 0$, the inequality becomes

$$\left(\frac{k+k_0}{k+n-1}\right)^2 \ge \frac{k_0}{n}, \qquad \frac{k+k_0}{k+n-1} \ge \frac{\sqrt{n}-1}{\sqrt{n}}, \qquad k \ge k_0.$$

To show that k_0 is the least positive value of k, it suffices to prove the inequality for $k = k_0$, that is to show that $F(a_1, a_2, \ldots, a_n) \ge 0$, where

$$F(a_1, a_2, \ldots, a_n) = n(ka_1 + a_2 + \cdots + a_n)^2 - (k+n-1)^2(a_1a_2 + a_2a_3 + \cdots + a_na_1).$$

We will show that

$$F(a_1, a_2, a_3, \ldots, a_n) \ge F(a_2, a_2, a_3, \ldots, a_n) \ge \cdots$$

$$\ge F(a_{n-1}, a_{n-1}, \ldots, a_{n-1}, a_n) \ge F(a_n, a_n, \ldots, a_n, a_n) = 0,$$

that is

$$F(a_i, \ldots, a_i, a_{i+1}, \ldots, a_n) \ge F(a_{i+1}, \ldots, a_{i+1}, a_{i+2}, \ldots, a_n), \qquad i \in \{1, 2, \ldots, n-1\}.$$

Write this inequality as follows:

$$n[(i+k-1)a_i + a_{i+1} + \cdots + a_n]^2 - (k+n-1)^2[(i-1)a_i^2 + a_ia_{i+1} + \cdots + a_na_i] \ge$$

$$\ge n[(i+k)a_{i+1} + a_{i+2} + \cdots + a_n]^2 - (k+n-1)2[ia_{i+1}^2 + a_{i+1}a_{i+2} + \cdots + a_na_{i+1}],$$

$$n(i+k-1)(a_i - a_{i+1})[(i+k-1)a_i + (i+k+1)a_{i+1} + 2a_{i+2} + \cdots + 2a_n]$$
$$-(k+n-1)^2[(i-1)(a_i^2 - a_{i+1}^2) + a_{i+1}(a_i - a_{i+1}) + a_n(a_i - a_{i+1})] \ge 0,$$

$$(a_i - a_{i+1})E_i \ge 0,$$

with

$$E_i = A_ia_i + A_{i+1}a_{i+1} + 2n(i+k-1))(a_{i+2} + \cdots + a_{n-1}) + 2n(i-k-1)a_n,$$

where

$$A_i = n(i+k-1)^2 - (i-1)(k+n-1)^2, \quad A_{i+1} = n(i+k-1)(i+k+1) - i(k+n-1)^2.$$

Since

$$(k+n-1)^2 = 4(n-\sqrt{n})^2 = 4nk,$$

we have

$$A_i = n(i+k-1)^2 - 4nk(i-1) = n(i-k-1)^2 \geq 0,$$

$$A_{i+1} = n(i+k-1)(i+k+1) - 4nik = n(i-k-1)(i-k+1).$$

Since

$$A_i a_i + A_{i+1}a_{i+1} \geq (A_i + A_{i+1})a_{i+1} = 2n(i-k-1)(i-k) \geq 0,$$

it suffices to show that

$$2n(i+k-1)(a_{i+2} + \cdots + a_{n-1}) + 2n(i-k-1)a_n \geq 0.$$

This is true for $i \geq k+1$, while for $i \leq k$ we have

$$2n(i+k-1)(a_{i+2}+\cdots+a_{n-1}) + 2n(i-k-1)a_n \geq 2n(i+k-1)(n-i-2)a_n + 2n(i-k-1)a_n$$

$$\geq 2n(i+k-1)(n-k-2)a_n + 2n(i-k-1)a_n = 2n[(i-1)(n-k-1) + k(n-k-3)]a_n$$

$$\geq 2nk(n-k-3)a_n = 4nk(\sqrt{n}-2) \geq 0.$$

For $k = k_0$, the equality occurs when $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = \cdots = a_{k+1}$ and $a_{k+2} = \cdots = a_n = 0$.

□

P 2.115. *Prove that* $\frac{3}{4}$ *is the least positive value of* k *such that*

$$\left(\frac{ka+b+c+d}{k+3}\right)^2 \geq \frac{ab+bc+cd+de+ea}{5}$$

whenever $a \geq b \geq c \geq d \geq e \geq 0$.

