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Contents

Chapter 1

Half Convex Function Method

1.1 Theoretical Basis

Half Convex Function Theorem (*Vasile Cirtoaje*, 2004). *Let $f(u)$ be a function defined on a real interval $\mathbb{I}$ and convex on $\mathbb{I}_{u\geq s}$ or $\mathbb{I}_{u\leq s}$, where $s \in \mathbb{I}$. The inequality*

$$f(a_1)+f(a_2)+\cdots+f(a_n) \geq nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying $a_1+a_2+\cdots+a_n = ns$ if and only if

$$f(x)+(n-1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ such that $x+(n-1)y = ns$.

Proof (for the *right convexity*). The necessity is obvious. Without loss of generality, assume that $a_1 \leq a_2 \leq \cdots \leq a_n$. If $a_1 \geq s$, then the required inequality follows from Jensen's inequality for convex functions. Otherwise, there exists $k \in \{1, 2, \ldots, n-1\}$ such that

$$a_1 \leq \cdots \leq a_k < s \leq a_{k+1} \leq \cdots \leq a_n.$$

Since f is convex on $\mathbb{I}_{u\geq s}$, we can apply Jensen's inequality to get

$$f(a_{k+1})+\cdots+f(a_n) \geq (n-k)f(z), \quad z = \frac{a_{k+1}+\cdots+a_n}{n-k}.$$

Thus, it suffices to show that

$$f(a_1)+\cdots+f(a_k)+(n-k)f(z) \geq nf(s).$$

Let $b_1, \ldots, b_k \in \mathbb{I}$ defined by

$$a_i+(n-1)b_i = ns, \quad i = 1, \ldots, k.$$

We claim that

$$z \ge b_1 \ge \cdots \ge b_k > s.$$

Indeed, we have

$$b_1 \ge \cdots \ge b_k,$$

$$b_k - s = \frac{s - a_k}{n-1} > 0,$$

$$\begin{aligned}(n-1)b_1 = ns - a_1 = (a_2 + \cdots + a_k) + a_{k+1} + \cdots + a_n \le (k-1)s + a_{k+1} + \cdots + a_n \\ = (k-1)s + (n-k)z \le (n-1)z.\end{aligned}$$

By hypothesis, we have

$$f(a_1) + (n-1)f(b_1) \ge nf(s),$$

$$\cdots$$

$$f(a_k) + (n-1)f(b_k) \ge nf(s),$$

hence

$$f(a_1) + \cdots + f(a_k) + (n-1)[f(b_1) + \cdots + f(b_k)] \ge knf(s).$$

Consequently, it suffices to show that

$$knf(s) - (n-1)[f(b_1) + \cdots + f(b_k)] + (n-k)f(z) \ge nf(s),$$

which is equivalent to

$$pf(z) + (k-p)f(s) \ge f(b_1) + \cdots + f(b_k), \quad p = \frac{n-k}{n-1} \le 1.$$

By Jensen's inequality, we have

$$pf(z) + (1-p)f(s) \ge f(w), \quad w = pz + (1-p)s \ge s.$$

Thus, we only need to show that

$$f(w) + (k-1)f(s) \ge f(b_1) + \cdots + f(b_k).$$

Since the decreasingly ordered vector $\vec{A_k} = (w, s, \ldots, s)$ majorizes the decreasingly ordered vector $\vec{B_k} = (b_1, b_2, \ldots, b_k)$, this inequality follows from Karamata's inequality for convex functions.

The proof for the *left convexity* is similar.

Remark 1. Let us denote

$$g(u) = \frac{f(u) - f(s)}{u - s}, \quad h(x, y) = \frac{g(x) - g(y)}{x - y}.$$

In many applications, it is useful to replace the hypothesis $f(x)+(n-1)f(y) \geq nf(s)$ in Half Convex Function Theorem (HCF Theorem) by the equivalent condition

$$h(x,y) \geq 0 \;\; \textit{for all} \;\; x,y \in \mathbb{I} \;\; \textit{such that} \;\; x+(n-1)y = ns.$$

This equivalence is true since

$$\begin{aligned} f(x)+(n-1)f(y)-nf(s) &= [f(x)-f(s)]+(n-1)[f(y)-f(s)] \\ &= (x-s)g(x)+(n-1)(y-s)g(y) \\ &= \frac{n-1}{n}(x-y)[g(x)-g(y)] \\ &= \frac{n-1}{n}(x-y)^2 h(x,y). \end{aligned}$$

Remark 2. Assume that f is differentiable on $\mathbb{I}$, and let

$$H(x,y) = \frac{f'(x)-f'(y)}{x-y}.$$

Then, the desired inequality of Jensen's type in HCF Theorem holds true by replacing the hypothesis

$$f(x)+(n-1)f(y) \geq nf(s)$$

with the more restrictive condition

$$H(x,y) \geq 0 \;\; \textit{for all} \;\; x,y \in \mathbb{I} \;\; \textit{such that} \;\; x+(n-1)y = ns.$$

To prove this claim, we will show that the new condition implies $f(x)+(n-1)f(y) \geq nf(s)$ for all $x,y \in \mathbb{I}$ such that $x+(n-1)y = ns$. Write this inequality as $f_1(x) \geq nf(s)$, where

$$f_1(x) = f(x)+(n-1)f(y) = f(x)+(n-1)f\left(\frac{ns-x}{n-1}\right).$$

From

$$f_1'(x) = f'(x)-f'\left(\frac{ns-x}{n-1}\right) = f'(x)-f'(y) = \frac{n}{n-1}(x-s)H(x,y),$$

it follows that f_1 is decreasing for $x \leq s$ and increasing for $x \geq s$; therefore,

$$f_1(x) \geq f_1(s) = nf(s).$$

Remark 3. The inequality in HCF Theorem becomes an equality for

$$a_1 = a_2 = \cdots = a_n = s$$

and also for

$$a_1 = x, \quad a_2 = \cdots = a_n = y$$

(or any cyclic permutation), where $x, y \in \mathbb{I}$ satisfy the equations

$$x + (n-1)y = ns, \quad f(x) + (n-1)f(y) = nf(s).$$

For $x \neq y$, these equations are equivalent to

$$x + (n-1)y = ns, \quad h(x, y) = 0.$$

Remark 4. HCF Theorem is also valid in the case when $\mathbb{I} = [a, b] \setminus \{u_0\}$ or $\mathbb{I} = (a, b) \setminus \{u_0\}$, where a, b, u_0 are real numbers such that $a < u_0 < b$. Clearly, two cases are possible:

(1) $u_0 < s$ - when f is right convex, i.e. convex on $\mathbb{I}_{u \geq s}$;
(2) $u_0 > s$ - when f is left convex, i.e. convex on $\mathbb{I}_{u \leq s}$.

Remark 5. In HCF Theorem for the *right convexity* (when HCF Theorem is called RHCF Theorem), it suffices to consider

$$x \leq s \leq y.$$

This claim is true because in the proof of RHCF Theorem, the hypothesis

$$f(x) + (n-1)f(y) \geq nf(s)$$

is used to get the inequalities

$$f(a_i) + (n-1)f(b_i) \geq nf(s), \quad i = 1, 2, \cdots, k,$$

where $a_i \leq s \leq b_i$.

Similarly, in HCF Theorem for the *left convexity* (when HCF Theorem is called LHCF Theorem), it suffices to consider

$$x \geq s \geq y.$$

The following theorem (LCRCF-Theorem) is also useful to prove some symmetric inequalities.

Left Convex-Right Concave Function Theorem (*Vasile Cirtoaje*, 2004). *Let $a \leq c$ be real numbers, let f be a continuous function on $\mathbb{I} = [a, \infty)$, strictly convex on $[a, c]$ and strictly concave on $[c, \infty)$, and let*

$$E(a_1, a_2, \ldots, a_n) = f(a_1) + f(a_2) + \cdots + f(a_n).$$

If $a_1, a_2, \ldots, a_n \in \mathbb{I}$ such that

$$a_1 + a_2 + \cdots + a_n = S = constant,$$

then

(a) *E is minimum for $a_1 = a_2 = \cdots = a_{n-1} \leq a_n$;*

(b) *E is maximum for either* $a_1 = a$ or $a < a_1 \le a_2 = \cdots = a_n$.

Proof. Without loss of generality, assume that $a_1 \le a_2 \le \ldots \le a_n$. Since $E(a_1, a_2, \ldots, a_n)$ is a continuous function on the compact set

$$\Lambda = \{(a_1, a_2, \ldots, a_n):\ a_1 + a_2 + \cdots + a_n = S,\ a_1, a_2, \ldots, a_n \in \mathbb{I}\},$$

E attains its minimum and maximum values.

(a) For the sake of contradiction, suppose that E is minimum at a point $(b_1, b_2, \ldots, b_n)$ with $b_1 \le b_2 \le \ldots \le b_n$ and $b_1 < b_{n-1}$. For $b_{n-1} \le c$, by Jensen's inequality for strictly convex functions, we have

$$f(b_1) + f(b_{n-1}) > 2f\left(\frac{b_1 + b_{n-1}}{2}\right),$$

while for $b_{n-1} > c$, by Karamata's inequality for strictly concave functions, we have

$$f(b_{n-1}) + f(b_n) > f(c) + f(b_{n-1} + b_n - c).$$

The both results contradict the assumption that E is minimum at $(b_1, b_2, \ldots, b_n)$.

(b) For the sake of contradiction, suppose that E is maximum at a point $(b_1, b_2, \ldots, b_n)$ with $a < b_1 \le b_2 \le \ldots \le b_n$ and $b_2 < b_n$. There are three cases to consider.

Case 1: $b_2 \ge c$. By Jensen's inequality for strictly concave functions, we have

$$f(b_2) + f(b_n) < 2f\left(\frac{b_2 + b_n}{2}\right).$$

Case 2: $b_2 < c$ and $b_1 + b_2 - a \le c$. By Karamata's inequality for strictly convex functions, we have

$$f(b_1) + f(b_2) < f(a) + f(b_1 + b_2 - a).$$

Case 3: $b_2 < c$ and $b_1 + b_2 - c \ge a$. By Karamata's inequality for strictly convex functions, we have

$$f(b_1) + f(b_2) < f(b_1 + b_2 - c) + f(c).$$

Clearly, the result of each case contradicts the assumption that E is msximum at $(b_1, b_2, \ldots, b_n)$.

Remark 6. The part (a) in LCRCF-Theorem is also true in the case where $\mathbb{I} = (a, \infty)$ and $\lim_{x \to a} f(x) = \infty$.

1.2 Applications

1.1. If a, b, c are real numbers such that $a+b+c=3$, then

$$3(a^4+b^4+c^4)+a^2+b^2+c^2+6 \ge 6(a^3+b^3+c^3).$$

1.2. If $a_1, a_2, \ldots, a_n \ge \dfrac{1-2n}{n-2}$ such that $a_1+a_2+\cdots+a_n=n$, then

$$a_1^3+a_2^3+\cdots+a_n^3 \ge n.$$

1.3. If $a_1, a_2, \ldots, a_n \ge \dfrac{-n}{n-2}$ such that $a_1+a_2+\cdots+a_n=n$, then

$$a_1^3+a_2^3+\cdots+a_n^3 \ge a_1^2+a_2^2+\cdots+a_n^2.$$

1.4. If $a_1, a_2, \ldots, a_n$ are real numbers such that $a_1+a_2+\cdots+a_n=n$, then

$$(n^2-3n+3)(a_1^4+a_2^4+\cdots+a_n^4-n) \ge 2(n^2-n+1)(a_1^2+a_2^2+\cdots+a_n^2-n).$$

1.5. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$, then

$$(n^2+n+1)(a_1^3+a_2^3+\cdots+a_n^3-n) \ge (n+1)(a_1^4+a_2^4+\cdots+a_n^4-n).$$

1.6. If a, b, c are real numbers such that $a+b+c=3$, then

(a) $$a^4+b^4+c^4-3+2(7+3\sqrt{7})(a^3+b^3+c^3-3) \ge 0;$$

(b) $$a^4+b^4+c^4-3+2(7-3\sqrt{7})(a^3+b^3+c^3-3) \ge 0.$$

1.7. Let $k \ge 3$ and $n \ge 3$ be integer numbers such that $k \le n+1$. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$, then

$$\frac{a_1^k+a_2^k+\cdots+a_n^k-n}{a_1^2+a_2^2+\cdots+a_n^2-n} \ge (n-1)\left[\left(\frac{n}{n-1}\right)^{k-1}-1\right].$$

1.8. Let $k \geq 3$ and $n \geq 3$ be integer numbers. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then

$$\frac{a_1^k + a_2^k + \cdots + a_n^k - n}{a_1^2 + a_2^2 + \cdots + a_n^2 - n} \leq \frac{n^{k-1} - 1}{n-1}.$$

1.9. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then

$$n^2\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n\right) \geq 4(n-1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n).$$

1.10. If $a_1, a_2, \ldots, a_8$ are positive real numbers such that $a_1 + a_2 + \cdots + a_8 = n$, then

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_8^2} \geq a_1^2 + a_2^2 + \cdots + a_8^2.$$

1.11. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = n$, then

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n \geq 2\left(1 + \frac{\sqrt{n-1}}{n}\right)(a_1 + a_2 + \cdots + a_n - n).$$

1.12. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{3a+b+c} + \frac{1}{3b+c+a} + \frac{1}{3c+a+b} \leq \frac{2}{5}\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right).$$

1.13. If $a, b, c, d \geq 3 - \sqrt{7}$ such that $a + b + c + d = 4$, then

$$\frac{1}{2+a^2} + \frac{1}{2+b^2} + \frac{1}{2+c^2} + \frac{1}{2+d^2} \geq \frac{4}{3}.$$

1.14. If $a_1, a_2, \ldots, a_n \in [0, n-2]$ such that $a_1 + a_2 + \cdots + a_n = n$, then

$$\frac{1}{n+a_1^2} + \frac{1}{n+a_2^2} + \cdots + \frac{1}{n+a_n^2} \leq \frac{n}{n+1}.$$

1.15. If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then

$$\frac{3-a}{9+a^2} + \frac{3-b}{9+b^2} + \frac{3-c}{9+c^2} \geq \frac{3}{5}.$$

1.16. If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then

$$\frac{1}{5+a+a^2} + \frac{1}{5+b+b^2} + \frac{1}{5+c+c^2} \geq \frac{3}{7}.$$

1.17. If a, b, c, d are nonnegative real numbers such that $a + b + c + d = 4$, then

$$\frac{1}{10+a+a^2} + \frac{1}{10+b+b^2} + \frac{1}{10+c+c^2} + \frac{1}{10+d+d^2} \leq \frac{1}{3}.$$

1.18. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $k \geq 1 - \frac{1}{n}$, then

$$\frac{1}{1+ka_1^2} + \frac{1}{1+ka_2^2} + \cdots + \frac{1}{1+ka_n^2} \geq \frac{n}{1+k}.$$

1.19. Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $0 < k \leq \frac{n-1}{n^2-n+1}$, then

$$\frac{1}{1+ka_1^2} + \frac{1}{1+ka_2^2} + \cdots + \frac{1}{1+ka_n^2} \leq \frac{n}{1+k}.$$

1.20. Let $a_1, a_2, \ldots, a_n$ be nonnegative numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $k \geq \frac{n^2}{4(n-1)}$, then

$$\frac{a_1(a_1-1)}{a_1^2+k} + \frac{a_2(a_2-1)}{a_2^2+k} + \cdots + \frac{a_n(a_n-1)}{a_n^2+k} \geq 0.$$

1.21. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then

$$\frac{1-a_1}{(n-2a_1)^2} + \frac{1-a_2}{(n-2a_2)^2} + \cdots + \frac{1-a_n}{(n-2a_n)^2} \leq 0.$$

1.22. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $k \ge 1 + \dfrac{n}{\sqrt{n-1}}$, then

$$\frac{a_1^2 - 1}{(a_1 + k)^2} + \frac{a_2^2 - 1}{(a_2 + k)^2} + \cdots + \frac{a_n^2 - 1}{(a_n + k)^2} \ge 0.$$

1.23. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $0 < k \le 1 + \sqrt{\dfrac{2n-1}{n-1}}$, then

$$\frac{a_1^2 - 1}{(a_1 + k)^2} + \frac{a_2^2 - 1}{(a_2 + k)^2} + \cdots + \frac{a_n^2 - 1}{(a_n + k)^2} \le 0.$$

1.24. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $k \ge \dfrac{(n-1)(2n-1)}{n^2}$, then

$$\frac{1}{1 + ka_1^3} + \frac{1}{1 + ka_2^3} + \cdots + \frac{1}{1 + ka_n^3} \ge \frac{n}{1+k}.$$

1.25. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $0 < k \le \dfrac{n-1}{n^2 - 2n + 2}$, then

$$\frac{1}{1 + ka_1^3} + \frac{1}{1 + ka_2^3} + \cdots + \frac{1}{1 + ka_n^3} \le \frac{n}{1+k}.$$

1.26. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $k \ge \dfrac{n^2}{n-1}$, then

$$\sqrt{\frac{a_1}{k - a_1}} + \sqrt{\frac{a_2}{k - a_2}} + \cdots + \sqrt{\frac{a_n}{k - a_n}} \le \frac{n}{\sqrt{k-1}}.$$