(Vasile Cîrtoaje, Math. Reflections, 3, 2024)

Solution. Setting $b = c = d = e = 1$, which involves and $a \geq 1$, the inequality becomes

$$\left(\frac{ka+3}{k+3}\right)^2 \geq \frac{2a+3}{5}, \qquad \left(\frac{ka+3}{k+3}\right)^2 - 1 \geq \frac{2a+3}{5} - 1,$$

$$(a-1)(5k^2a + 3k^2 + 18k - 18) \geq 0.$$

It is true for $a \geq 1$ only if

$$5k^2a + 3k^2 + 18k - 18 \geq 0.$$

Setting $a = 1$, we get the necessary condition $(4k-3)(k+3) \geq 0$, i.e. $k \geq \frac{3}{4}$. To show that $\frac{3}{4}$ is the least positive value of k, we need to prove the original inequality for $k = \frac{3}{4}$. Since

$$ab + bc + cd + de + ea \leq ab + bc + cd + d^2 + da,$$

it suffices to prove that $F(a, b, c, d) \geq 0$, where

$$F(a, b, c, d) = (3a + 4b + 4c + 4d)^2 - 45(ab + bc + cd + d^2 + da).$$

We will show that

$$F(a, b, c, d) \geq F(a, b, s, s) \geq 0,$$

where $s = \dfrac{c+d}{2}$, $s \geq \sqrt{cd} \geq d$. We have

$$\frac{F(a, b, c, d) - F(a, b, s, s)}{45} = (ab + bs + s^2 + s^2 + sa) - (ab + bc + cd + d^2 + da)$$

$$= b(s-c) + (s^2 - cd) + (s^2 - d^2) + a(s-d) \geq b(s-c) + a(s-d) = \frac{(c-d)(a-b)}{2} \geq 0.$$

Next, for fixed a and b, we write the inequality $F(a, b, s, s) \geq 0$ as $f(s) \geq 0$, where

$$f(s) = (3a + 4b + 8s)^2 - 45[2s^2 + (a+b)s + ab].$$

Since $f(s)$ is concave and $s \in [0, b]$, it suffices to show that $f(0) \geq 0$ and $f(b) \geq 0$. Indeed,

$$f(0) = (3a + 4b)^2 - 45ab = (3a - 4b)^2 + 3ab > 0$$

and

$$f(b) = (3a + 12b)^2 - 45(2ab + 3b^2) = 9(a-b)^2 \geq 0.$$

For $k = \frac{3}{4}$, the equality occurs when $a = b = c = d = e \geq 0$.

□

P 2.116. *If* $a_1 \geq a_2 \geq \cdots \geq a_8 \geq 0$, *then*

$$(2a_1 + a_2 + \cdots + a_7)^2 \geq 8(a_1a_2 + a_2a_3 + \cdots + a_8a_1).$$

(Vasile Cîrtoaje, Mathproblems, 1, 2024)

Solution. Since

$$a_1a_2 + a_2a_3 + \cdots + a_8a_1 \le a_1a_2 + a_2a_3 + \cdots + a_6a_7 + a_7^2 + a_7a_1,$$

it suffices to prove that $F(a_1, a_2, \ldots, a_7) \ge 0$, where

$$F(a_1, a_2, \ldots, a_7) = (2a_1 + a_2 + \cdots + a_7)^2 - 8(a_1a_2 + a_2a_3 + \cdots + a_6a_7 + a_7^2 + a_7a_1).$$

We will show that

$$F(a_1, a_2, a_3, a_4, a_5, a_6, a_7) \ge F(a_2, a_2, a_3, a_4, a_5, a_6, a_7) \ge \cdots \ge F(a_6, a_6, a_6, a_6, a_6, a_6, a_7) \ge 0.$$