1.27. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then

$$n^{-a_1^2} + n^{-a_2^2} + \cdots + n^{-a_n^2} \ge 1.$$

1.28. If a,b,c,d,e are nonnegative real numbers such that $a+b+c+d+e=5$, then

$$(a^2+1)(b^2+1)(c^2+1)(d^2+1)(e^2+1) \ge (a+1)(b+1)(c+1)(d+1)(e+1).$$

1.29. If $a_1,a_2,\ldots,a_{10}$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_{10}=10$, then

$$(1-a_1+a_1^2)(1-a_2+a_2^2)\cdots(1-a_{10}+a_{10}^2) \ge 1.$$

1.30. If $a_1,a_2,\ldots,a_n$ $(n \ge 3)$ are positive real numbers such that $a_1+a_2+\cdots+a_n=1$, then

$$\left(\frac{1}{\sqrt{a_1}}-\sqrt{a_1}\right)\left(\frac{1}{\sqrt{a_2}}-\sqrt{a_2}\right)\cdots\left(\frac{1}{\sqrt{a_n}}-\sqrt{a_n}\right) \ge \left(\sqrt{n}-\frac{1}{\sqrt{n}}\right)^n.$$

1.31. Let $a_1,a_2,\ldots,a_n$ be positive real numbers such that $a_1+a_2+\cdots+a_n=n$. If

$$k \le \left(1+\frac{2\sqrt{n-1}}{n}\right)^2,$$

then

$$\left(ka_1+\frac{1}{a_1}\right)\left(ka_2+\frac{1}{a_2}\right)\cdots\left(ka_n+\frac{1}{a_n}\right) \ge (k+1)^n.$$

1.32. If a,b,c,d are nonzero real numbers such that

$$a,b,c,d \ge \frac{-1}{2}, \quad a+b+c+d=4,$$

then

$$3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right)+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \ge 16.$$

1.33. If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers such that $a_1^2+a_2^2+\cdots+a_n^2=n$, then

$$a_1^3+a_2^3+\cdots+a_n^3-n+\sqrt{\frac{n}{n-1}}\,(a_1+a_2+\cdots+a_n-n) \ge 0.$$

1.34. If a,b,c,d,e are nonnegative real numbers such that $a^2+b^2+c^2+d^2+e^2=5$, then

$$\frac{1}{7-2a}+\frac{1}{7-2b}+\frac{1}{7-2c}+\frac{1}{7-2d}+\frac{1}{7-2e}\le 1.$$

1.35. If $0\le a_1,a_2,\dots,a_n<k$ such that $a_1^2+a_2^2+\cdots+a_n^2=n$, where $1<k\le 1+\sqrt{\frac{n}{n-1}}$, then

$$\frac{1}{k-a_1}+\frac{1}{k-a_2}+\cdots+\frac{1}{k-a_n}\ge\frac{n}{k-1}.$$

1.36. If a,b,c are nonnegative real numbers, no two of which are zero, then

$$\sqrt{1+\frac{48a}{b+c}}+\sqrt{1+\frac{48b}{c+a}}+\sqrt{1+\frac{48c}{a+b}}\ge 15.$$

1.37. If a,b,c are nonnegative real numbers, then

$$\sqrt{\frac{3a^2}{7a^2+5(b+c)^2}}+\sqrt{\frac{3b^2}{7b^2+5(c+a)^2}}+\sqrt{\frac{3c^2}{7c^2+5(a+b)^2}}\le 1.$$

1.38. If a,b,c are nonnegative real numbers, then

$$\sqrt{\frac{a^2}{a^2+2(b+c)^2}}+\sqrt{\frac{b^2}{b^2+2(c+a)^2}}+\sqrt{\frac{c^2}{c^2+2(a+b)^2}}\ge 1.$$

1.39. Let a,b,c be nonnegative real numbers, no two of which are zero. If

$$k\ge k_0,\quad k_0=\frac{\ln 3}{\ln 2}-1\approx 0.585,$$

then

$$\left(\frac{2a}{b+c}\right)^k+\left(\frac{2b}{c+a}\right)^k+\left(\frac{2c}{a+b}\right)^k\ge 3.$$

1.40. If $a,b,c\in[1,7+4\sqrt{3}]$, then

$$\sqrt{\frac{2a}{b+c}}+\sqrt{\frac{2b}{c+a}}+\sqrt{\frac{2c}{a+b}}\ge 3.$$

1.41. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. If

$$0<k\le k_0,\quad k_0=\frac{\ln 2}{\ln 3-\ln 2}\approx 1.71,$$

then

$$a^k(b+c)+b^k(c+a)+c^k(a+b)\le 6.$$

1.42. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$. If $k\ge\frac{n^2}{4(n-1)}$, then

$$(a_1^2+k)(a_2^2+k)\cdots(a_n^2+k)\ge(1+k)^n.$$

1.43. Let a, b, c be nonnegative real numbers such $a+b+c=3$. If $k\ge k_0$, where

$$k_0=\frac{\sqrt{6}-2}{\sqrt{6}-\sqrt{2}-1}=(2+\sqrt{2})(2+\sqrt{3})\approx 12.74\ ,$$

then

$$\sqrt{a}+\sqrt{b}+\sqrt{c}-3\ge k\left(\sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}}-3\right).$$

1.44. Let a, b, c be nonnegative real numbers such $a+b+c=3$. If $k\le k_1$, where

$$k_1=(\sqrt{3}-1)(\sqrt{3}+\sqrt{2})\approx 2.303\ ,$$

then

$$\sqrt{a}+\sqrt{b}+\sqrt{c}-3\le k\left(\sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}}-3\right).$$

1.45. If a, b, c are positive real numbers such that $abc=1$, then

$$a^2+b^2+c^2-3\ge 18(a+b+c-ab-bc-ca).$$

1.46. If a, b, c are positive real numbers such that $abc=1$, then

$$\sqrt{a^2-a+1}+\sqrt{b^2-b+1}+\sqrt{c^2-c+1}\ge a+b+c.$$

1.47. If $a,b,c,d \ge \frac{1}{1+\sqrt{6}}$ such that $abcd = 1$, then

$$\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}+\frac{1}{d+2} \le \frac{4}{3}.$$

1.48. If $a_1,a_2,\ldots,a_n$ are positive real numbers such that $a_1a_2\cdots a_n = 1$, then

$$a_1^2+a_2^2+\cdots+a_n^2-n \ge 6\sqrt{3}\left(a_1+a_2+\cdots+a_n-\frac{1}{a_1}-\frac{1}{a_2}-\cdots-\frac{1}{a_n}\right).$$

1.49. If $a_1,a_2,\ldots,a_n$ $(n\ge 4)$ are positive real numbers such that $a_1a_2\cdots a_n = 1$, then

$$(n-1)(a_1^2+a_2^2+\cdots+a_n^2)+n(n+3) \ge (2n+2)(a_1+a_2+\cdots+a_n).$$

1.50. Let a,b,c,d be positive real numbers such that $abcd = 1$. If p and q are nonnegative real numbers such that $p+q=3$, then then

$$\frac{1}{1+pa+qa^3}+\frac{1}{1+pb+qb^3}+\frac{1}{1+pc+qc^3}+\frac{1}{1+pd+qd^3} \ge 1.$$

1.51. Let $a_1,a_2,\ldots,a_n$ $(n\ge 3)$ be positive real numbers such that $a_1a_2\ldots a_n = 1$. If p and q are nonnegative real numbers such that $p+q\ge n-1$, then

$$\frac{1}{1+pa_1+qa_1^2}+\frac{1}{1+pa_2+qa_2^2}+\cdots+\frac{1}{1+pa_n+qa_n^2} \ge \frac{n}{1+p+q}.$$

1.52. Let $a_1,a_2,\ldots,a_n$ be positive real numbers such that $a_1a_2\ldots a_n = 1$. If $k\ge n^2-1$, then

$$\frac{1}{\sqrt{1+ka_1}}+\frac{1}{\sqrt{1+ka_2}}+\cdots+\frac{1}{\sqrt{1+ka_n}} \ge \frac{n}{\sqrt{1+k}}.$$

1.53. If $a_1,a_2,\ldots,a_n$ are positive real numbers such that $a_1a_2\ldots a_n = 1$, then

$$\frac{1}{1+a_1+\cdots+a_1^{n-1}}+\frac{1}{1+a_2+\cdots+a_2^{n-1}}+\cdots+\frac{1}{1+a_n+\cdots+a_n^{n-1}} \ge 1.$$

1.54. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \ldots a_n = 1$. If $p, q \geq 0$ such that $0 < p + q \leq \dfrac{1}{n-1}$, then

$$\frac{1}{1+pa_1+qa_1^2}+\frac{1}{1+pa_2+qa_2^2}+\cdots+\frac{1}{1+pa_n+qa_n^2} \leq \frac{n}{1+p+q}.$$

1.55. Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be positive real numbers such that $a_1 a_2 \ldots a_n = 1$. If

$$0 < k \leq \left(\frac{n}{n-1}\right)^2 - 1,$$

then

$$\frac{1}{\sqrt{1+ka_1}}+\frac{1}{\sqrt{1+ka_2}}+\cdots+\frac{1}{\sqrt{1+ka_n}} \leq \frac{n}{\sqrt{1+k}}.$$

1.56. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 a_2 \ldots a_n = 1$, then

$$a_1^{n-1}+a_2^{n-1}+\cdots+a_n^{n-1}+n(n-2) \geq (n-1)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right).$$

1.57. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \ldots a_n = 1$. If $k \geq n$, then

$$a_1^k+a_2^k+\cdots+a_n^k+kn \geq (k+1)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right).$$

1.58. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 a_2 \ldots a_n = 1$, then

$$\left(1-\frac{1}{n}\right)^{a_1}+\left(1-\frac{1}{n}\right)^{a_2}+\cdots+\left(1-\frac{1}{n}\right)^{a_n} \leq n-1.$$

1.59. If a, b, c are positive real numbers such that $abc = 1$, then

$$\frac{1}{1+\sqrt{1+3a}}+\frac{1}{1+\sqrt{1+3b}}+\frac{1}{1+\sqrt{1+3c}} \leq 1.$$

1.60. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 a_2 \cdots a_n = 1$, then

$$\frac{1}{1+\sqrt{1+4n(n-1)a_1}}\frac{1}{1+\sqrt{1+4n(n-1)a_2}}+\cdots+\frac{1}{1+\sqrt{1+4n(n-1)a_n}} \geq \frac{1}{2}.$$

1.61. If a,b,c are positive real numbers such that $abc=1$, then

$$\frac{a^6}{1+2a^5}+\frac{b^6}{1+2b^5}+\frac{c^6}{1+2c^5}\ge 1.$$

1.62. If a,b,c are positive real numbers such that $abc=1$, then

$$\sqrt{25a^2+144}+\sqrt{25b^2+144}+\sqrt{25c^2+144}\le 5(a+b+c)+24.$$

1.63. If a,b,c are positive real numbers such that $abc=1$, then

$$\sqrt{16a^2+9}+\sqrt{16b^2+9}+\sqrt{16c^2+9}\ge 4(a+b+c)+3.$$

1.64. If a,b,c,d are real numbers such that $a+b+c+d=4$, then

$$\frac{a}{a^2-a+4}+\frac{b}{b^2-b+4}+\frac{c}{c^2-c+4}+\frac{d}{d^2-d+4}\le 1.$$

1.65. If a,b,c are nonnegative real numbers such that $a+b+c=12$, then

$$(a^2+10)(b^2+10)(c^2+10)\ge 13310.$$

1.66. Let a,b,c be nonnegative real numbers. If

$$k_0\le k\le 3, \quad k_0=\frac{\ln 2}{\ln 3-\ln 2}\approx 1.71,$$

then

$$a^k(b+c)+b^k(c+a)+c^k(a+b)\le 2\left(\frac{a+b+c}{2}\right)^{k+1}.$$

1.67. If $a_1,a_2,\ldots,a_n$ are positive real numbers such that $a_1+a_2+\cdots+a_n=n$, then

$$(n+1)^2\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)\ge 4(n+2)(a_1^2+a_2^2+\cdots+a_n^2)+n(n^2-3n+6).$$

1.3 Solutions

P 1.1. *If a, b, c are real numbers such that $a+b+c=3$, then*

$$3(a^4+b^4+c^4)+a^2+b^2+c^2+6 \geq 6(a^3+b^3+c^3).$$

(Vasile Cirtoaje, 2006)

Solution. Write the inequality as

$$f(a)+f(b)+f(c) \geq 3f(s), \quad s=\frac{a+b+c}{3}=1,$$

where

$$f(u)=3u^4-6u^3+u^2, \quad u \in \mathbb{R}.$$

From

$$f''(u)=2(18u^2-18u+1),$$

it follows that $f''(u)>0$ for $u \geq 1$, hence f is convex for $u \geq 1$. By HCF Theorem, it suffices to show that $f(x)+2f(y) \geq 3f(1)$ for all real x, y such that $x+2y=3$. Using Remark 1, we only need to show that $h(x,y) \geq 0$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y}, \quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u)=3(u^3+u^2+u+1)-6(u^2+u+1)+u+1=3u^3-3u^2-2u-2,$$

$$h(x,y)=3(x^2+xy+y^2)-3(x+y)-2=(3y-4)^2 \geq 0.$$

From $x+2y=3$ and $h(x,y)=0$, we get $x=1/3$, $y=4/3$. Therefore, in accordance with Remark 3, the equality holds for $a=b=c=1$, and also for $a=1/3$ and $b=c=4/3$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$, then*

$$(a_1^2-a_1)^2+(a_2^2-a_2)^2+\cdots+(a_n^2-a_n)^2 \geq \frac{n-1}{n^2-3n+3}(a_1^2+a_2^2+\cdots+a_n^2-n),$$

with equality for $a_1=a_2=\cdots=a_n=1$, and also for $a_1=\dfrac{1}{n^2-3n+3}$ and $a_2=a_3=\cdots=a_n=1+\dfrac{n-2}{n^2-3n+3}$ (or any cyclic permutation).

□

P 1.2. *If* $a_1, a_2, \ldots, a_n \geq \dfrac{1-2n}{n-2}$ *such that* $a_1 + a_2 + \cdots + a_n = n$, *then*

$$a_1^3 + a_2^3 + \cdots + a_n^3 \geq n.$$

(Vasile Cirtoaje, 2000)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = u^3, \quad u \geq \frac{1-2n}{n-2}.$$

From $f''(u) = 6u$, it follows that f is convex for $u \geq 1$. According to HCF Theorem and Remark 1, it suffices to show that $h(x,y) \geq 0$ for all $x, y \geq \dfrac{1-2n}{n-2}$ such that $x + (n-1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u-1} = u^2 + u + 1,$$

$$h(x,y) = \frac{g(x) - g(y)}{x-y} = x + y + 1 = \frac{(n-2)x + 2n - 1}{n-1} \geq 0.$$

From $x + (n-1)y = n$ and $h(x,y) = 0$, we get $x = \dfrac{1-2n}{n-2}$, $y = \dfrac{n+1}{n-2}$. Therefore, in accordance with Remark 3, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{1-2n}{n-2}, \quad a_2 = \cdots = a_n = \frac{n+1}{n-2}$$

(or any cyclic permutation).

□

P 1.3. *If* $a_1, a_2, \ldots, a_n \geq \dfrac{-n}{n-2}$ *such that* $a_1 + a_2 + \cdots + a_n = n$, *then*

$$a_1^3 + a_2^3 + \cdots + a_n^3 \geq a_1^2 + a_2^2 + \cdots + a_n^2.$$

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = u^3 - u^2, \quad u \geq \frac{-n}{n-2}.$$

From $f''(u) = 6u - 2$, it follows that f is convex for $u \geq 1$. According to HCF Theorem and Remark 1, it suffices to show that $h(x,y) \geq 0$ for all $x,y \geq \dfrac{1-2n}{n-2}$ such that $x + (n-1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u-1} = u^2,$$

$$h(x,y) = \frac{g(x)-g(y)}{x-y} = x + y = \frac{(n-2)x+n}{n-1} \geq 0.$$

From $x + (n-1)y = n$ and $h(x,y) = 0$, we get $x = \dfrac{-n}{n-2}$, $y = \dfrac{n}{n-2}$. Therefore, in accordance with Remark 3, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{-n}{n-2}, \quad a_2 = \cdots = a_n = \frac{n}{n-2}$$

(or any cyclic permutation).