Since

$$F(a_6, a_6, a_6, a_6, a_6, a_6, a_7) = (7a_6 + a_7)^2 - 8(5a_6^2 + 2a_6a_7 + a_7^2) = (a_6 - a_7)(9a_6 + 7a_7) \ge 0,$$

we only need to show that

$$F(a_i, \ldots, a_i, a_{i+1}, a_{i+2}, \ldots, a_7) \ge F(a_{i+1}, \ldots, a_{i+1}, a_{i+1}, a_{i+2}, \ldots, a_7)$$

for $i = 1, \ldots, 5$. Write the inequality as follows:

$$[(i+1)a_i + a_{i+1} + a_{i+2} + \cdots + a_7]^2 - 8[(i-1)a_i^2 + a_ia_{i+1} + a_{i+1}a_{i+2} + \cdots + a_6a_7 + a_7^2 + a_7a_i] \ge$$

$$\ge [(i+2)a_{i+1} + a_{i+2} + \cdots + a_7]^2 - 8[ia_{i+1}^2 + a_{i+1}a_{i+2} + \cdots + a_6a_7 + a_7^2 + a_7a_{i+1}],$$

$$(i+1)(a_i - a_{i+1})[(i+1)a_i + (i+3)a_{i+1} + 2a_{i+2} + \cdots + 2a_7] \ge 8(a_i - a_{i+1})[(i-1)a_i + ia_{i+1} + a_7].$$

Since $a_i - a_{i+1} \ge 0$, the inequality holds if

$$(i+1)[(i+1)a_i + (i+3)a_{i+1} + 2a_{i+2} + \cdots + 2a_7] \ge 8[(i-1)a_i + ia_{i+1} + a_7],$$

that is

$$(i-3)^2a_i + (i-1)(i-3)a_{i+1} + 2(i+1)(a_{i+2} + \cdots + a_6) + 2(i-3)a_7 \ge 0.$$

This inequality is clearly true for $i \ge 3$. It also holds for $i = 1$ and $i = 2$, because

$$(i-3)^2a_i + (i-1)(i-3)a_{i+1} \ge (i-3)^2a_{i+1} + (i-1)(i-3)a_{i+1} = 2(2-i)(3-i)a_{i+1} \ge 0$$

and

$$2(i+1)a_6 + 2(i-3)a_7 \ge 2(i+1)a_7 + 2(i-3)a_7 = 4(i-1)a_7 \ge 0.$$

The equality occurs when $a_1 = a_2 = \cdots = a_8$, and also when $a_1 = a_2 = a_3$ and $a_4 = \cdots = a_8 = 0$.

Remark. Similarly, we can prove the following generalization:

• *Let $n \ge 3$ be an integer such that $2n$ is a perfect square. Then, $k_0 = (\sqrt{n}-2)^2$ is the least positive value of the constant k such that*

$$\left(\frac{ka_1 + a_2 + \cdots + a_{n-1}}{k+n-2}\right)^2 \ge \frac{a_1a_2 + a_2a_3 + \cdots + a_na_1}{n}$$

whenever $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$.

Setting $a_1 = a_2 = \cdots = a_{k_0+1} = 1$ and $a_{k_0+2} = \cdots = a_n = 0$, the desired inequality becomes

$$\left(\frac{k+k_0}{k+n-2}\right)^2 \ge \frac{k_0}{n}, \qquad \frac{k+k_0}{k+n-2} \ge \frac{\sqrt{n}-\sqrt{2}}{\sqrt{n}}, \qquad k \ge k_0.$$

To show that k_0 is the smallest positive value of k, we need to show that the original inequality holds for $k = k_0$. Since

$$(k+n-2)^2 = 4kn$$

and

$$a_1a_2 + a_2a_3 + \cdots + a_na_1 \le a_1a_2 + a_2a_3 + \cdots + a_{n-2}a_{n-1} + a_{n-1}^2 + a_{n-1}a_1,$$

it suffices to prove that $F(a_1, a_2, \ldots, a_{n-1}) \ge 0$, where

$$F(a_1, a_2, \ldots, a_{n-1}) = (ka_1 + a_2 + \cdots + a_{n-2} + a_{n-1})^2 - 4k(a_1a_2 + a_2a_3 + \cdots + a_{n-2}a_{n-1} + a_{n-1}^2 + a_{n-1}a_1).$$

We will show that

$$F(a_1, a_2, a_3, \ldots, a_{n-1}) \ge F(a_2, a_2, a_3, \ldots, a_{n-1}) \ge \cdots \ge F(a_{n-2}, a_{n-2}, \cdots, a_{n-2}, a_{n-1}) \ge 0.$$

Since

$$F(a_{n-2}, a_{n-2}, \cdots, a_{n-2}, a_{n-1}) = [(n-3+k)a_{n-2} + a_{n-1}]^2 - 4k[(n-3)a_{n-2}^2 + 2a_{n-2}a_{n-1} + a_{n-1}^2]$$

$$= (a_{n-2} - a_{n-1})[(n-3-k)^2a_{n-2} + (4k-1)a_{n-1}] \ge 0,$$

we only need to show that

$$F(a_i, \cdots, a_i, a_{i+1}, a_{i+2}, \ldots, a_{n-1}) \ge F(a_{i+1}, \cdots, a_{i+1}, a_{i+1}, a_{i+2}, \ldots, a_{n-1})$$

for $i = 1, \ldots, n-3$. Write the inequality as follows:

$$[(i-1+k)a_i + a_{i+1} + a_{i+2} + \cdots + a_{n-1}]^2 - 4k[(i-1)a_i^2 + a_ia_{i+1} + a_{i+1}a_{i+2} + \cdots + a_{n-2}a_{n-1} + a_{n-1}^2 + a_{n-1}a_i] \ge$$

$$\ge [(i+k)a_{i+1} + a_{i+2} + \cdots + a_{n-1}]^2 - 4k[ia_{i+1}^2 + a_{i+1}a_{i+2} + \cdots + a_{n-2}a_{n-1} + a_{n-1}^2 + a_{n-1}a_{i+1}],$$

$$(i-1+k)(a_i - a_{i+1})[(i-1+k)a_i + (i+1+k)a_{i+1} + 2a_{i+2} + \cdots + 2a_{n-1}]^2 \ge 4k(a_i - a_{i+1})[(i-1)(a_i + a_{i+1}) + a_{n-1}],$$

$$(a_i - a_{i+1})[(i-1-k)^2a_i + (i-1-k)(i+1-k)a_{i+1} + 2(i-1+k)a_{i+2} + \cdots + 2(i-1+k)a_{n-2} + 2(i-1-k)a_{n-1}] \ge 0.$$

Since

$$(i-1-k)^2a_i+(i-1-k)(i+1-k)a_{i+1} \ge (i-1-k)^2a_{i+1}+(i-1-k)(i+1-k)a_{i+1} = 2(i-k-1)(i-k)a_{i+1} \ge 0$$

and

$$2(i-1+k)a_{n-2}+2(i-1-k)a_{n-1} \ge 2(i-1+k)a_{n-1}+2(i-1-k)a_{n-1} = 4(i-1)a_{n-1},$$

the last inequality is true and the proof is completed. For $k = k_0$, the equality occurs when $a_1 = a_2 = \cdots = a_n$, and also when $a_1 = a_2 = \cdots = a_{k+1}$ and $a_{k+2} = \cdots = a_n = 0$.

□

P 2.117. *Let a, b, c, d be nonnegative real numbers such that $ab + bc + cd = 7$. Prove that*

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}+\frac{1}{d+1} \ge \frac{3}{2}.$$

(Vasile Cîrtoaje, Crux Mathematicorum, 1, 2025)

Solution. By the Cauchy-Schwarz inequality, we have

$$[(a+1)b+(d+1)c]\left(\frac{1}{a+1}+\frac{1}{d+1}\right) \ge (\sqrt{b}+\sqrt{c})^2,$$

$$\frac{1}{a+1}+\frac{1}{d+1} \ge \frac{b+c+2\sqrt{bc}}{b+c-bc+7}.$$

So, it suffices to show that

$$\frac{1}{b+1}+\frac{1}{c+1}+\frac{b+c+2\sqrt{bc}}{b+c-bc+7} \ge \frac{3}{2}.$$

Let

$$s = \frac{b+c}{2}, \qquad p = \sqrt{bc}, \qquad s \ge p.$$

We need to show that

$$\frac{2s+2}{2s+p^2+1}+\frac{2s+2p}{2s-p^2+7} \ge \frac{3}{2},$$

which is equivalent to $F \ge 0$, where

$$F = 4s^2+8(p-1)s+3p^4+4p^3-22p^2+4p+7$$

$$= 4(s+p-1)^2+3p^4+4p^3-26p^2+12p+3.$$

For $p \le \dfrac{1}{2}$, we have

$$F \ge 3p^4+4p^3-26p^2+12p+3 > -28p^2+12p+1 = (1-2p)(1+14p) \ge 0.$$

For $p \ge \frac{1}{2}$, we have $s + p - 1 \ge 2p - 1 \ge 0$, therefore

$$F \ge 4(2p-1)^2 + 3p^4 + 4p^3 - 26p^2 + 12p + 3$$

$$= 3p^4 + 4p^3 - 10p^2 - 4p + 7 = (p-1)^2(p+1)(3p+7) \ge 0.$$

The equality occurs for $a = d = 3$ and $b = c = 1$.