□

P 1.4. *If $a_1, a_2, \ldots, a_n$ are real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then*

$$(n^2 - 3n + 3)(a_1^4 + a_2^4 + \cdots + a_n^4 - n) \geq 2(n^2 - n + 1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n).$$

(Vasile Cirtoaje, 2009)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = (n^2 - 3n + 3)u^4 - 2(n^2 - n + 1)u^2, \quad u \in \mathbb{R}.$$

For $u \geq 1$, we have

$$\frac{1}{4}f''(u) = 3(n^2-3n+3)u^2 - (n^2-n+1) \geq 3(n^2-3n+3) - (n^2-n+1) = 2(n-2)^2 \geq 0.$$

By HCF Theorem and Remark 1, it suffices to show that $h(x,y) \geq 0$ for $x,y \in \mathbb{R}$ such that $x + (n-1)y = n$, where

$$h(x,y) = \frac{g(x)-g(y)}{x-y}, \quad g(u) = \frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u) = (n^2 - 3n + 3)(u^3 + u^2 + u + 1) - 2(n^2 - n + 1)(u + 1)$$

and

$$h(x,y)=(n^2-3n+3)(x^2+xy+y^2+x+y+1)-2(n^2-n+1)=[(n^2-3n+3)y-n^2+n+1]^2\geq 0.$$

In accordance with Remark 3, the equality holds for $a_1=a_2=\cdots=a_n=1$, and also for $a_1=-1+\dfrac{2}{n^2-3n+3}$ and $a_2=a_3=\cdots=a_n=1+\dfrac{2n-4}{n^2-3n+3}$ (or any cyclic permutation). □

P 1.5. *If $a_1,a_2,\cdots,a_n$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$, then*

$$(n^2+n+1)(a_1^3+a_2^3+\cdots+a_n^3-n)\geq(n+1)(a_1^4+a_2^4+\cdots+a_n^4-n).$$

(Vasile Cirtoaje, 2009)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\geq nf(s),\quad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=(n^2+n+1)u^3-(n+1)u^4,\quad u\in[0,n].$$

For $u\in[0,1]$, we have

$$f''(u)=6u[n^2+n+1-2(n+1)u]\geq 6u[n^2+n+1-2(n+1)]=6(n^2-n-1)u\geq 0.$$

By HCF Theorem and Remark 1, it suffices to show that $h(x,y)\geq 0$ for $x,y\geq 0$ such that $x+(n-1)y=n$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y},\quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u)=(n^2+n+1)(u^2+u+1)-(n+1)(u^3+u^2+u+1)=-(n+1)u^3+n^2(u^2+u+1)$$

and

$$\begin{aligned}h(x,y)&=-(n+1)(x^2+xy+y^2)+n^2(x+y+1)\\&=y[-(n+1)(n^2-3n+3)y+n(n^2+n-3)]\\&\geq y\left[-(n+1)(n^2-3n+3)\frac{n}{n-1}+n(n^2+n-3)\right]\\&=\frac{2n^2(n-2)y}{n-1}\geq 0.\end{aligned}$$

In accordance with Remark 3, the equality holds for $a_1=a_2=\cdots=a_n=1$, and also for $a_1=n$ and $a_2=a_3=\cdots=a_n=0$ (or any cyclic permutation).

□

P 1.6. *If a, b, c are real numbers such that $a+b+c=3$, then*

(a) $$a^4+b^4+c^4-3+2(7+3\sqrt{7})(a^3+b^3+c^3-3)\ge 0;$$

(b) $$a^4+b^4+c^4-3+2(7-3\sqrt{7})(a^3+b^3+c^3-3)\ge 0.$$

(Vasile Cirtoaje, 2009)

Solution. Setting

$$p=2(7\pm 3\sqrt{7}),$$

we can write the desired inequalities as

$$f(a)+f(b)+f(c)\ge 3f(s),\quad s=\frac{a+b+c}{3}=1,$$

where

$$f(u)=u^4+pu^3,\quad u\in\mathbb{R}.$$

From

$$f''(u)=6u(2u^2+p),$$

it follows that $f''(u)>0$ for $u\ge 1$, hence f is convex for $u\ge 1$. By HCF Theorem, it suffices to show that $f(x)+2f(y)\ge 3f(1)$ for all real x, y such that $x+2y=3$. Using Remark 1, we only need to show that $h(x,y)\ge 0$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y},\quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u)=u^3+u^2+u+1+p(u^2+u+1)+u+1=u^3+(p+1)(u^2+u+1),$$

$$\begin{aligned}h(x,y)&=x^2+xy+y^2)+(p+1)(x+y+1)=3y^2-(10+p)y+13+4p\\&=3\left(y-\frac{10+p}{6}\right)^2\ge 0.\end{aligned}$$

In accordance with Remark 3, the equality holds for $a=b=c=1$, and also for $a=-(1+p)/3$ and $b=c=(10+p)/6$ (or any cyclic permutation).

(a) Since $p=2(7+3\sqrt{7})$, the equality holds for $a=b=c=1$, and also for $a=-5-2\sqrt{7}$ and $b=c=4+\sqrt{7}$ (or any cyclic permutation).

(b) Since $p=2(7-3\sqrt{7})$, the equality holds for $a=b=c=1$, and also for $a=-5+2\sqrt{7}$ and $b=c=4-\sqrt{7}$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- *If $a_1, a_2, \ldots, a_n$ are real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then*

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n + p(a_1^2 + a_2^2 + \cdots + a_n^2 - n) \geq 0,$$

where

$$p = \frac{2(n^2 - n + 1) \pm 2\sqrt{3(n^2 - n + 1)(n^2 - 3n + 3)}}{(n-2)^2}.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{-2(n^2 - 3n + 1) - (n-1)(n-2)p}{2(n^2 - 3n + 3)}, \quad a_2 = a_3 = \cdots = a_n = \frac{2(n^2 - n - 1) + (n-2)p}{2(n^2 - 3n + 3)}$$

(or any cyclic permutation).

□

P 1.7. *Let $k \geq 3$ and $(n \geq 3)$ be integer numbers such that $k \leq n + 1$. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then*

$$\frac{a_1^k + a_2^k + \cdots + a_n^k - n}{a_1^2 + a_2^2 + \cdots + a_n^2 - n} \geq (n-1)\left[\left(\frac{n}{n-1}\right)^{k-1} - 1\right].$$

(Vasile Cirtoaje, 2012)

Solution. Denote

$$m = (n-1)\left[\left(\frac{n}{n-1}\right)^{k-1} - 1\right] = \left(\frac{n}{n-1}\right)^{k-2} + \left(\frac{n}{n-1}\right)^{k-3} + \cdots + 1,$$

and write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = u^k - mu^2, \quad u \in [0, n].$$

We will show that $f(u)$ is convex for $u \in [1, n]$. Since

$$f''(u) = k(k-1)u^{k-2} - 2m \geq k(k-1) - 2m,$$

we need to show that

$$\frac{k(k-1)}{2} \geq \left(\frac{n}{n-1}\right)^{k-2} + \left(\frac{n}{n-1}\right)^{k-3} + \cdots + 1.$$

Since $n \ge k-1$, this inequality is true if

$$\frac{k(k-1)}{2} \ge \left(\frac{k-1}{k-2}\right)^{k-2} + \left(\frac{k-1}{k-2}\right)^{k-3} + \cdots + 1.$$

By Bernoulli's inequality, we have

$$\left(\frac{k-1}{k-2}\right)^{j} = \frac{1}{\left(1-\frac{1}{k-1}\right)^{j}} \le \frac{1}{1-\frac{j}{k-1}} = \frac{k-1}{k-j-1}, \quad 0 \le j \le k-2.$$

Therefore, it suffices to show that

$$\frac{k(k-1)}{2} \ge (k-1)\left(1+\frac{1}{2}+\cdots+\frac{1}{k-1}\right).$$

This is true if

$$\frac{k}{2} \ge 1+\frac{1}{2}+\cdots+\frac{1}{k-1},$$

which can be easily proved by induction. According to HCF Theorem and Remark 1, we only need to show that $h(x,y) \ge 0$ for $x,y \ge 0$ such that $x+(n-1)y = n$, where

$$h(x,y) = \frac{g(x)-g(y)}{x-y}, \quad g(u) = \frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u) = \frac{(u^k-1)-m(u^2-1)}{u-1} = (u^{k-1}+u^{k-2}+\cdots+1)-m(u+1),$$

$$h(x,y) = \left(\frac{x^{k-1}-y^{k-1}}{x-y} + \frac{x^{k-2}-y^{k-1}}{x-y} + \cdots + 1\right) - m$$

$$= \left[\frac{x^{k-1}-y^{k-1}}{x-y} - \left(\frac{n}{n-1}\right)^{k-2}\right] + \left[\frac{x^{k-2}-y^{k-2}}{x-y} - \left(\frac{n}{n-1}\right)^{k-3}\right] + \cdots + \left(\frac{x^2-y^2}{x-y} - \frac{n}{n-1}\right).$$

It suffices to show that

$$\frac{x^{j+1}-y^{j+1}}{x-y} \ge \left(\frac{n}{n-1}\right)^{j}$$

for $x \ne y$ and $j = 1,2,\cdots,k-2$. This is true if $f_j(y) \ge 0$ for $y \in \left[0, \frac{n}{n-1}\right]$, where

$$f_j(y) = x^j + x^{j-1}y + \cdots + xy^{j-1} + y^j - \left(\frac{n}{n-1}\right)^{j}, \quad x = n-(n-1)y.$$

Since $x' = -(n-1)$ and $n-1 \ge k-2 \ge j$, we get

$$f_j'(y) = -(n-1)[jx^{j-1} + (j-1)x^{j-2}y + \cdots + y^{j-1}] + x^{j-1} + 2x^{j-2}y + \cdots + jy^{j-1}$$

$$\leq -j[jx^{j-1}+(j-1)x^{j-2}y+\cdots+y^{j-1}]+x^{j-1}+2x^{j-2}y+\cdots+jy^{j-1}$$
$$=-(j\cdot j-1)x^{j-1}-[j\cdot(j-1)-2]x^{j-2}y-\cdots-(j\cdot 2-j+1)xy^{j-2}\leq 0.$$

Therefore, $f_j(y)$ is decreasing, hence $f_j(y)$ is minimal for $y=\frac{n}{n-1}$, when $x=0$. Thus,

$$f_j(y)\geq f_j\left(\frac{n}{n-1}\right)=0.$$

This completes the proof. From $x+(n-1)y=n$ and $h(x,y)=0$, we get $x=0$, $y=\frac{n}{n-1}$. Therefore, in accordance with Remark 3, the equality holds for $a_1=0$ and $a_2=a_3=\cdots=a_n=\frac{n}{n-1}$ (or any cyclic permutation).

Remark. For $k=3$ and $k=4$, we get the following statements (*Vasile Cirtoaje*, 2002):

- *If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$, then*

$$(n-1)(a_1^3+a_2^3+\cdots+a_n^3-n)\geq(2n-1)(a_1^2+a_2^2+\cdots+a_n^2-n),$$

with equality for $a_1=a_2=\cdots=a_n=1$, and also for $a_1=0$ and $a_2=a_3=\cdots=a_n=\frac{n}{n-1}$ (or any cyclic permutation).

- *If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$, then*

$$(n-1)^2(a_1^4+a_2^4+\cdots+a_n^4-n)\geq(3n^2-3n+1)(a_1^2+a_2^2+\cdots+a_n^2-n),$$

with equality for $a_1=a_2=\cdots=a_n=1$, and also for $a_1=0$ and $a_2=a_3=\cdots=a_n=\frac{n}{n-1}$ (or any cyclic permutation).

□

P 1.8. *Let $k\geq 3$ and $n\geq 3$ be integer numbers. If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$, then*

$$\frac{a_1^k+a_2^k+\cdots+a_n^k-n}{a_1^2+a_2^2+\cdots+a_n^2-n}\leq\frac{n^{k-1}-1}{n-1}.$$

(Vasile Cirtoaje, 2012)

Solution. Denote

$$m=\frac{n^{k-1}-1}{n-1}=n^{k-2}+n^{k-3}+\cdots+1,$$

and write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\geq nf(s),\quad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u) = mu^2 - u^k, \quad u \in [0, n].$$

We will show that $f(u)$ is convex for $u \in [0, 1]$. Since

$$f''(u) = 2m - k(k-1)u^{k-2} \geq 2m - k(k-1),$$

we need to show that

$$n^{k-2} + n^{k-3} + \cdots + 1 \geq \frac{k(k-1)}{2}.$$

This is true if

$$3^{k-2} + 3^{k-3} + \cdots + 1 \geq \frac{k(k-1)}{2}.$$

By Bernoulli's inequality, we have

$$3^{k-2} + 3^{k-3} + \cdots + 1 \geq [1 + 2(k-2)] + [1 + 2(k-3)] + \cdots + [1 + 2 \cdot 0]$$

$$= (k-1) + (k-2)(k-1) = (k-1)^2 > \frac{k(k-1)}{2}.$$

According to HCF Theorem and Remark 1, we only need to show that $h(x, y) \geq 0$ for $x, y \geq 0$ such that $x + (n-1)y = n$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = \frac{m(u^2 - 1) - (u^k - 1)}{u - 1} = m(u + 1) - (u^{k-1} + u^{k-2} + \cdots + 1),$$

$$h(x, y) = m - \frac{x^{k-1} - y^{k-1}}{x - y} - \frac{x^{k-2} - y^{k-1}}{x - y} - \cdots - 1$$

$$= \left(n^{k-2} - \frac{x^{k-1} - y^{k-1}}{x - y}\right) + \left(n^{k-3} - \frac{x^{k-2} - y^{k-2}}{x - y}\right) + \cdots + \left(n - \frac{x^2 - y^2}{x - y}\right).$$

It suffices to show that

$$n^j \geq \frac{x^{j+1} - y^{j+1}}{x - y}$$

for $x \neq y$ and $j = 1, 2, \cdots, k-2$. We will show that

$$n^j \geq (x + y)^j \geq \frac{x^{j+1} - y^{j+1}}{x - y}.$$

The left inequality is true since

$$n - (x + y) = x + (n-1)y - (x + y) = (n-2)y \geq 0.$$

The right inequality is also true since

$$(x+y)^j = x^j + \binom{j}{1}x^{j-1}y + \cdots + \binom{j}{j-1}xy^{j-1} + y^j$$

and

$$\frac{x^{j+1}-y^{j+1}}{x-y} = x^j + x^{j-1}y + \cdots + xy^{j-1} + y^j.$$

In accordance with Remark 3, the equality holds for $a_1 = n$ and $a_2 = a_3 = \cdots = a_n = 0$ (or any cyclic permutation).

Remark. For $k = 3$ and $k = 4$, we get the following statements (*Vasile Cirtoaje*, 2002):

- *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then*

$$a_1^3 + a_2^3 + \cdots + a_n^3 - n \le (n+1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n),$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = n$ and $a_2 = a_3 = \cdots = a_n = 0$ (or any cyclic permutation).

- *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then*

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \le (n^2+n+1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n),$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = n$ and $a_2 = a_3 = \cdots = a_n = 0$ (or any cyclic permutation).

□

P 1.9. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then*

$$n^2\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n\right) \ge 4(n-1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n).$$

(Vasile Cirtoaje, 2004)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{n^2}{u} - 4(n-1)u^2, \quad u \in (0,n).$$

For $u \in (0,1]$, we have

$$f''(u) = \frac{2n^2}{u^3} - 8(n-1) \ge 2n^2 - 8(n-1) = 2(n-2)^2 \ge 0.$$

Thus, f is convex on $(0,1]$. By HCF Theorem and Remark 1, it suffices to show that $h(x,y) \geq 0$ for $x,y > 0$ such that $x+(n-1)y = n$, where

$$h(x,y) = \frac{g(x)-g(y)}{x-y}, \quad g(u) = \frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u) = \frac{-n^2}{u} - 4(n-1)(u+1)$$

and

$$h(x,y) = \frac{n^2}{xy} - 4(n-1) = \frac{[n-(2n-2)y]^2}{xy} \geq 0.$$

In accordance with Remark 3, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = n/2$ and $a_2 = a_3 = \cdots = a_n = n/(2n-2)$ (or any cyclic permutation).

□

P 1.10. *If $a_1, a_2, \ldots, a_8$ are positive real numbers such that $a_1 + a_2 + \cdots + a_8 = n$, then*

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_8^2} \geq a_1^2 + a_2^2 + \cdots + a_8^2.$$

(Vasile Cirtoaje, 2007)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_8) \geq 8f(s), \quad s = \frac{a_1 + a_2 + \cdots + a_8}{8} = 1,$$

where

$$f(u) = \frac{1}{u^2} - u^2, \quad u \in (0,n).$$

For $u \in (0,1]$, we have

$$f''(u) = \frac{6}{u^4} - 2 \geq 6 - 2 > 0.$$

Thus, f is convex on $(0,1]$. By HCF Theorem and Remark 1, it suffices to show that $h(x,y) \geq 0$ for $x,y > 0$ such that $x + 7y = 8$, where

$$h(x,y) = \frac{g(x)-g(y)}{x-y}, \quad g(u) = \frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u) = -u - 1 - \frac{1}{u} - \frac{1}{u^2}$$

and

$$h(x,y) = -1 + \frac{1}{xy} + \frac{x+y}{x^2y^2}.$$

From $8 = x + 7y \geq 2\sqrt{7xy}$, we get $xy \leq 16/7$. Therefore,

$$h(x,y) \geq -1 + \frac{1}{xy} + \frac{7(x+y)}{16xy} = \frac{112y^2 - 170y + 72}{16xy}$$
$$> \frac{112y^2 - 176y + 72}{16xy} = \frac{14y^2 - 22y + 9}{2xy} > 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_8 = 1$.

Remark. Similarly, we can prove the following generalization.

- *If $a_1, a_2, \ldots, a_n$ ($n \geq 4$) are positive real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then*

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} + 8 - n \geq \frac{8}{n}\left(a_1^2 + a_2^2 + \cdots + a_n^2\right).$$

□

P 1.11. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = n$, then*

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n \geq 2\left(1 + \frac{\sqrt{n-1}}{n}\right)(a_1 + a_2 + \cdots + a_n - n).$$

(Vasile Cirtoaje, 2006)

Solution. Replacing each a_i by $1/a_i$, we need to prove that

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{1}{u^2} - 2\left(1 + \frac{\sqrt{n-1}}{n}\right)\frac{1}{u}, \quad u \in (0,n).$$

For $u \in (0,1]$, we have

$$f''(u) = \frac{6 - 4\left(1 + \frac{\sqrt{n-1}}{n}\right)u}{u^4} \geq \frac{6 - 4\left(1 + \frac{\sqrt{n-1}}{n}\right)}{u^4} = \frac{2(\sqrt{n-1} - 1)^2}{nu^4} \geq 0.$$

Thus, f is convex on $(0,1]$. By HCF Theorem and Remark 1, it suffices to show that $h(x,y) \geq 0$ for $x, y > 0$ such that $x + (n-1)y = n$, where

$$h(x,y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = \frac{-1}{u^2} + \left(1 + \frac{2\sqrt{n-1}}{n}\right)\frac{1}{u}$$

and

$$h(x,y) = \frac{1}{xy}\left(\frac{1}{x} + \frac{1}{y} - 1 - \frac{2\sqrt{n-1}}{n}\right).$$

We only need to show that

$$\frac{1}{x} + \frac{1}{y} \geq 1 + \frac{2\sqrt{n-1}}{n}.$$

Indeed, using the Cauchy-Schwarz inequality, we get

$$\frac{1}{x} + \frac{1}{y} \geq \frac{(1+\sqrt{n-1})^2}{x+(n-1)y} = \frac{(1+\sqrt{n-1})^2}{n} = 1 + \frac{2\sqrt{n-1}}{n}.$$

In accordance with Remark 3, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = \dfrac{1+\sqrt{n-1}}{n}$ and $a_2 = a_3 = \cdots = a_n = \dfrac{n-1+\sqrt{n-1}}{n}$ (or any cyclic permutation).

□

P 1.12. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{3a+b+c} + \frac{1}{3b+c+a} + \frac{1}{3c+a+b} \leq \frac{2}{5}\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right).$$

(Vasile Cirtoaje, 2006)

Solution. Due to homogeneity, we may assume that $a+b+c=3$. So, we need to show that

$$f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a+b+c}{3} = 1,$$

where

$$f(u) = \frac{2}{3-u} - \frac{5}{2u+3}, \quad u \in [0,3).$$

For $u \in [1,3)$, we have

$$f''(u) = \frac{4}{(3-u)^3} - \frac{40}{(2u+3)^3} = \frac{36[2u^3+3u^2+9(u-1)(3-u)]}{(3-u)^3(2u+3)^3} > 0;$$

therefore, f is convex on $[1,3)$. By HCF Theorem and Remark 1, it suffices to show that $h(x,y) \geq 0$ for $x, y \geq 0$ such that $x + 2y = 3$, where

$$h(x,y) = \frac{g(x)-g(y)}{x-y}, \quad g(u) = \frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u) = \frac{1}{3-u} + \frac{2}{2u+3}$$

and

$$h(x,y) = \frac{1}{(3-x)(3-y)} - \frac{2}{(2x+3)(2y+3)} = \frac{9(2x+2y-3)}{(3-x)(3-y)(2x+3)(2y+3)}$$
$$= \frac{9x}{(3-x)(3-y)(2x+3)(2y+3)} \ge 0.$$

The equality holds for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 1.13. *If $a, b, c, d \ge 3 - \sqrt{7}$ such that $a + b + c + d = 4$, then*

$$\frac{1}{2+a^2} + \frac{1}{2+b^2} + \frac{1}{2+c^2} + \frac{1}{2+d^2} \ge \frac{4}{3}.$$

(Vasile Cirtoaje, 2008)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \ge 4f(s), \quad s = \frac{a+b+c+d}{4} = 1,$$

where

$$f(u) = \frac{1}{2+u^2}, \quad u \ge 3 - \sqrt{7}.$$

For $u \ge 1$, we have

$$f''(u) = \frac{3(3u^2-2)}{(2+u^2)^3} > 0.$$

Thus, f is convex for $u \ge 1$. By HCF Theorem and Remark 1, it suffices to show that $h(x,y) \ge 0$ for $x, y \ge 0$ such that $x + 3y = 4$. We have

$$g(u) = \frac{f(u) - f(1)}{u-1} = \frac{-1-u}{3(2+u^2)}$$

and

$$h(x,y) = \frac{g(x) - g(y)}{x-y} = \frac{xy + x + y - 2}{3(2+x^2)(2+y^2)} \ge 0$$

since

$$xy + x + y - 2 = \frac{-x^2 + 6x - 2}{3} = \frac{(3+\sqrt{7}-x)(x-3+\sqrt{7})}{3} \ge 0.$$

In accordance with Remark 3, the equality holds for $a = b = c = d = 1$, and also for $a = 3-\sqrt{7}$ and $b = c = d = \dfrac{1+\sqrt{7}}{3}$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization.

- *If $a_1, a_2, \ldots, a_n \ge n-1-\sqrt{n^2-3n+3}$ such that $a_1 + a_2 + \cdots + a_n = n$, then*

$$\frac{1}{2+a_1^2} + \frac{1}{2+a_2^2} + \cdots + \frac{1}{2+a_n^2} \ge \frac{n}{3},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = n-1-\sqrt{n^2-3n+3}$ and $a_2 = a_3 = \cdots = a_n = \dfrac{1+\sqrt{n^2-3n+3}}{n-1}$ (or any cyclic permutation).

□

P 1.14. *If $a_1, a_2, \ldots, a_n \in [0, n-2]$ such that $a_1 + a_2 + \cdots + a_n = n$, then*

$$\frac{1}{n+a_1^2} + \frac{1}{n+a_2^2} + \cdots + \frac{1}{n+a_n^2} \le \frac{n}{n+1}.$$

(Vasile Cirtoaje, 2008)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{-1}{n+u^2}, \quad u \in [0, n-2], \quad n \ge 3.$$

For $u \in [0,1]$, we have

$$f''(u) = \frac{2(n-3u^2)}{(n+u^2)^3} \ge 0.$$

Thus, f is convex on $[0,1]$. By HCF Theorem and Remark 1, it suffices to show that $h(x,y) \ge 0$ for $x, y \ge 0$ such that $x + (n-1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u-1} = \frac{u+1}{(n+1)(n+u^2)}$$

and

$$h(x,y) = \frac{g(x) - g(y)}{x-y} = \frac{n-x-y-xy}{(n+1)(n+x^2)(n+y^2)}.$$

We need to show that

$$n - x - y - xy \ge 0.$$

Indeed, we have

$$n-x-y-xy=\frac{(n-x)(n-2-x)}{n-1}\geq 0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$, and also for $a_1=n-2$ and $a_2=a_3=\cdots=a_n=\dfrac{2}{n-1}$ (or any cyclic permutation).

□

P 1.15. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$\frac{3-a}{9+a^2}+\frac{3-b}{9+b^2}+\frac{3-c}{9+c^2}\geq\frac{3}{5}.$$

(Vasile Cirtoaje, 2013)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)\geq 3f(s),\quad s=\frac{a+b+c}{3}=1,$$

where

$$f(u)=\frac{3-u}{9+u^2},\quad u\in[0,3].$$

For $u\in[1,3]$, we have

$$\frac{1}{2}f''(u)=\frac{u^2(9-u)+27(u-1)}{(9+u^2)^3}>0.$$

Thus, f is convex on $[1,3]$. By HCF Theorem and Remark 1, it suffices to show that $h(x,y)\geq 0$ for $x,y\geq 0$ such that $x+2y=3$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y},\quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u)=\frac{-(6+u)}{5(9+u^2)}$$

and

$$h(x,y)=\frac{xy+6x+6y-9}{5(9+x^2)(9+y^2)}=\frac{x(9-x)}{10(9+x^2)(9+y^2)}\geq 0.$$

The equality holds for $a=b=c=1$, and also for $a=0$ and $b=c=3/2$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization.

• *If $a_1, a_2, \dots, a_n$ $(n \ge 3)$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_n = n$, then*

$$\frac{n-a_1}{n^2+(n^2-3n+1)a_1^2}+\frac{n-a_2}{n^2+(n^2-3n+1)a_2^2}+\cdots+\frac{n-a_n}{n^2+(n^2-3n+1)a_n^2} \ge \frac{n}{2n-1},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = 0$ and $a_2 = a_3 = \cdots = a_n = \dfrac{n}{n-1}$ (or any cyclic permutation).

□

P 1.16. *If a, b, c are nonnegative real numbers such that $a+b+c = 3$, then*

$$\frac{1}{5+a+a^2}+\frac{1}{5+b+b^2}+\frac{1}{5+c+c^2} \ge \frac{3}{7}.$$

(Vasile Cirtoaje, 2008)

Solution. Write the inequality as

$$f(a)+f(b)+f(c) \ge 3f(s), \quad s = \frac{a+b+c}{3} = 1,$$

where

$$f(u) = \frac{1}{5+u+u^2}, \quad u \in [0,3].$$

For $u \ge 1$, we have

$$f''(u) = \frac{2(3u^2+3u-4)}{(5+u+u^2)^3} > 0.$$

Thus, f is convex on [1,3]. By HCF Theorem and Remark 1, it suffices to show that $h(x,y) \ge 0$ for $x,y \ge 0$ such that $x+2y = 3$. We have

$$g(u) = \frac{f(u)-f(1)}{u-1} = \frac{-2-u}{7(5+u+u^2)}$$

and

$$h(x,y) = \frac{g(x)-g(y)}{x-y} = \frac{xy+2(x+y)-3}{7(5+x+x^2)(5+y+y^2)} = \frac{x(5-x)}{14(5+x+x^2)(5+y+y^2)} \ge 0.$$

In accordance with Remark 3, the equality holds for $a = b = c = 1$, and also for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization.

- *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $0 < k \le \frac{2(2n-1)}{n-1}$, then*

$$\frac{1}{k + a_1 + a_1^2} + \frac{1}{k + a_2 + a_2^2} + \cdots + \frac{1}{k + a_n + a_n^2} \ge \frac{n}{k+2},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$. If $k = \frac{2(2n-1)}{n-1}$, then the equality holds also for $a_1 = 0$ and $a_2 = a_3 = \cdots = a_n = \frac{n}{n-1}$ (or any cyclic permutation).

□

P 1.17. *If a, b, c, d are nonnegative real numbers such that $a + b + c + d = 4$, then*

$$\frac{1}{10 + a + a^2} + \frac{1}{10 + b + b^2} + \frac{1}{10 + c + c^2} + \frac{1}{10 + d + d^2} \le \frac{1}{3}.$$

(Vasile Cirtoaje, 2008)

Solution. Write the inequality as

$$f(a) + f(b) + f(c) + f(d) \ge 4f(s), \quad s = \frac{a+b+c+d}{4} = 1,$$

where

$$f(u) = \frac{-1}{10 + u + u^2}, \quad u \in [0, 4].$$

For $u \in [0, 1]$, we have

$$f''(u) = \frac{6(3 - u - u^2)}{(10 + u + u^2)^3} > 0.$$

Thus, f is convex on [0,1]. By HCF Theorem and Remark 1, it suffices to show that $h(x, y) \ge 0$ for $x, y \ge 0$ such that $x + 3y = 4$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{2 + u}{12(10 + u + u^2)}$$

and

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{8 - 2(x + y) - xy}{12(10 + x + x^2)(10 + y + y^2)}.$$

We need to show that

$$8 - 2(x + y) - xy \ge 0.$$

Indeed, we have

$$8 - 2(x + y) - xy = 3y^2 \ge 0.$$

The equality holds for $a = b = c = d = 1$, and also for $a = 4$ and $b = c = d = 0$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization.

- *Let $a_1, a_2, \ldots, a_n$ ($n \ge 4$) be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $k \ge 2n + 2$, then*

$$\frac{1}{k + a_1 + a_1^2} + \frac{1}{k + a_2 + a_2^2} + \cdots + \frac{1}{k + a_n + a_n^2} \le \frac{n}{k+2},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$. If $k = 2n + 2$, then the equality holds also for $a_1 = n$ and $a_2 = a_3 = \cdots = a_n = 0$ (or any cyclic permutation).

□

P 1.18. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $k \ge 1 - \frac{1}{n}$, then*

$$\frac{1}{1 + ka_1^2} + \frac{1}{1 + ka_2^2} + \cdots + \frac{1}{1 + ka_n^2} \ge \frac{n}{1+k}.$$

(Vasile Cirtoaje, 2005)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{1}{1 + ku^2}, \quad u \in [0, n].$$

For $u \in [1, n]$, we have

$$f''(u) = \frac{2k(3ku^2 - 1)}{(1 + ku^2)^3} > 0,$$

since

$$3ku^2 - 1 \ge 3k - 1 \ge 3\left(1 - \frac{1}{n}\right) - 1 = 2 - \frac{3}{n} > 0.$$

Thus, f is convex on $[1, n]$. By HCF Theorem and Remark 1, it suffices to show that $h(x, y) \ge 0$ for $x, y \ge 0$ such that $x + (n-1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-k(u+1)}{(1+k)(1+ku^2)}$$

and

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{k^2(x + y + xy) - k}{(1+k)(1+kx^2)(1+ky^2)}.$$

We need to show that

$$k(x+y+xy)-1 \geq 0.$$

Indeed, we have

$$k(x+y+xy)-1 \geq \left(1-\frac{1}{n}\right)(x+y+xy)-1 = \frac{x(2n-2-x)}{n} \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = 1-\frac{1}{n}$, then the equality holds also for $a_1 = 0$ and $a_2 = a_3 = \cdots = a_n = \frac{n}{n-1}$ (or any cyclic permutation).

□

P 1.19. *Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $0 < k \leq \frac{n-1}{n^2-n+1}$, then*

$$\frac{1}{1+ka_1^2} + \frac{1}{1+ka_2^2} + \cdots + \frac{1}{1+ka_n^2} \leq \frac{n}{1+k}.$$

(Vasile Cirtoaje, 2005)

Solution. Replacing all negative numbers a_i by $-a_i$, we need to show the same inequality for $a_1, a_2, \ldots, a_n \geq 0$ such that $a_1 + a_2 + \cdots + a_n \geq n$. Since the left side of the desired inequality is decreasing with respect to each a_i, is sufficient to consider that $a_1 + a_2 + \cdots + a_n = n$. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{-1}{1+ku^2}, \quad u \in [0, n].$$

For $u \in [0, 1]$, we have

$$f''(u) = \frac{2k(1-3ku^2)}{(1+ku^2)^3} \geq 0,$$

since

$$1-3ku^2 \geq 1-3k \geq 1-\frac{3(n-1)}{n^2-n+1} = \frac{(n-2)^2}{n^2-n+1} \geq 0.$$

Thus, f is convex on $[0, 1]$. By HCF Theorem and Remark 1, it suffices to show that $h(x, y) \geq 0$ for $x, y \geq 0$ such that $x + (n-1)y = n$. We have

$$g(u) = \frac{f(u)-f(1)}{u-1} = \frac{k(u+1)}{(1+k)(1+ku^2)}$$

and

$$h(x,y) = \frac{g(x)-g(y)}{x-y} = \frac{k-k^2(x+y+xy)}{(1+k)(1+kx^2)(1+ky^2)}.$$

We need to show that

$$1-k(x+y+xy) \geq 0.$$

Indeed, we have

$$1-k(x+y+xy) \geq 1-\frac{n-1}{n^2-n+1}(x+y+xy) = \frac{(x-n+1)^2}{n^2-n+1} \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = \dfrac{n-1}{n^2-n+1}$, then the equality holds also for $a_1 = n-1$ and $a_2 = a_3 = \cdots = a_n = \dfrac{1}{n-1}$ (or any cyclic permutation).

□

P 1.20. *Let $a_1, a_2, \ldots, a_n$ be nonnegative numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $k \geq \dfrac{n^2}{4(n-1)}$, then*

$$\frac{a_1(a_1-1)}{a_1^2+k} + \frac{a_2(a_2-1)}{a_2^2+k} + \cdots + \frac{a_n(a_n-1)}{a_n^2+k} \geq 0.$$

(Vasile Cîrtoaje, 2012)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n) \geq nf(s), \quad s = \frac{a_1+a_2+\cdots+a_n}{n} = 1,$$

where

$$f(u) = \frac{u(u-1)}{u^2+k}, \quad u \in [0,n].$$

From

$$f'(u) = \frac{u^2+2ku-k}{(u^2+k)^2}, \quad f''(u) = \frac{2(k^2-u^3)+6ku(1-u)}{(u^2+k)^3},$$

it follow that f is convex on $[0,1]$. By HCF Theorem and Remark 1, it suffices to show that $h(x,y) \geq 0$ for $x,y \geq 0$ such that $x+(n-1)y = n$, where

$$h(x,y) = \frac{g(x)-g(y)}{x-y}, \quad g(u) = \frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u) = \frac{u}{u^2+k}, \quad h(x,y) = \frac{k-xy}{(x^2+k)(y^2+k)} \geq \frac{n^2-4(n-1)xy}{4(n-1)(x^2+k)(y^2+k)}.$$

We only need to show that $n^2 \geq 4(n-1)xy$. Indeed, this follows by the AM-GM inequality, as follows:

$$n^2 = [x+(n-1)y]^2 \geq \left[2\sqrt{(n-1)xy}\,\right]^2 = 4(n-1)xy.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = n/2$ and $a_2 = a_3 = \cdots = a_n = n/(2n-2)$ (or any cyclic permutation).