□

Appendix A

Glosar

1. AM-GM (ARITHMETIC MEAN-GEOMETRIC MEAN) INEQUALITY

If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers, then

$$a_1 + a_2 + \cdots + a_n \ge n\sqrt[n]{a_1 a_2 \cdots a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

2. WEIGHTED AM-GM INEQUALITY

Let $p_1, p_2, \ldots, p_n$ be positive real numbers satisfying

$$p_1 + p_2 + \cdots + p_n = 1.$$

If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers, then

$$p_1 a_1 + p_2 a_2 + \cdots + p_n a_n \ge a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

3. AM-HM (ARITHMETIC MEAN-HARMONIC MEAN) INEQUALITY

If $a_1, a_2, \ldots, a_n$ are positive real numbers, then

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \ge n^2,$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

4. POWER MEAN INEQUALITY

The power mean of order k of positive real numbers $a_1, a_2, \ldots, a_n$,

$$M_k = \begin{cases} \left(\frac{a_1^k + a_2^k + \cdots + a_n^k}{n}\right)^{\frac{1}{k}}, & k \neq 0 \\ \sqrt[n]{a_1 a_2 \cdots a_n}, & k = 0 \end{cases},$$

is an increasing function with respect to $k \in \mathbb{R}$. For instant, $M_2 \geq M_1 \geq M_0 \geq M_{-1}$ is equivalent to

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}.$$

5. BERNOULLI'S INEQUALITY

For any real number $x \geq -1$, we have

a) $(1+x)^r \geq 1 + rx$ for $r \geq 1$ and $r \leq 0$;

b) $(1+x)^r \leq 1 + rx$ for $0 \leq r \leq 1$.

If $a_1, a_2, \ldots, a_n$ are real numbers such that either $a_1, a_2, \ldots, a_n \geq 0$ or

$$-1 \leq a_1, a_2, \ldots, a_n \leq 0,$$

then

$$(1+a_1)(1+a_2)\cdots(1+a_n) \geq 1 + a_1 + a_2 + \cdots + a_n.$$

6. SCHUR'S INEQUALITY

For any nonnegative real numbers a, b, c and any positive number k, the inequality holds

$$a^k(a-b)(a-c) + b^k(b-c)(b-a) + c^k(c-a)(c-b) \geq 0,$$

with equality for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation).
For $k = 1$, we get the third degree Schur's inequality, which can be rewritten as follows

$$a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + bc(b+c) + ca(c+a),$$

$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca),$$

$$a^2 + b^2 + c^2 + \frac{9abc}{a+b+c} \geq 2(ab+bc+ca),$$

$$(b-c)^2(b+c-a) + (c-a)^2(c+a-b) + (a-b)^2(a+b-c) \geq 0.$$

For $k = 2$, we get the fourth degree Schur's inequality, which holds for any real numbers a, b, c, and can be rewritten as follows

$$a^4 + b^4 + c^4 + abc(a + b + c) \geq ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2),$$

$$a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 \geq (ab + bc + ca)(a^2 + b^2 + c^2 - ab - bc - ca),$$

$$(b - c)^2(b + c - a)^2 + (c - a)^2(c + a - b)^2 + (a - b)^2(a + b - c)^2 \geq 0,$$

$$6abcp \geq (p^2 - q)(4q - p^2), \qquad p = a + b + c, \quad q = ab + bc + ca.$$

A generalization of the fourth degree Schur's inequality, which holds for any real numbers a, b, c and any real number m, is the following (*Vasile Cîrtoaje*, 2004)

$$\sum (a - mb)(a - mc)(a - b)(a - c) \geq 0,$$

where the equality holds for $a = b = c$, and for $a/m = b = c$ (or any cyclic permutation). This inequality is equivalent to

$$\sum a^4 + m(m + 2)\sum a^2b^2 + (1 - m^2)abc\sum a \geq (m + 1)\sum ab(a^2 + b^2),$$

$$\sum (b - c)^2(b + c - a - ma)^2 \geq 0.$$

A more general result is given by the following theorem (*Vasile Cîrtoaje*, 2004).