□

P 1.21. *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then*

$$\frac{1-a_1}{(n-2a_1)^2} + \frac{1-a_2}{(n-2a_2)^2} + \cdots + \frac{1-a_n}{(n-2a_n)^2} \leq 0.$$

(Vasile Cîrtoaje, 2012)

Solution. For $n = 2$, the inequality is an identity. Consider further $n \geq 3$ and write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{u-1}{(n-2u)^2}, \quad u \in \mathbb{I} = [0,n] \setminus \{n/2\}.$$

From

$$f'(u) = \frac{2u+n-4}{(n-2u)^3}, \quad f''(u) = \frac{8(u+n-3)}{(n-2u)^4},$$

it follow that f is convex on $[0,1]$. By HCF Theorem, Remark 1 and Remark 4, it suffices to show that $h(x,y) \geq 0$ for $x, y \in \mathbb{I}$ such that $x + (n-1)y = n$, where

$$h(x,y) = \frac{g(x)-g(y)}{x-y}, \quad g(u) = \frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u) = \frac{1}{(n-2u)^2}$$

and

$$h(x,y) = \frac{4(n-x-y)}{(n-2x)^2(n-2y)^2} = \frac{4(n-2)y}{(n-2x)^2(n-2y)^2} \geq 0.$$

In accordance with Remark 3, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = n$ and $a_2 = a_3 = \cdots = a_n = 0$ (or any cyclic permutation).

□

P 1.22. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $k \ge 1 + \dfrac{n}{\sqrt{n-1}}$, then*

$$\frac{a_1^2 - 1}{(a_1 + k)^2} + \frac{a_2^2 - 1}{(a_2 + k)^2} + \cdots + \frac{a_n^2 - 1}{(a_n + k)^2} \ge 0.$$

(Vasile Cirtoaje, 2008)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{u^2 - 1}{(u + k)^2}, \quad u \in [0, n].$$

For $u \in [0, 1]$, we have

$$f''(u) = \frac{2(k^2 - 3 - 2ku)}{(u + k)^4} \ge \frac{2(k^2 - 2k - 3)}{(u + k)^4} = \frac{2(k + 1)(k - 3)}{(u + k)^4} \ge 0.$$

Thus, f is convex on $[0, 1]$. By HCF Theorem and Remark 1, it suffices to show that $h(x, y) \ge 0$ for $x, y \ge 0$ such that $x + (n - 1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{u + 1}{(u + k)^2}$$

and

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{(k - 1)^2 - (1 + x)(1 + y)}{(x + k)^2 (y + k)^2}.$$

Since

$$(k - 1)^2 \ge \frac{n^2}{n - 1},$$

we need to show that

$$n^2 \ge (n - 1)(1 + x)(1 + y).$$

Indeed,

$$n^2 - (n - 1)(1 + x)(1 + y) = n^2 - (1 + x)(2n - 1 - x) = (x - n + 1)^2 \ge 0.$$

Thus, the proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = 1 + \dfrac{n}{\sqrt{n-1}}$, then the equality holds also for $a_1 = n - 1$ and $a_2 = a_3 = \cdots = a_n = \dfrac{1}{n-1}$ (or any cyclic permutation).

$\square$

P 1.23. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $0 < k \le 1 + \sqrt{\frac{2n-1}{n-1}}$, then*

$$\frac{a_1^2 - 1}{(a_1 + k)^2} + \frac{a_2^2 - 1}{(a_2 + k)^2} + \cdots + \frac{a_n^2 - 1}{(a_n + k)^2} \le 0.$$

(Vasile Cirtoaje, 2008)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{1 - u^2}{(u + k)^2}, \quad u \in [0, n].$$

For $u \ge 1$, we have

$$f''(u) = \frac{2(2ku - k^2 + 3)}{(u + k)^4} \ge \frac{2(2k - k^2 + 3)}{(u + k)^4} = \frac{2(1 + k)(3 - k)}{(u + k)^4} > 0.$$

Thus, f is convex on $[1, n]$. By HCF Theorem and Remark 1, it suffices to show that $h(x, y) \ge 0$ for $x, y \ge 0$ such that $x + (n-1)y = n$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = \frac{-u - 1}{(u + k)^2}$$

and

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = \frac{2k - k^2 + x + y + xy}{(x + k)^2(y + k)^2} \ge \frac{2k - k^2 + x + y}{(x + k)^2(y + k)^2}.$$

Since

$$x + y \ge \frac{x + (n-1)y}{n - 1} = \frac{n}{n - 1},$$

we get

$$2k - k^2 + x + y \ge 2k - k^2 + \frac{n}{n - 1} = -(k - 1)^2 + \frac{2n - 1}{n - 1} \ge 0,$$

hence $h(x, y) \ge 0$. Thus, the proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = 1 + \sqrt{\frac{2n-1}{n-1}}$, then the equality holds also for $a_1 = 0$ and $a_2 = a_3 = \cdots = a_n = \frac{n}{n-1}$ (or any cyclic permutation).

□

P 1.24. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $k \ge \frac{(n-1)(2n-1)}{n^2}$, then*

$$\frac{1}{1+ka_1^3} + \frac{1}{1+ka_2^3} + \cdots + \frac{1}{1+ka_n^3} \ge \frac{n}{1+k}.$$

(Vasile Cirtoaje, 2008)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{1}{1+ku^3}, \quad u \in [0, n].$$

For $u \in [1, n]$, we have

$$f''(u) = \frac{6ku(2ku^3 - 1)}{(1+ku^3)^3} \ge \frac{6ku(2k-1)}{(1+ku^3)^3} > 0.$$

Thus, f is convex on $[1, n]$. By HCF Theorem and Remark 1, it suffices to show that $h(x, y) \ge 0$ for $x, y \ge 0$ such that $x + (n-1)y = n$, where

$$h(x,y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = \frac{-k(u^2 + u + 1)}{(1+k)(1+ku^3)}$$

and

$$\frac{1}{k^2} h(x,y) = \frac{x^2y^2 + xy(x+y-1) + (x+y)^2 - (x+y+1)/k}{(1+k)(1+kx^3)(1+ky^3)}.$$

Since

$$x + y \ge \frac{x + (n-1)y}{n-1} = \frac{n}{n-1} > 1,$$

it suffices to show that $k(x+y)^2 \ge x + y + 1$. From $x + y \ge \frac{n}{n-1}$, we get

$$k(x+y) \ge \frac{2n-1}{n}$$

and

$$k(x+y)^2 - x - y \ge (x+y)[k(x+y) - 1] \ge \frac{n}{n-1}\left[\frac{2n-1}{n} - 1\right] = 1.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = \frac{(n-1)(2n-1)}{n^2}$, then the equality holds also for $a_1 = 0$ and $a_2 = a_3 = \cdots = a_n = \frac{n}{n-1}$ (or any cyclic permutation).

□

P 1.25. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $0 < k \le \dfrac{n-1}{n^2-2n+2}$, then*

$$\frac{1}{1+ka_1^3} + \frac{1}{1+ka_2^3} + \cdots + \frac{1}{1+ka_n^3} \le \frac{n}{1+k}.$$

(Vasile Cirtoaje, 2008)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{-1}{1+ku^3}, \quad u \in [0, n].$$

For $u \in [0,1]$, we have

$$f''(u) = \frac{6ku(1-2ku^3)}{(1+ku^3)^3} \ge \frac{6ku(1-2k)}{(1+ku^3)^3} > 0.$$

Thus, f is convex on $[0,1]$. By HCF Theorem and Remark 1, it suffices to show that $h(x,y) \ge 0$ for $x, y \ge 0$ such that $x + (n-1)y = n$, where

$$h(x,y) = \frac{g(x) - g(y)}{x-y}, \quad g(u) = \frac{f(u) - f(1)}{u-1}.$$

We have

$$g(u) = \frac{k(u^2+u+1)}{(1+k)(1+ku^3)}$$

and

$$\frac{1}{k^2}h(x,y) = \frac{(x+y+1)/k - x^2y^2 - xy(x+y-1) - (x+y)^2}{(1+k)(1+kx^3)(1+ky^3)}.$$

It suffices to show that

$$\frac{(n^2-2n+2)(x+y+1)}{n-1} - x^2y^2 - xy(x+y-1) - (x+y)^2 \ge 0,$$

which is equivalent to

$$[2 + ny - (n-1)y^2][1-(n-1)y]^2 \ge 0.$$

This is true since

$$2 + ny - (n-1)y^2 = 2 + y[n - (n-1)y] = 2 + xy > 0.$$

In accord with Remark 3, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = \dfrac{n-1}{n^2-2n+2}$, then the equality holds also for $a_1 = n-1$ and $a_2 = a_3 = \cdots = a_n = \dfrac{1}{n-1}$ (or any cyclic permutation).

□

P 1.26. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $k \ge \frac{n^2}{n-1}$, then*

$$\sqrt{\frac{a_1}{k-a_1}} + \sqrt{\frac{a_2}{k-a_2}} + \cdots + \sqrt{\frac{a_n}{k-a_n}} \le \frac{n}{\sqrt{k-1}}.$$

(Vasile Cirtoaje, 2008)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = -\sqrt{\frac{u}{k-u}}, \quad u \in [0, n].$$

For $u \in [0,1]$, we have

$$f''(u) = \frac{k(k-4u)}{4u^{3/2}(k-u)^{5/2}} \ge \frac{k(k-4)}{4u^{3/2}(k-u)^{5/2}} \ge 0.$$

Thus, f is convex on $[0,1]$. By HCF Theorem, it suffices to prove that $f(x) + (n-1)f(y) \ge nf(1)$ for all $x, y \ge 0$ such that $x + (n-1)y = n$. We write the inequality as

$$\sqrt{\frac{(k-1)x}{k-x}} + (n-1)\sqrt{\frac{(k-1)y}{k-y}} \le n,$$

$$\sqrt{1 + \frac{(n-1)k(1-y)}{(n-1)y + k - n}} \le 1 + (n-1)\left[1 - \sqrt{\frac{(k-1)y}{k-y}}\,\right].$$

Let

$$z = \sqrt{\frac{(k-1)y}{k-y}},$$

which yields

$$y = \frac{kz^2}{z^2 + k - 1},$$

$$1 - y = \frac{(k-1)(1-z^2)}{z^2 + k - 1}, \quad (n-1)y + k - n = \frac{(k-1)(nz^2 + k - n)}{z^2 + k - 1},$$

hence

$$\frac{k(1-y)}{(n-1)y + k - n} = \frac{k(1-z^2)}{k - n(1-z^2)} = \frac{1}{1/(1-z^2) - n/k}$$

$$\le \frac{1}{1/(1-z^2) - (n-1)/n} = \frac{n(1-z^2)}{(n-1)z^2 + 1}.$$

Therefore, it suffices to show that

$$\sqrt{1+\frac{n(n-1)(1-z^2)}{(n-1)z^2+1}} \le 1+(n-1)(1-z).$$

By squaring, we get the obvious inequality

$$(z-1)^2[(n-1)z-1]^2 \ge 0.$$

So, we only need to show that $1+(n-1)(1-z) \ge 0$, that is, $z \le n/(n-1)$. Since

$$y = \frac{n-x}{n-1} \le \frac{n}{n-1}$$

and

$$\frac{1}{k-1} \le \frac{1}{n^2/(n-1)-1} = \frac{n-1}{n^2-n+1},$$

we have

$$z = \sqrt{\frac{k-1}{k/y-1}} \le \sqrt{\frac{k-1}{k(n-1)/n-1}} = \sqrt{\frac{n}{n-1-1/(k-1)}}$$

$$\le \sqrt{\frac{n}{n-1-(n-1)/(n^2-n+1)}} = \frac{\sqrt{n^2-n+1}}{n-1} < \frac{n}{n-1}.$$

The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = \frac{n^2}{n-1}$, then the equality holds also for $a_1 = \frac{n(n-1)^2}{n^2-2n+2}$ and $a_2 = a_3 = \cdots = a_n = \frac{n}{(n-1)(n^2-2n+2)}$ (or any cyclic permutation).

□

P 1.27. *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then*

$$n^{-a_1^2} + n^{-a_2^2} + \cdots + n^{-a_n^2} \ge 1.$$

(Vasile Cîrtoaje, 2006)

Solution. Let $k = \ln n$. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n) \ge nf(s), \quad s = \frac{a_1+a_2+\cdots+a_n}{n} = 1,$$

where

$$f(u) = n^{-u^2}, \quad u \in [0,n].$$

For $u \ge 1$, we have

$$f''(u) = 2kn^{-u^2}(2ku^2-1) \ge 2kn^{-u^2}(2k-1) \ge 2kn^{-u^2}(2\ln 2-1) > 0;$$

therefore, f is convex on $[1,n]$. By HCF Theorem and Remark 5, it suffices to show that $f(x)+(n-1)f(y)\ge nf(1)$ for $0\le x\le 1\le y$ and $x+(n-1)y=n$. The desired inequality is equivalent to $g(x)\ge 0$, where

$$g(x)=n^{-x^2}+(n-1)n^{-y^2},\quad y=\frac{n-x}{n-1},\ \ 0\le x\le 1.$$

Since $y'=-1/(n-1)$, we get

$$g'(x)=-2xkn^{-x^2}-2(n-1)kyy'n^{-y^2}=2k(yn^{-y^2}-xn^{-x^2}).$$

The derivative $g'(x)$ has the same sign as $g_1(x)$, where

$$g_1(x)=\ln(yn^{-y^2})-\ln(xn^{-x^2})=\ln y-\ln x+k(x^2-y^2).$$

From

$$g_1'(x)=\frac{y'}{y}-\frac{1}{x}+2k(x-yy')=n\left[\frac{-1}{x(n-x)}+\frac{2k+2(n-2)kx}{(n-1)^2}\right],$$

we see that $g_1'(x)$ has for $0<x\le 1$ the same sign as

$$h(x)=\frac{-(n-1)^2}{2k}+x(n-x)(1+nx-2x).$$

Since

$$\begin{aligned}h'(x)&=n+2(n^2-2n-1)x-3(n-2)x^2\ge nx+2(n^2-2n-1)x-3(n-2)x\\&=2(n-1)(n-2)x\ge 0,\end{aligned}$$

h is strictly increasing on $[0,1]$. From

$$h(0)=\frac{-(n-1)^2}{2k}<0,\quad h(1)=(n-1)^2\left(1-\frac{1}{2k}\right)>0,$$

it follows that there is $x_1\in(0,1)$ such that $h(x)<0$ for $x\in[0,x_1)$, $h(x_1)=0$ and $h(x)>0$ for $x\in(x_1,1]$. Therefore, g_1 is strictly decreasing on $(0,x_1]$ and strictly increasing on $[x_1,1]$. Since $\lim_{x\to 0}g_1(x)=\infty$ and $g_1(1)=0$, there is $x_2\in(0,x_1)$ such that $g_1(x)>0$ for $x\in(0,x_2)$, $g_1(x_2)=0$ and $g_1(x)<0$ for $x\in(x_2,1)$. Consequently, g is strictly increasing on $[0,x_2]$ and strictly decreasing on $[x_2,1]$. From $g(0)>0$ and $g(1)=0$, it follows that $g(x)\ge 0$ for $x\in[0,1]$. The proof is completed. The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 1.28. *If a,b,c,d,e are nonnegative real numbers such that $a+b+c+d+e=5$, then then*

$$(a^2+1)(b^2+1)(c^2+1)(d^2+1)(e^2+1)\ge(a+1)(b+1)(c+1)(d+1)(e+1).$$

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)+f(e) \ge nf(s), \quad s=\frac{a+b+c+d+e}{5}=1,$$

where

$$f(u)=\ln(u^2+1)-\ln(u+1), \quad u \in [0,5].$$

For $u \in [0,1]$, we have

$$f''(u)=\frac{2(1-u^2)}{(u^2+1)^2}+\frac{1}{(u+1)^2}=\frac{(u^2-u^4)+4u(1-u^2)+u^2+3}{(u^2+1)^2(u+1)^2}>0.$$

Therefore, f is convex on $[0,1]$. By HCF Theorem and Remark 2, we only need to show that $H(x,y) \ge 0$ for $x,y \ge 0$ such that $x+4y=5$, where

$$H(x,y)=\frac{f'(x)-f'(y)}{x-y}=\frac{2(1-xy)}{(x^2+1)(y^2+1)}+\frac{1}{(x+1)(y+1)};$$

that is,

$$(x^2+1)(y^2+1)H(x,y)=2(1-xy)+\frac{(x^2+1)(y^2+1)}{(x+1)(y+1)}.$$

Since

$$\frac{x^2+1}{x+1} \ge \frac{x+1}{2}, \quad \frac{y^2+1}{y+1} \ge \frac{y+1}{2},$$

it suffices to prove that

$$2(1-xy)+\frac{(x+1)(y+1)}{4} \ge 0,$$

which is equivalent to $x+y+9-7xy \ge 0$. Indeed,

$$x+y+9-7xy=28x^2-38x+14=28(x-19/28)^2+31/28>0.$$

The proof is completed. The equality holds for $a=b=c=d=e=1$.