Theorem. *Let*

$$f_4(a, b, c) = \sum a^4 + \alpha\sum a^2b^2 + \beta abc\sum a - \gamma\sum ab(a^2 + b^2),$$

where α, β, γ are real constants such that $1 + \alpha + \beta = 2\gamma$. Then,

(a) $f_4(a, b, c) \geq 0$ *for all $a, b, c \in \mathbb{R}$ if and only if*

$$1 + \alpha \geq \gamma^2;$$

(b) $f_4(a, b, c) \geq 0$ *for all $a, b, c \geq 0$ if and only if*

$$\alpha \geq (\gamma - 1)\max\{2, \gamma + 1\}.$$

7. CAUCHY-SCHWARZ INEQUALITY

If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, then

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2,$$

with equality for

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}.$$

Notice that the equality conditions are also valid for $a_i = b_i = 0$, where $1 \leq i \leq n$.

8. HÖLDER'S INEQUALITY

If x_{ij} $(i = 1, 2, \cdots, m; j = 1, 2, \cdots n)$ are nonnegative real numbers, then

$$\prod_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij}\right) \geq \left(\sum_{j=1}^{n} \sqrt[m]{\prod_{i=1}^{m} x_{ij}}\right)^{m}.$$

9. CHEBYSHEV'S INEQUALITY

Let $a_1 \geq a_2 \geq \cdots \geq a_n$ be real numbers.

a) If $b_1 \geq b_2 \geq \cdots b_n$, then

$$n\sum_{i=1}^{n} a_i b_i \geq \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right);$$

b) If $b_1 \leq b_2 \leq \cdots \leq b_n$, then

$$n\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right).$$

10. REARRANGEMENT INEQUALITY

(1) If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are two increasing (or decreasing) real sequences, and $(i_1, i_2, \cdots, i_n)$ is an arbitrary permutation of $(1, 2, \cdots, n)$, then

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \geq a_1b_{i_1} + a_2b_{i_2} + \cdots + a_nb_{i_n}.$$

(2) If $a_1, a_2, \ldots, a_n$ is decreasing and $b_1, b_2, \ldots, b_n$ is increasing, then

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \leq a_1b_{i_1} + a_2b_{i_2} + \cdots + a_nb_{i_n}.$$

(3) Let $b_1, b_2, \ldots, b_n$ and $c_1, c_2, \ldots, c_n$ be two real sequences such that

$$b_1 + \cdots + b_k \geq c_1 + \cdots + c_k, \quad k = 1, 2, \cdots, n.$$

If $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, then

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \geq a_1c_1 + a_2c_2 + \cdots + a_nc_n.$$

11. CONVEX FUNCTIONS

A function f defined on a real interval $\mathbb{I}$ is said to be *convex* if

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$$

for all $x,\ y \in \mathbb{I}$ and any $\alpha,\ \beta \ge 0$ with $\alpha + \beta = 1$. If the inequality is reversed, then f is said to be concave.
If f is differentiable on $\mathbb{I}$, then f is (strictly) convex if and only if the derivative f' is (strictly) increasing. If $f'' \ge 0$ on $\mathbb{I}$, then f is convex on $\mathbb{I}$. Also, if $f'' \ge 0$ on $(a,\ b)$ and f is continuous on $[a,\ b]$, then f is convex on $[a,\ b]$.

Jensen's inequality. *Let $p_1, p_2, \dots, p_n$ be positive real numbers. If f is a convex function on a real interval $\mathbb{I}$, then for any $a_1,\ a_2,\ \dots,\ a_n \in \mathbb{I}$, the inequality holds*

$$\frac{p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n)}{p_1 + p_2 + \cdots + p_n} \ge f\left(\frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n}\right).$$

For $p_1 = p_2 = \cdots = p_n$, Jensen's inequality becomes

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge n f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right).$$

12. SQUARE PRODUCT INEQUALITY

Let a, b, c be real numbers, and let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc,$$
$$s = \sqrt{p^2 - 3q} = \sqrt{a^2 + b^2 + c^2 - ab - bc - ca}.$$

From the identity

$$(a-b)^2(b-c)^2(c-a)^2 = -27r^2 + 2(9pq - 2p^3)r + p^2q^2 - 4q^3,$$

it follows that

$$\frac{-2p^3 + 9pq - 2(p^2 - 3q)\sqrt{p^2 - 3q}}{27} \le r \le \frac{-2p^3 + 9pq + 2(p^2 - 3q)\sqrt{p^2 - 3q}}{27},$$

which is equivalent to

$$\frac{p^3 - 3ps^2 - 2s^3}{27} \le r \le \frac{p^3 - 3ps^2 + 2s^3}{27}.$$

Therefore, for constant p and q, the product r is minimal and maximal when two of a, b, c are equal.

13. KARAMATA'S MAJORIZATION INEQUALITY

Let f be a convex function on a real interval $\mathbb{I}$. If a decreasingly ordered sequence

$$A = (a_1, a_2, \dots, a_n), \quad a_i \in \mathbb{I},$$

majorizes a decreasingly ordered sequence

$$B = (b_1, b_2, \ldots, b_n), \quad b_i \in \mathbb{I},$$

then

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge f(b_1) + f(b_2) + \cdots + f(b_n).$$

We say that a sequence $A = (a_1, a_2, \ldots, a_n)$ with $a_1 \ge a_2 \ge \cdots \ge a_n$ majorizes a sequence $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \ge b_2 \ge \cdots \ge b_n$, and write it as

$$A \succ B,$$

if

$$\begin{gathered} a_1 \ge b_1, \\ a_1 + a_2 \ge b_1 + b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_1 + a_2 + \cdots + a_{n-1} \ge b_1 + b_2 + \cdots + b_{n-1}, \\ a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n. \end{gathered}$$

14. VASC'S CYCLIC INEQUALITY

The following theorem gives Vasc's cyclic inequality (*Vasile Cîrtoaje*, 1991).

Theorem 1. *If a, b, c are real numbers, then*

$$(a^2 + b^2 + c^2)^2 \ge 3(a^3b + b^3c + c^3a),$$

with equality for $a = b = c$, and also for

$$\frac{a}{\sin^2 \frac{4\pi}{7}} = \frac{b}{\sin^2 \frac{2\pi}{7}} = \frac{c}{\sin^2 \frac{\pi}{7}}$$

(or any cyclic permutation).

A generalization of Vasc's inequality is the following (*Vasile Cîrtoaje*, 2009).

Theorem 2. *Let*

$$f_4(a, b, c) = \sum a^4 + A \sum a^2b^2 + Babc \sum a + C \sum a^3b + D \sum ab^3,$$

where A, B, C, D are real constants such that

$$1 + A + B + C + D = 0.$$

The inequality $f_4(a, b, c) \ge 0$ holds for all real numbers a, b, c if and only if

$$3(1 + A) \ge C^2 + CD + D^2.$$

Notice that

$$\frac{4}{S}f_4(a,b,c) = (U+V+C+D)^2 + 3\left(U-V+\frac{C-D}{3}\right)^2 + \frac{4}{3}(3+3A-C^2-CD-D^2),$$

where

$$S = \sum a^2b^2 - \sum a^2bc,$$

$$U = \frac{\sum a^3b - \sum a^2bc}{S},$$

$$V = \frac{\sum ab^3 - \sum a^2bc}{S}.$$

For $A = B = 0$, $C = -2$ and $D = 1$, we get the following inequality

$$a^4 + b^4 + c^4 + ab^3 + bc^3 + ca^3 \ge 2(a^3b + b^3c + c^3a),$$

with equality for $a = b = c$, and also for

$$\frac{a}{\sin\frac{\pi}{9}} = \frac{b}{\sin\frac{7\pi}{9}} = \frac{c}{\sin\frac{13\pi}{9}}$$

(or any cyclic permutation) - *Vasile Cîrtoaje*, 1991.

15. CYCLIC INEQUALITIES OF DEGREE THREE AND FOUR

Consider the third degree cyclic homogeneous polynomial

$$f_3(a,b,c) = \sum a^3 + Babc + C\sum a^2b + D\sum ab^2,$$

where B, C, D are real constants. The following theorem holds.

Theorem 1. *The cyclic inequality $f_3(a,b,c) \ge 0$ holds for all nonnegative numbers a, b, c if and only if*

$$f_3(1,1,1) \ge 0$$

and

$$f_3(a,1,0) \ge 0$$

for all $a \ge 0$.