□

P 1.29. *If $a_1,a_2,\cdots,a_{10}$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_{10}=10$, then*

$$(1-a_1+a_1^2)(1-a_2+a_2^2)\cdots(1-a_{10}+a_{10}^2) \ge 1.$$

(Vasile Cîrtoaje, 2006)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_{10}) \ge 10f(s), \quad s=\frac{a_1+a_2+\cdots+a_{10}}{10}=1,$$

where

$$f(u)=\ln(1-u+u^2), \quad u\in[0,10].$$

From

$$f''(u)=\frac{1+2u(1-u)}{(1-u+u^2)^2},$$

it follows that $f''(u)>0$ for $u\in[0,1]$, hence f is convex on $[0,1]$. According to HCF Theorem, we only need to show that $f(x)+9f(y)\ge 10f(1)$ for all $x,y\ge 0$ such that $x+9y=10$. Using Remark 2, it suffices to prove that $H(x,y)\ge 0$, where

$$H(x,y)=\frac{f'(x)-f'(y)}{x-y}.$$

Since

$$H(x,y)=\frac{1+x+y-2xy}{(1-x+x^2)(1-y+y^2)},$$

we need to show that

$$1+x+y-2xy\ge 0.$$

Indeed,

$$1+x+y-2xy=18y^2-28y+11=18\left(y-\frac{7}{9}\right)^2+\frac{1}{9}>0.$$

The proof is completed. The equality holds for $a_1=a_2=\cdots=a_{10}=1$.

□

P 1.30. *If $a_1,a_2,\ldots,a_n$ $(n\ge 3)$ are positive real numbers such that $a_1+a_2+\cdots+a_n=1$, then*

$$\left(\frac{1}{\sqrt{a_1}}-\sqrt{a_1}\right)\left(\frac{1}{\sqrt{a_2}}-\sqrt{a_2}\right)\cdots\left(\frac{1}{\sqrt{a_n}}-\sqrt{a_n}\right)\ge\left(\sqrt{n}-\frac{1}{\sqrt{n}}\right)^n.$$

(Vasile Cîrtoaje, 2006)

Solution. Apply HCF Theorem to the function

$$f(u)=\ln\left(\frac{1}{\sqrt{u}}-\sqrt{u}\right)=\ln(1-u)-\frac{1}{2}\ln u, \quad u\in(0,1),$$

for $s=1/n$. From

$$f'(u)=\frac{-1}{1-u}-\frac{1}{2u}, \quad f''(u)=\frac{1-2u-u^2}{2u^2(1-u)^2},$$

it follows that f is convex on $(0, \sqrt{2}-1]$. Since

$$s = \frac{1}{n} \le \frac{1}{3} < \sqrt{2}-1,$$

f is also convex on $(0, s]$.

First Solution. By HCF Theorem, it suffices to show that $f(x)+(n-1)f(y) \ge nf(1/n)$ for all $x, y > 0$ such that $x+(n-1)y = 1$; that is, to show that

$$\left(\frac{1}{\sqrt{x}} - \sqrt{x}\right)\left(\frac{1}{\sqrt{y}} - \sqrt{y}\right)^{n-1} \ge \left(\sqrt{n} - \frac{1}{\sqrt{n}}\right)^{n}.$$

Write this inequality as

$$n^{n/2}(1-y)^{n-1} \ge (n-1)^{n-1}x^{1/2}y^{(n-3)/2}.$$

By squaring, this inequality becomes as follows

$$n^{n}(1-y)^{2n-2} \ge (n-1)^{2n-2}xy^{n-3},$$

$$(2-2y)^{2n-2} \ge \frac{(2n-2)^{2n-2}}{n^{n}}xy^{n-3},$$

$$\left[n\cdot\frac{1}{n} + x + (n-3)y\right]^{2n-2} \ge [n+1+(n-3)]^{n+1+(n-3)}\cdot\frac{1}{n^{n}}\cdot x\cdot y^{n-3}.$$

Clearly, the last inequality follows from the AM-GM inequality. The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1/n$.

Second Solution. By HCF Theorem and Remark 2, it suffices to prove that $H(x,y) \ge 0$ for $x, y > 0$ such that $x+(n-1)y = 1$, where

$$H(x,y) = \frac{f'(x)-f'(y)}{x-y}.$$

We have

$$H(x,y) = \frac{1-x-y-xy}{2xy(1-x)(1-y)} = \frac{(n-1)(1+y)-2}{2x(1-x)(1-y)}$$
$$\ge \frac{2(1+y)-2}{2x(1-x)(1-y)} = \frac{y}{x(1-x)(1-y)} > 0.$$

Remark 1. We may write the inequality in P 1.30 in the form

$$\prod_{i=1}^{n}\left(\frac{1}{\sqrt{a_i}} - 1\right)\cdot\prod_{i=1}^{n}(1+\sqrt{a_i}) \ge \left(\sqrt{n} - \frac{1}{\sqrt{n}}\right)^{n}.$$

On the other hand, by the AM-GM inequality and the Cauchy-Schwarz inequality, we have

$$\prod_{i=1}^{n}(1+\sqrt{a_i}) \le \left(1+\frac{1}{n}\sum_{i=1}^{n}\sqrt{a_i}\right)^n \le \left(1+\sqrt{\frac{1}{n}\sum_{i=1}^{n}a_i}\right)^n = \left(1+\frac{1}{\sqrt{n}}\right)^n.$$

Thus, the following statement follows.

- *If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are positive real numbers such that $a_1 + a_2 + \cdots + a_n = 1$, then*

$$\left(\frac{1}{\sqrt{a_1}}-1\right)\left(\frac{1}{\sqrt{a_2}}-1\right)\cdots\left(\frac{1}{\sqrt{a_n}}-1\right) \ge (\sqrt{n}-1)^n,$$

with equality for $a_1 = a_2 = \cdots = a_n = 1/n$.

Remark 2. By squaring, the inequality in P 1.30 becomes

$$\prod_{i=1}^{n}\frac{(1-a_i)^2}{a_i} \ge \frac{(n-1)^{2n}}{n^n}.$$

On the other hand, since the function $f(x) = \ln\frac{1+x}{1-x}$ is convex on $(0,1)$, by Jensen's inequality,

$$\prod_{i=1}^{n}\left(\frac{1+a_i}{1-a_i}\right) \ge \left(\frac{1+\frac{a_1+a_2+\cdots+a_n}{n}}{1-\frac{a_1+a_2+\cdots+a_n}{n}}\right)^n = \left(\frac{n+1}{n-1}\right)^n.$$

Multiplying these inequality yields the following result (*Kee-Wai Lau*, 2000).

- *If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are positive real numbers such that $a_1 + a_2 + \cdots + a_n = 1$, then*

$$\left(\frac{1}{a_1}-a_1\right)\left(\frac{1}{a_2}-a_2\right)\cdots\left(\frac{1}{a_n}-a_n\right) \ge \left(n-\frac{1}{n}\right)^n,$$

with equality for $a_1 = a_2 = \cdots = a_n = 1/n$.

□

P 1.31. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If*

$$0 < k \le \left(1+\frac{2\sqrt{n-1}}{n}\right)^2,$$

then

$$\left(ka_1+\frac{1}{a_1}\right)\left(ka_2+\frac{1}{a_2}\right)\cdots\left(ka_n+\frac{1}{a_n}\right) \ge (k+1)^n.$$

(Vasile Cîrtoaje, 2006)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s), \quad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=\ln\left(ku+\frac{1}{u}\right), \quad u\in(0,n).$$

We have

$$f'(u)=\frac{ku^2-1}{u(ku^2+1)}, \quad f''(u)=\frac{1+4ku^2-k^2u^4}{u^2(ku^2+1)^2}.$$

For $u\in(0,1]$, we have $f''(u)>0$, since

$$1+4ku^2-k^2u^4>ku^2(4-ku^2)\ge ku^2(4-k)\ge 0.$$

Therefore, f is convex on $(0,1]$. By HCF Theorem and Remark 2, it suffices to prove that $H(x,y)\ge 0$ for $x,y>0$ such that $x+(n-1)y=n$, where

$$H(x,y)=\frac{f'(x)-f'(y)}{x-y}.$$

Since

$$H(x,y)=\frac{1+k(x+y)^2-k^2x^6y^2}{xy(kx^2+1)(ky^2+1)}>\frac{k[(x+y)^2-kx^6y^2]}{xy(kx^2+1)(ky^2+1)},$$

it suffices to show that

$$x+y\ge\sqrt{k}\,xy.$$

Indeed, by the Cauchy-Schwarz inequality, we have

$$(x+y)[(n-1)y+x]\ge(\sqrt{n-1}+1)^2xy,$$

hence

$$x+y\ge\frac{1}{n}(\sqrt{n-1}+1)^2xy\ge\sqrt{k}\,xy.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 1.32. *If a,b,c,d are nonzero real numbers such that*

$$a,b,c,d\ge\frac{-1}{2}, \quad a+b+c+d=4,$$

then

$$3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right)+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\ge 16.$$

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)\geq 4f(s),\quad s=\frac{a+b+c+d}{4}=1,$$

where

$$f(u)=\frac{3}{u^2}+\frac{1}{u},\quad u\in\mathbb{I}=\left[\frac{-1}{2},\frac{11}{2}\right]\setminus\{0\}.$$

Clearly, f is convex for $u\in\mathbb{I}$, $u\geq s$. Therefore, by HCF Theorem and Remark 4, it suffices to prove that $f(x)+3f(y)\geq 4f(1)$ for all $x,y\in\mathbb{I}$ such that $x+3y=4$. According to Remark 1, we only need to show that $h(x,y)\geq 0$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y},\qquad g(u)=\frac{f(u)-f(1)}{u-1}.$$

Indeed, we have

$$g(u)=-\frac{4}{u}-\frac{3}{u^2},$$

$$h(x,y)=\frac{4xy+3x+3y}{x^2y^2}=\frac{2(1+2x)(6-x)}{3x^2y^2}\geq 0.$$

The proof is completed. In accord with Remark 3, the equality holds for $a=b=c=d=1$, and also for $a=\frac{-1}{2}$ and $b=c=d=\frac{3}{2}$ (or any cyclic permutation).

□

P 1.33. *If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers such that $a_1^2+a_2^2+\cdots+a_n^2=n$, then*

$$a_1^3+a_2^3+\cdots+a_n^3-n+\sqrt{\frac{n}{n-1}}\,(a_1+a_2+\cdots+a_n-n)\geq 0.$$

(Vasile Cirtoaje, 2007)

Solution. Replacing each a_i by $\sqrt{a_i}$, we have to prove that

$$f(a_1)+f(a_2)+\cdots+f(a_n)\geq nf(s),\qquad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=u\sqrt{u}+k\sqrt{u},\quad k=\sqrt{\frac{n}{n-1}},\qquad u\in[0,n].$$

For $u\geq 1$, we have

$$f''(u)=\frac{3u-k}{4u\sqrt{u}}\geq\frac{3-k}{4u\sqrt{u}}>0.$$

Therefore, f is convex for $u \ge 1$. According to HCF Theorem and Remark 1, it suffices to show that $h(x,y) \ge 0$ for $x,y \ge 0$ such that $x+(n-1)y = n$. Since

$$g(u) = \frac{f(u)-f(1)}{u-1} = 1 + \frac{u+k}{\sqrt{u}+1}$$

and

$$h(x,y) = \frac{g(x)-g(y)}{x-y} = \frac{\sqrt{x}+\sqrt{y}+\sqrt{xy}-k}{(\sqrt{x}+\sqrt{y})(\sqrt{x}+1)(\sqrt{y}+1)},$$

we need to show that

$$\sqrt{x}+\sqrt{y}+\sqrt{xy} \ge k.$$

This is true if

$$\sqrt{x+y} \ge \sqrt{\frac{n}{n-1}}.$$

Indeed, we have

$$x+y \ge \frac{x}{n-1} + y = \frac{n}{n-1}.$$

The proof is completed. In accord with Remark 3, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = 0$ and $a_2 = \cdots = a_n = \sqrt{\dfrac{n}{n-1}}$ (or any cyclic permutation).

□

P 1.34. *If a,b,c,d,e are nonnegative real numbers such that $a^2+b^2+c^2+d^2+e^2 = 5$, then*

$$\frac{1}{7-2a} + \frac{1}{7-2b} + \frac{1}{7-2c} + \frac{1}{7-2d} + \frac{1}{7-2e} \le 1.$$

(Vasile Cirtoaje, 2010)

Solution. Replacing a,b,c,d,e by $\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}, \sqrt{e}$, we have to prove that

$$f(a)+f(b)+f(c)+f(d)+f(e) \ge 5f(s), \quad s = \frac{a+b+c+d+e}{5} = 1,$$

where

$$f(u) = \frac{1}{2\sqrt{u}-7}, \quad u \in [0,5].$$

For $u \in [0,1]$, we have

$$f''(u) = \frac{7-6\sqrt{u}}{2u\sqrt{u}(7-2\sqrt{u})^3} > 0.$$

Therefore, f is convex for $u \in [0,1]$. According to HCF Theorem and Remark 1, it suffices to show that $h(x,y) \ge 0$ for $x,y \ge 0$ such that $x+4y = 5$. Since

$$g(u) = \frac{f(u)-f(1)}{u-1} = \frac{-2}{5(7-2\sqrt{u})(1+\sqrt{u})}$$

and

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{2(5-2\sqrt{x}-2\sqrt{y})}{(\sqrt{x}+\sqrt{y})(1+\sqrt{x})(1+\sqrt{y})(7-2\sqrt{x})(7-2\sqrt{y})},$$

we need to show that

$$\sqrt{x}+\sqrt{y}\le\frac{5}{2}.$$

Indeed, by the Cauchy-Schwarz inequality, we have

$$(\sqrt{x}+\sqrt{y})^2\le\left(1+\frac{1}{4}\right)(x+4y)=\frac{25}{4}.$$

The proof is completed. The equality holds for $a=b=c=d=e=1$, and also for $a=2$ and $b=c=d=e=\frac{1}{2}$ (or any cyclic permutation).

Remark Similarly, we can prove the following generalization.

- *Let $a_1,a_2,\dots,a_n$ be nonnegative real numbers such that $a_1^2+a_2^2+\cdots+a_n^2=n$. If $k\ge 1+\frac{n}{\sqrt{n-1}}$, then*

$$\frac{1}{k-a_1}+\frac{1}{k-a_2}+\cdots+\frac{1}{k-a_n}\le\frac{n}{k-1},$$

with equality for $a_1=a_2=\cdots=a_n=1$. If $k=1+\frac{n}{\sqrt{n-1}}$, then the equality holds also for $a_1=\sqrt{n-1}$ and $a_2=\cdots=a_n=\frac{1}{\sqrt{n-1}}$.

□

P 1.35. *If $0\le a_1,a_2,\dots,a_n<k$ such that $a_1^2+a_2^2+\cdots+a_n^2=n$, where $1<k\le 1+\sqrt{\frac{n}{n-1}}$, then*

$$\frac{1}{k-a_1}+\frac{1}{k-a_2}+\cdots+\frac{1}{k-a_n}\ge\frac{n}{k-1}.$$

(Vasile Cirtoaje, 2010)

Solution. Replacing $a_1,a_2,\dots,a_n$ by $\sqrt{a_1},\sqrt{a_2},\dots,\sqrt{a_n}$, we have to prove that

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s),\quad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=\frac{1}{k-\sqrt{u}},\quad u\in[0,k^2).$$

From

$$f''(u) = \frac{3\sqrt{u}-k}{4u\sqrt{u}(k-\sqrt{u})^3},$$

it follows that f is convex for $u \ge 1$, since

$$3\sqrt{u}-k \ge 3-k \ge 2-\sqrt{\frac{n}{n-1}} > 0.$$

According to HCF Theorem and Remark 1, it suffices to show that $h(x,y) \ge 0$ for all $0 \le x, y < k^2$ such that $x+(n-1)y = n$. Since

$$g(u) = \frac{f(u)-f(1)}{u-1} = \frac{1}{(k-1)(k-\sqrt{u})(1+\sqrt{u})}$$

and

$$h(x,y) = \frac{g(x)-g(y)}{x-y} = \frac{\sqrt{x}+\sqrt{y}+1-k}{(k-1)(\sqrt{x}+\sqrt{y})(1+\sqrt{x})(1+\sqrt{y})(k-\sqrt{x})(k-\sqrt{y})},$$

we need to show that

$$\sqrt{x}+\sqrt{y} \ge k-1.$$

Indeed,

$$\sqrt{x}+\sqrt{y} \ge \sqrt{x+y} \ge \sqrt{\frac{x}{n-1}+y} = \sqrt{\frac{n}{n-1}} \ge k-1.$$

The proof is completed. In accord with Remark 3, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = 0$ and $a_2 = \cdots = a_n = \sqrt{\frac{n}{n-1}}$ (or any cyclic permutation).

□

P 1.36. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\sqrt{1+\frac{48a}{b+c}} + \sqrt{1+\frac{48b}{c+a}} + \sqrt{1+\frac{48c}{a+b}} \ge 15.$$

(Vasile Cirtoaje, 2005)

Solution. Due to homogeneity, we may assume that $a+b+c=1$. Thus, we need to show that

$$f(a)+f(b)+f(c) \ge 3f(s), \quad s = \frac{a+b+c}{3} = \frac{1}{3},$$

where

$$f(u) = \sqrt{\frac{1+47u}{1-u}}, \quad u \in [0,1).$$

From

$$f''(u) = \frac{48(47u-11)}{\sqrt{(1-u)^5(1+47u)^3}},$$

it follows that f is convex on $[1/3,1)$. By HCF Theorem, it suffices to show that $f(x)+2f(y) \geq 3f(1/3)$ for $x,y \geq 0$ such that $x+2y=1$; that is,

$$\sqrt{\frac{1+47x}{1-x}} + 2\sqrt{\frac{49-47x}{1+x}} \geq 15.$$

Setting

$$t = \sqrt{\frac{49-47x}{1+x}}, \quad 1 < t \leq 7,$$

the inequality turns into

$$\sqrt{\frac{1175-23t^2}{t^2-1}} \geq 15-2t.$$

By squaring, this inequality becomes

$$350-15t-61t^2+15t^3-t^4 \geq 0,$$

$$(5-t)^2(2+t)(7-t) \geq 0.$$

The last inequality is clearly true. From $x+2y=1$ and $f(x)+2f(y)=3f(1/3)$, we get $x=y=1/3$ and $x=0$, $y=1/2$. Therefore, in accordance with Remark 3, the equality in the case $a+b+c=1$ holds for $a=b=c=1/3$, and also for $a=0$ and $b=c=1/2$ (or any cyclic permutation). For the original inequality, the equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.37. *If a,b,c are nonnegative real numbers, then*

$$\sqrt{\frac{3a^2}{7a^2+5(b+c)^2}} + \sqrt{\frac{3b^2}{7b^2+5(c+a)^2}} + \sqrt{\frac{3c^2}{7c^2+5(a+b)^2}} \leq 1.$$

(Vasile Cirtoaje, 2008)

Solution. Due to homogeneity, we may assume that $a+b+c=3$. Thus, we need to show that

$$f(a)+f(b)+f(c) \geq 3f(s), \quad s = \frac{a+b+c}{3} = 1,$$

where

$$f(u) = -\sqrt{\frac{3u^2}{7u^2+5(3-u)^2}} = \frac{-u}{\sqrt{4u^2-10u+15}}, \quad u \in [0,3].$$

From

$$f''(u) = \frac{5(-8u^2+41u-30)}{(4u^2-10u+15)^{5/2}} \ge \frac{5(-8u^2+38u-30)}{(4u^2-10u+15)^{5/2}} = \frac{10(u-1)(15-4u)}{(4u^2-10u+15)^{5/2}},$$

it follows that f is convex on $[1,3]$. By HCF Theorem, it suffices to prove the original homogeneous inequality for $b=c=1$; that is

$$\sqrt{\frac{3a^2}{7a^2+20}} + 2\sqrt{\frac{3}{5a^2+10a+12}} \le 1.$$

By squaring two times, the inequality becomes

$$a(5a^3+10a^2+16a+50) \ge 3a\sqrt{(7a^2+20)(5a^2+10a+12)},$$

$$a^2(5a^6+20a^5-11a^4+38a^3-80a^2-40a+68) \ge 0,$$

$$a^2(a-1)^2(5a^4+30a^3+44a^2+96a+68) \ge 0.$$

The last inequality is clearly true. The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.38. *If a,b,c are nonnegative real numbers, then*

$$\sqrt{\frac{a^2}{a^2+2(b+c)^2}} + \sqrt{\frac{b^2}{b^2+2(c+a)^2}} + \sqrt{\frac{c^2}{c^2+2(a+b)^2}} \ge 1.$$

(Vasile Cirtoaje, 2008)

Solution. Due to homogeneity, we may assume that $a+b+c=3$. Thus, we need to show that

$$f(a)+f(b)+f(c) \ge 3f(s), \quad s=\frac{a+b+c}{3}=1,$$

where

$$f(u) = \sqrt{\frac{3u^2}{u^2+2(3-u)^2}} = \frac{u}{\sqrt{u^2-4u+6}}, \quad u \in [0,3].$$

From

$$f''(u) = \frac{2(2u^2-11u+12)}{(u^2-4u+6)^{5/2}} \ge \frac{2(-11u+12)}{(u^2-4u+6)^{5/2}},$$

it follows that f is convex on $[0,1]$. By HCF Theorem, it suffices to prove the original homogeneous inequality for $b=c=1$; that is

$$\frac{a}{\sqrt{a^2+8}} + \frac{2}{\sqrt{2a^2+4a+3}} \ge 1.$$

By squaring, the inequality becomes

$$a\sqrt{(a^2+8)(2a^2+4a+3)} \geq 3a^2+8a-2.$$

For the nontrivial case $3a^2+8a-2>0$, squaring both sides, we get

$$a^6+2a^5+5a^4-8a^3-14a^2+16a-2 \geq 0,$$

$$(a-1)^2[a^4+4a^3+9a^2+4a+(3a^2+8a-2)] \geq 0.$$

The last inequality is clearly true. The equality holds for $a=b=c$.

□

P 1.39. *Let a,b,c be nonnegative real numbers, no two of which are zero. If*

$$k \geq k_0, \quad k_0 = \frac{\ln 3}{\ln 2} - 1 \approx 0.585,$$

then

$$\left(\frac{2a}{b+c}\right)^k + \left(\frac{2b}{c+a}\right)^k + \left(\frac{2c}{a+b}\right)^k \geq 3.$$

(Vasile Cirtoaje, 2005)

Solution. For $k=1$, the inequality is just the well known Nesbitt's inequality

$$\frac{2a}{b+c}+\frac{2b}{c+a}+\frac{2c}{a+b} \geq 3,$$

while for $k \geq 1$, the inequality follows from Jensens's inequality applied to the convex function $f(u)=u^k$:

$$\left(\frac{2a}{b+c}\right)^k + \left(\frac{2b}{c+a}\right)^k + \left(\frac{2c}{a+b}\right)^k \geq 3\left(\frac{\frac{2a}{b+c}+\frac{2b}{c+a}+\frac{2c}{a+b}}{3}\right)^k \geq 3.$$

Consider now that $k_0 \leq k < 1$. Due to homogeneity, we may assume that $a+b+c=1$. Thus, we need to show that

$$f(a)+f(b)+f(c) \geq 3f(s), \quad s=\frac{a+b+c}{3}=\frac{1}{3},$$

where

$$f(u)=\left(\frac{2u}{1-u}\right)^k, \quad u \in [0,1).$$

From

$$f''(u)=\frac{4k}{(1-u)^4}\left(\frac{2u}{1-u}\right)^{k-2}(2u+k-1),$$

it follows that f is convex on $[1/3,1)$ (since $u\ge 1/3$ involves $2u+k-1\ge 2/3+k-1=k-1/3>0$). By HCF Theorem, it suffices to prove the original homogeneous inequality for $b=c=1$; that is, to show that $h(a)\ge 3$ for all $a\ge 0$, where

$$h(a)=a^k+2\left(\frac{2}{a+1}\right)^k.$$

For $a>0$, the derivative

$$h'(a)=ka^{k-1}-k\left(\frac{2}{a+1}\right)^{k+1}$$

has the same sign as

$$g(a)=(k-1)\ln a-(k+1)\ln\frac{2}{a+1}.$$

From

$$g'(a)=\frac{2ka+k-1}{a(a+1)},$$

it follows that $g'(a)=0$ for $a_0=(1-k)/(2k)<1$, $g'(a)<0$ for $a\in(0,a_0)$ and $g'(a)>0$ for $a\in(a_0,\infty)$. Then, g is strictly decreasing on $(0,a_0]$ and strictly increasing on (a_0,∞). Since $\lim_{a\to 0}g(a)=\infty$ and $g(1)=0$, there exists $a_1\in(0,a_0)$ such that $g(a_1)=0$, $g(a)>0$ for $a\in(0,a_1)\cup(1,\infty)$ and $g(a)<0$ for $a\in(a_1,1)$; hence, $h(a)$ is strictly increasing on $[0,a_1]\cup[1,\infty)$ and strictly decreasing on $[a_1,1]$. Consequently,

$$h(a)\ge\min\{h(0),h(1)\}.$$

Since $h(0)=2^{k+1}\ge 3$ and $h(1)=3$, we get $h(a)\ge 3$. The proof is completed. The equality holds for $a=b=c$. If $k=k_0$, then the equality holds also for $a=0$ and $b=c$ (or any cyclic permutation).

Remark. For $k=2/3$, we can give the following solution (based on the AM-GM inequality):

$$\begin{aligned}\sum\left(\frac{2a}{b+c}\right)^{2/3}&=\sum\frac{2a}{\sqrt[3]{2a\cdot(b+c)\cdot(b+c)}}\\&\ge\sum\frac{6a}{2a+(b+c)+(b+c)}=3.\end{aligned}$$

□

P 1.40. *If $a,b,c\in[1,7+4\sqrt{3}]$, then*

$$\sqrt{\frac{2a}{b+c}}+\sqrt{\frac{2b}{c+a}}+\sqrt{\frac{2c}{a+b}}\ge 3.$$

(Vasile Cirtoaje, 2007)

Solution. Denoting

$$s = \frac{a+b+c}{3},$$

we need to show that

$$f(a)+f(b)+f(c) \geq 3f(s),$$

where

$$f(u) = \sqrt{\frac{2u}{3s-u}}, \quad 1 \leq u < 3s.$$

For $u \geq s$, we have

$$f''(u) = 3s\left(\frac{3s-u}{2u}\right)^{3/2}\frac{4u-3s}{(3s-u)^4} > 0.$$

Therefore, f is convex for $u \geq s$. By HCF Theorem, it suffices to prove the original inequality for $b = c$; that is,

$$\sqrt{\frac{a}{b}} + 2\sqrt{\frac{2b}{a+b}} \geq 3.$$

Putting $t = \sqrt{\dfrac{b}{a}}$, the condition $a, b \in [1, 7+4\sqrt{3}]$ involves

$$2-\sqrt{3} \leq t \leq 2+\sqrt{3}.$$

We need to show that

$$2\sqrt{\frac{2t^2}{t^2+1}} \geq 3 - \frac{1}{t}.$$

This is true if

$$\frac{8t^2}{t^2+1} \geq \left(3-\frac{1}{t}\right)^2.$$

This is equivalent to the obvious inequality

$$(t-1)^2(t^2-4t+1) \geq 0.$$

The proof is completed. In accord with Remark 3, the equality holds for $a = b = c$, and also for $a = 1$ and $b = c = 7+4\sqrt{3}$ (or any cyclic permutation).

□

P 1.41. *Let a, b, c be nonnegative real numbers such that $a+b+c = 3$. If*

$$0 < k \leq k_0, \quad k_0 = \frac{\ln 2}{\ln 3 - \ln 2} \approx 1.71,$$

then

$$a^k(b+c)+b^k(c+a)+c^k(a+b) \leq 6.$$

Solution. For $0 < k \le 1$, the inequality follows from Jensens's inequality applied to the convex function $f(u) = u^k$:

$$(b+c)a^k + (c+a)b^k + (a+b)c^k \le 2(a+b+c)\left[\frac{(b+c)a+(c+a)b+(a+b)c}{2(a+b+c)}\right]^k$$

$$= 6\left(\frac{ab+bc+ca}{3}\right)^k \le 6\left(\frac{a+b+c}{3}\right)^{2k} = 6.$$

Consider now that $1 < k \le k_0$, and write the inequality as

$$f(a) + f(b) + f(c) \ge 3f(s), \quad s = \frac{a+b+c}{3} = 1,$$

where

$$f(u) = u^k(u-3), \quad u \in [0,3].$$

For $u \ge 1$, we have

$$f''(u) = ku^{k-2}[(k+1)u - 3k + 3] \ge ku^{k-2}[(k+1) - 3k + 3] = 2k(2-k)u^{k-2} > 0;$$

therefore, f is convex for $u \ge 1$. By HCF Theorem, it suffices to consider the case where two of a, b, c are equal. Write the desired inequality in the homogeneous form

$$a^k(b+c) + b^k(c+a) + c^k(a+b) \le 6\left(\frac{a+b+c}{3}\right)^{k+1}.$$

Since this inequality is trivial for $b = c = 0$, we may consider that $b = c = 1$. So, we need to show that $g(a) \ge 0$ for $a \ge 0$, where

$$g(a) = 3\left(\frac{a+2}{3}\right)^{k+1} - a^k - a - 1.$$

We have

$$g'(a) = (k+1)\left(\frac{a+2}{3}\right)^k - ka^{k-1} - 1, \quad \frac{1}{k}g''(a) = \frac{k+1}{3}\left(\frac{a+2}{3}\right)^{k-1} - \frac{k-1}{a^{2-k}}.$$

Since g'' is strictly increasing, $\lim_{a\to 0} g(a) = -\infty$ and $g''(1) = 2k(2-k)/3 > 0$, there exists $a_1 \in (0,1)$ such that $g''(a_1) = 0$, $g''(a) < 0$ for $a \in (0, a_1)$, and $g''(a) > 0$ for $a > 1$. Therefore, g' is strictly decreasing on $[0, a_1]$ and strictly increasing on $[a_1, \infty)$. Since

$$g'(0) = (k+1)(2/3)^k - 1 \ge (k+1)(2/3)^{k_0} - 1 = \frac{k+1}{2} - 1 = \frac{k-1}{2} > 0, \quad g'(1) = 0,$$

there exists $a_2 \in (0, a_1)$ such that $g'(a_2) = 0$, $g'(a) > 0$ for $a \in [0, a_2) \cup (1, \infty)$, and $g'(a) < 0$ for $a \in (a_2, 1)$. Thus, g is strictly increasing on $[0, a_2] \cup [1, \infty)$ and strictly decreasing on $[a_2, 1]$. Consequently,

$$g(a) \ge \min\{g(0), g(1)\},$$

and from

$$g(0) = 3(2/3)^{k+1} - 1 \geq 3(2/3)^{k_0+1} - 1 = 1 - 1 = 0, \quad g(1) = 0,$$

we get $g(a) \geq 0$. This completes the proof. The equality holds for $a = b = c = 1$. If $k = k_0$, then the equality holds also for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

Remark 1. Using the Cauchy-Schwarz inequality, we get

$$\sum \frac{a}{b^k + c^k} \geq \frac{(a+b+c)^2}{\sum a(b^k + c^k)} = \frac{9}{\sum a^k(b+c)} \geq \frac{3}{2}.$$

Thus, the following statement holds.

- *Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. If*

$$0 < k \leq k_0, \quad k_0 = \frac{\ln 2}{\ln 3 - \ln 2} \approx 1.71,$$

then

$$\frac{a}{b^k + c^k} + \frac{b}{c^k + a^k} + \frac{c}{a^k + b^k} \geq \frac{3}{2},$$

with equality for $a = b = c = 1$. If $k = k_0$, then the equality holds also for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

Remark 2. Also, the following statement holds.