Consider now the fourth degree cyclic homogeneous polynomial

$$f_4(a,b,c) = \sum a^4 + A\sum a^2b^2 + Babc\sum a + C\sum a^3b + D\sum ab^3,$$

where A, B, C, D are real constants.

The following theorem states the necessary and sufficient conditions that $f_4(a,b,c) \ge 0$ for all real numbers a,b,c.

Theorem 2 (*Vasile Cîrtoaje*, 2012). *The inequality $f_4(a,b,c) \ge 0$ holds for all real numbers a,b,c if and only if $g_4(t) \ge 0$ for all $t \ge 0$, where*

$$g_4(t) = 3(2+A-C-D)t^4 - Ft^3 + 3(4-B+C+D)t^2 + 1 + A + B + C + D,$$

$$F = \sqrt{27(C-D)^2 + E^2}, \qquad E = 8 - 4A + 2B - C - D.$$

Note that in the special case $f_4(1,1,1) = 0$ (when $1+A+B+C+D = 0$), Theorem 1 yields Theorem 0 from the preceding section 21.

The following theorem states some strong sufficient conditions that $f_4(a,b,c) \ge 0$ for all real numbers a,b,c.

Theorem 3 (*Vasile Cîrtoaje*, 2012). *The inequality $f_4(a,b,c) \ge 0$ holds for all real numbers a,b,c if the following two conditions are satisfied*:

(a) $1+A+B+C+D \ge 0$;

(b) *there exists a real number $t \in (-\sqrt{3}, \sqrt{3})$ such that $f(t) \ge 0$, where*

$$f(t) = 2Gt^3 - (6+2A+B+3C+3D)t^2 + 2(1+C+D)Gt + H,$$

$$G = \sqrt{1+A+B+C+D}, \qquad H = 2 + 2A - B - C - D - C^2 - CD - D^2.$$

The following theorem states the necessary and sufficient conditions that $f_4(a,b,c) \ge 0$ for all $a,b,c \ge 0$.

Theorem 4 (*Vasile Cîrtoaje*, 2013). *Let*

$$E = 8 - 4A + 2B - C - D, \qquad F = \sqrt{27(C-D)^2 + E^2},$$

$$g_4(t) = 3(2+A-C-D)t^4 - Ft^3 + 3(4-B+C+D)t^2 + 1 + A + B + C + D,$$

$$g_3(t) = \frac{2E}{F}t^3 + 3t^2 - 1.$$

For $F = 0$, the inequality $f_4(a,b,c) \ge 0$ holds for all $a,b,c \ge 0$ if and only if $g_4(t) \ge 0$ for all $t \in [0,1]$.

For $F \ne 0$, the inequality $f_4(a,b,c) \ge 0$ holds for all $a,b,c \ge 0$ if and only if the following two conditions are satisfied:

(a) *$g_4(t) \ge 0$ for all $t \in [0,t_1]$, where $t_1 \in [1/2,1]$ such that $g_3(t_1) = 0$;*

(b) *$f_4(a,1,0) \ge 0$ for all $a \ge 0$.*

The following theorem states some strong sufficient conditions that $f_4(a,b,c) \ge 0$ for all $a,b,c \ge 0$.

Theorem 5 (*Vasile Cîrtoaje*, 2013). *The inequality $f_4(a,b,c) \ge 0$ holds for all $a,b,c \ge 0$ if*

$$1 + A + B + C + D \ge 0$$

and one of the following two conditions is satisfied:

(*a*) $3(1+A) \ge C^2 + CD + D^2$;

(*b*) $3(1+A) < C^2 + CD + D^2$, *and there exists* $t \ge 0$ *such that*
$$(C+2D)t^2 + 6t + 2C + D \ge 2\sqrt{(t^4+t^2+1)(C^2+CD+D^2-3-3A)}.$$

16. VASC'S EXPONENTIAL INEQUALITY

Let $0 < k \le e$.

(a) *If* $a, b > 0$, *then (Vasile Cîrtoaje, 2006)*

$$a^{ka} + b^{kb} \ge a^{kb} + b^{ka};$$

(b) *If* $a, b \in (0,1]$, *then (Vasile Cîrtoaje, 2010)*

$$2\sqrt{a^{ka}b^{kb}} \ge a^{kb} + b^{ka}.$$

Appendix B

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