- *Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. If*

$$k \geq k_1, \quad k_1 = \frac{\ln 9 - \ln 8}{\ln 3 - \ln 2} \approx 0.2905,$$

then

$$\frac{a^k}{b+c} + \frac{b^k}{c+a} + \frac{c^k}{a+b} \geq \frac{3}{2},$$

with equality for $a = b = c = 1$. If $k = k_1$, then the equality holds also for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

For $k_1 \leq k \leq 2$, the inequality can be proved using the Cauchy-Schwarz inequality, as follows:

$$\sum \frac{a^k}{b+c} \geq \frac{(a+b+c)^2}{\sum a^{2-k}(b+c)} = \frac{9}{\sum a^{2-k}(b+c)} = \frac{3}{2}.$$

For $k > 2$, the inequality can be deduced from the Cauchy-Schwarz inequality and Bernoulli's inequality, as follows:

$$\sum \frac{a^k}{b+c} \ge \frac{\left(\sum a^{k/2}\right)^2}{\sum(b+c)} = \frac{\left(\sum a^{k/2}\right)^2}{6},$$

$$\sum a^{k/2} \ge \sum \left[1 + \frac{k}{2}(a-1)\right] = 3.$$

□

P 1.42. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $k \ge \dfrac{n^2}{4(n-1)}$, then*

$$(a_1^2 + k)(a_2^2 + k)\cdots(a_n^2 + k) \ge (1+k)^n.$$

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \ln(u^2 + k), \quad u \in [0, n].$$

We have

$$f'(u) = \frac{2u}{u^2 + k}, \quad f''(u) = \frac{2(k - u^2)}{(u^2 + k)^2}.$$

For $u \in [0, 1]$, we have $f''(u) > 0$, since

$$k - u^2 \ge k - 1 \ge \frac{n^2}{4(n-1)} - 1 \ge 0.$$

Therefore, f is convex on $[0, 1]$. By HCF Theorem and Remark 2, it suffices to prove that $H(x, y) \ge 0$ for $x, y \ge 0$ such that $x + (n-1)y = n$, where

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$

Since

$$H(x, y) = \frac{2(k - xy)}{(x^2 + k)(y^2 + k)} \ge \frac{2[n^2 - 4(n-1)xy]}{4(n-1)(x^2 + k)(y^2 + k)},$$

we only need to show that

$$n^2 \ge 4(n-1)xy.$$

Indeed, we have

$$n^2 = [x + (n-1)y]^2 \ge 4(n-1)xy.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.43. *Let a,b,c be nonnegative real numbers such $a+b+c=3$. If $k\ge k_0$, where*

$$k_0=\frac{\sqrt{6}-2}{\sqrt{6}-\sqrt{2}-1}=(2+\sqrt{2})(2+\sqrt{3})\approx 12.74\ ,$$

then

$$\sqrt{a}+\sqrt{b}+\sqrt{c}-3\ge k\left(\sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}}-3\right).$$

(Vasile Cirtoaje, 2008)

Solution. By the Cauchy-Schwarz inequality

$$(1+1+1)[(a+b)+(b+c)+(c+a)]\ge\left(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\right)^2,$$

we get

$$\sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}}\le 3.$$

Therefore, it suffices to consider the case $k=k_0$. Write the inequality as

$$f(a)+f(b)+f(c)\ge 3f(s),\quad s=\frac{a+b+c}{3}=1,$$

where

$$f(u)=\sqrt{u}-k_0\sqrt{\frac{3-u}{2}},\quad u\in[0,3).$$

For $u\ge 1$, we have

$$4f''(u)=-u^{-3/2}+\frac{k_0}{4}\left(\frac{3-u}{2}\right)^{-3/2}\ge -1+\frac{k_0}{4}>0.$$

Therefore, f is convex on $[1,3]$. By HCF Theorem and Remark 5, it suffices to consider only the case $a\le 1\le b=c$. Write the original inequality in the homogeneous form

$$\sqrt{a}+\sqrt{b}+\sqrt{c}-3\sqrt{\frac{a+b+c}{3}}\ge k_0\left(\sqrt{\frac{a+b}{2}}+\sqrt{\frac{b+c}{2}}+\sqrt{\frac{c+a}{2}}-3\sqrt{\frac{a+b+c}{3}}\right).$$

Due to homogeneity, we may assume that $b=c=1$. Moreover, it is convenient to use the substitution $x=\sqrt{a}$. Thus, we need to show that $g(x)\ge 0$ for $x\in[0,1]$, where

$$g(x)=x+2-k_0+3(k_0-1)\sqrt{\frac{x^2+2}{3}}-2k_0\sqrt{\frac{x^2+1}{2}}.$$

We have

$$g'(x)=1+(k_0-1)x\sqrt{\frac{3}{x^2+2}}-k_0x\sqrt{\frac{2}{x^2+1}},$$

$$g''(x) = \frac{k_0}{2}\left(\frac{2}{x^2+1}\right)^{3/2}\left[\left(m\cdot\frac{x^2+1}{x^2+2}\right)^{3/2} - 1\right],$$

where

$$m = \sqrt[3]{6\left(1-\frac{1}{k_0}\right)^2} \approx 1.72.$$

Clearly, $g''(x)$ has the same sign as $h(x)$, where

$$h(x) = m\cdot\frac{x^2+1}{x^2+2} - 1.$$

Since

$$h(0) = \frac{m}{2} - 1 < 0, \quad h(1) = \frac{2m}{3} - 1 > 0,$$

there is $x_1 \in (0,1)$ such that $h(x) < 0$ for $x \in [0, x_1)$, $h(x_1) = 0$ and $h(x) > 0$ for $x \in (x_1, 1]$. Therefore, g' is strictly decreasing on $[0, x_1]$ and strictly increasing on $[x_1, 1]$. Since $g'(0) = 1$ and $g'(1) = 0$, there is $x_2 \in (0, x_1)$ such that $g'(x) > 0$ for $x \in [0, x_2)$, $g'(x_2) = 0$ and $g'(x) < 0$ for $x \in (x_2, 1)$. Thus, $g(x)$ is strictly increasing on $[0, x_2]$ and strictly decreasing on $[x_2, 1]$. From $g(0) = 2 - k_0 + (k_0 - 1)\sqrt{6} - k_0\sqrt{2} = 0$ and $g(1) = 0$, it follows that $g(x) \geq 0$ for $x \in [0,1]$. This completes the proof. The equality holds for $a = b = c$. If $k = k_0$, then the equality holds also for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

□

P 1.44. *Let a, b, c be nonnegative real numbers such $a + b + c = 3$. If $k \leq k_1$, where*

$$k_1 = (\sqrt{3} - 1)(\sqrt{3} + \sqrt{2}) \approx 2.303\ ,$$

then

$$\sqrt{a} + \sqrt{b} + \sqrt{c} - 3 \leq k\left(\sqrt{\frac{a+b}{2}} + \sqrt{\frac{b+c}{2}} + \sqrt{\frac{c+a}{2}} - 3\right).$$

(Vasile Cirtoaje, 2008)

Solution. By the Cauchy-Schwarz inequality

$$(1+1+1)[(a+b)+(b+c)+(c+a)] \geq \left(\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a}\right)^2,$$

we get

$$\sqrt{\frac{a+b}{2}} + \sqrt{\frac{b+c}{2}} + \sqrt{\frac{c+a}{2}} \leq 3.$$

Therefore, it suffices to consider the case $k = k_1$. Write the inequality as

$$f(a) + f(b) + f(c) \geq 3f(s), \quad s = \frac{a+b+c}{3} = 1,$$

where

$$f(u) = -\sqrt{u} + k_1\sqrt{\frac{3-u}{2}}, \quad u \in [0,3).$$

For $0 \leq u \leq 1$, we have

$$4f''(u) = u^{-3/2} - \frac{k_1}{4}\left(\frac{3-u}{2}\right)^{-3/2} \geq 1 - \frac{k_1}{4} > 0.$$

Therefore, f is convex on $[0,1]$. By HCF Theorem and Remark 5, it suffices to consider only the case $a \geq 1 \geq b = c$. Write the original inequality in the homogeneous form

$$\sqrt{a} + \sqrt{b} + \sqrt{c} - 3\sqrt{\frac{a+b+c}{3}} \leq k_1\left(\sqrt{\frac{a+b}{2}} + \sqrt{\frac{b+c}{2}} + \sqrt{\frac{c+a}{2}} - 3\sqrt{\frac{a+b+c}{3}}\right).$$

Due to homogeneity, we may assume that $b = c = 1$. Moreover, it is convenient to use the substitution $x = \sqrt{a}$. Thus, we need to show that $g(x) \leq 0$ for $x \geq 1$, where

$$g(x) = x + 2 - k_1 + 3(k_1 - 1)\sqrt{\frac{x^2+2}{3}} - 2k_1\sqrt{\frac{x^2+1}{2}}.$$

We have

$$g'(x) = 1 + (k_1 - 1)x\sqrt{\frac{3}{x^2+2}} - k_1 x\sqrt{\frac{2}{x^2+1}},$$

$$g''(x) = \frac{k_1}{2}\left(\frac{2}{x^2+1}\right)^{3/2}\left[\left(m \cdot \frac{x^2+1}{x^2+2}\right)^{3/2} - 1\right],$$

where

$$m = \sqrt[3]{6\left(1 - \frac{1}{k_1}\right)^2} \approx 1.2431 .$$

Clearly, $g''(x)$ has the same sign as $h(x)$, where

$$h(x) = m \cdot \frac{x^2+1}{x^2+2} - 1.$$

Since h is strictly increasing on $[1,\infty)$ and

$$h(1) = \frac{2m}{3} - 1 < 0, \quad \lim_{x\to\infty} h_1(x) = m - 1 > 0,$$

there is $x_1 \in (1,\infty)$ such that $h(x) < 0$ for $x \in [1,x_1)$, $h(x_1) = 0$ and $h(x) > 0$ for $x \in (x_1,\infty)$. Therefore, g' is strictly decreasing on $[1,x_1]$ and strictly increasing on $[x_1,\infty)$. Since $g'(1) = 0$ and $\lim_{x\to\infty} g'(x) = 0$, it follows that $g'(x) < 0$ for $x \in (1,\infty)$. Thus, $g(x)$ is strictly decreasing on $[1,\infty)$, hence $g(x) \le g(1) = 0$. This completes the proof. The equality holds for $a = b = c = 1$. If $k = k_0$, then the equality holds also for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

□

P 1.45. *If a,b,c are positive real numbers such that $abc = 1$, then*

$$a^2 + b^2 + c^2 - 3 \ge 18(a + b + c - ab - bc - ca).$$

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x) + f(y) + f(z) \ge 3f(s), \quad s = \frac{x+y+z}{3} = 0,$$

where

$$f(u) = e^{2u} - 1 - 18(e^u - e^{-u}), \quad u \in \mathbb{R}.$$

For $u \le 0$, we have

$$f''(u) = 4e^{2u} + 18(e^{-u} - e^u) > 0,$$

hence f is convex on $(-\infty,0]$. By HCF Theorem, it suffices to prove the original inequality for $b = c := t$ and $a = 1/t^2$, where $t > 0$. Since

$$a^2 + b^2 + c^2 - 3 = \frac{1}{t^4} + 2t^2 - 3 = \frac{(t^2-1)^2(2t^2+1)}{t^4}$$

and

$$a + b + c - ab - bc - ca = \frac{-(t^4 - 2t^3 + 2t - 1)}{t^2} = \frac{-(t-1)^3(t+1)}{t^2},$$

we get

$$a^2 + b^2 + c^2 - 3 - 18(a + b + c - ab - bc - ca) = \frac{(t-1)^2(2t-1)^2(t+1)(5t+1)}{t^4} \ge 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 4$ and $b = c = 1/2$ (or any cyclic permutation).

□

P 1.46. *If a, b, c are positive real numbers such that $abc = 1$, then*

$$\sqrt{a^2-a+1}+\sqrt{b^2-b+1}+\sqrt{c^2-c+1} \ge a+b+c.$$

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x)+f(y)+f(z) \ge 3f(s), \quad s = \frac{x+y+z}{3} = 0,$$

where

$$f(u) = \sqrt{e^{2u}-e^u+1}-e^u, \quad u \in \mathbb{R}.$$

We claim that f is convex for $u \ge 0$. Since

$$e^{-u}f''(u) = \frac{4e^{3u}-6e^{2u}+9e^u-2}{4(e^{2u}-e^u+1)^{3/2}}-1,$$

we need to show that

$$(4x^3-6x^2+9x-2)^2 \ge 16(x^2-x+1)^3,$$

where $x = e^u \ge 1$. Indeed,

$$(4x^3-6x^2+9x-2)^2-16(x^2-x+1)^3 = 12x^3(x-1)+9x^2+12(x-1) > 0.$$

By HCF Theorem, it suffices to prove the original inequality for $b = c := t$ and $a = 1/t^2$, where $t > 0$; that is,

$$\frac{\sqrt{t^4-t^2+1}}{t^2}+2\sqrt{t^2-t+1} \ge \frac{1}{t^2}+2t,$$

$$\frac{t^2-1}{\sqrt{t^4-t^2+1}+1}+\frac{2(1-t)}{\sqrt{t^2-t+1}+t} \ge 0.$$

Since

$$\frac{t^2-1}{\sqrt{t^4-t^2+1}} \ge \frac{t^2-1}{t^2+1},$$

it suffices to show that

$$\frac{t^2-1}{t^2+1}+\frac{2(1-t)}{\sqrt{t^2-t+1}+t} \ge 0,$$

which is equivalent to

$$(t-1)\left[\frac{t+1}{t^2+1}-\frac{2}{\sqrt{t^2-t+1}+t}\right] \ge 0,$$

$$(t-1)\left[(t+1)\sqrt{t^2-t+1}-t^2+t-2\right]\ge 0,$$

$$\frac{(t-1)^2(3t^2-2t+3)}{(t+1)\sqrt{t^2-t+1}+t^2-t+2}\ge 0.$$

Clearly, the last inequality is true. The equality holds for $a=b=c=1$.

□

P 1.47. *If* $a,b,c,d\ge \dfrac{1}{1+\sqrt{6}}$ *such that* $abcd=1$*, then*

$$\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}+\frac{1}{d+2}\le \frac{4}{3}.$$

(Vasile Cirtoaje, 2005)

Solution. Using the substitution

$$a=e^x,\quad b=e^y,\quad c=e^z,\quad d=e^w,$$

we need to show that

$$f(x)+f(y)+f(z)+f(w)\ge 4f(s),\qquad s=\frac{x+y+z+w}{4}=0,$$

where

$$f(u)=\frac{-1}{e^u+2},\quad u\in\mathbb{R}.$$

For $u\le 0$, we have

$$f''(u)=\frac{e^u(2-e^u)}{(e^u+2)^3}>0,$$

hence f is convex on $(-\infty,0]$. By HCF Theorem, it suffices to prove the original inequality for $b=c=d:=t$ and $a=1/t^3$, where $t\ge \dfrac{1}{1+\sqrt{6}}$; that is,

$$\frac{t^3}{2t^3+1}+\frac{3}{t+2}\le\frac{4}{3},$$

which is equivalent to the obvious inequality

$$(t-1)^2(5t^2+2t-1)\ge 0.$$

In accord with Remark 3, the equality holds for $a=b=c=d=1$, and also for $a=19+9\sqrt{6}$ and $b=c=d=\dfrac{1}{1+\sqrt{6}}$ (or any cyclic permutation).

□

P 1.48. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 a_2 \cdots a_n = 1$, then*

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n \geq 6\sqrt{3}\left(a_1 + a_2 + \cdots + a_n - \frac{1}{a_1} - \frac{1}{a_2} - \cdots - \frac{1}{a_n}\right).$$

Solution. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

where

$$f(u) = e^{2u} - 1 - 6\sqrt{3}\,(e^u - e^{-u}), \quad u \in \mathbb{R}.$$

For $u \leq 0$, we have

$$f''(u) = 4e^{2u} + 6\sqrt{3}(e^{-u} - e^u) > 0,$$

hence f is convex on $(-\infty, 0]$. By HCF Theorem and Remark 2, it suffices to show that $H(x, y) \geq 0$ for $x, y \in \mathbb{R}$ such that $x + (n-1)y = 0$, where

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$

From

$$f'(u) = 2e^{2u} - 6\sqrt{3}\,(e^u + e^{-u}),$$

we get

$$H(x, y) = \frac{2(e^x - e^y)}{x - y}\left(e^x + e^y - 3\sqrt{3} + 3\sqrt{3}\,e^{-x}e^{-y}\right).$$

Since $(e^x - e^y)/(x - y) > 0$, we need to prove that

$$e^x + e^y + 3\sqrt{3}\,e^{-x}e^{-y} \geq 3\sqrt{3}.$$

Indeed, by the AM-GM inequality, we have

$$e^x + e^y + 3\sqrt{3}\,e^{-x}e^{-y} \geq 3\sqrt[3]{e^x \cdot e^y \cdot 3\sqrt{3}\,e^{-x}e^{-y}} = 3\sqrt{3}.$$

The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.49. *If $a_1, a_2, \ldots, a_n$ $(n \geq 4)$ are positive real numbers such that $a_1 a_2 \cdots a_n = 1$, then*

$$(n-1)(a_1^2 + a_2^2 + \cdots + a_n^2) + n(n+3) \geq (2n+2)(a_1 + a_2 + \cdots + a_n).$$

Solution. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

where

$$f(u) = (n-1)e^{2u} - (2n+2)e^u, \quad u \in \mathbb{R}.$$

For $u \ge 0$, we have

$$\begin{aligned} f''(u) &= 4(n-1)e^{2u} - (2n+2)e^u = 2e^u[2(n-1)e^u - n - 1] \\ &\ge 2e^u[2(n-1) - n - 1] = 2(n-3)e^u > 0. \end{aligned}$$

Therefore, f is convex on $[0, \infty)$. By HCF Theorem and Remark 2, it suffices to show that $H(x, y) \ge 0$ for $x, y \in \mathbb{R}$ such that $x + (n-1)y = 0$, where

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$

From

$$f'(u) = 2(n-1)e^{2u} - (2n+2)e^u,$$

we get

$$H(x, y) = \frac{2(e^x - e^y)}{x - y}\,[(n-1)(e^x + e^y) - (n+1)].$$

Since $(e^x - e^y)/(x - y) > 0$, we need to prove that $(n-1)(e^x + e^y) \ge n + 1$. Using the AM-GM inequality, we have

$$\begin{aligned} (n-1)(e^x + e^y) &= (n-1)e^x + e^y + e^y + \cdots + e^y \ge n\sqrt[n]{(n-1)e^x \cdot e^y \cdot e^y \cdots e^y} \\ &= n\sqrt[n]{(n-1)e^{x+(n-1)y}} = n\sqrt[n]{n-1}. \end{aligned}$$

Thus, it suffices to show that

$$n\sqrt[n]{n-1} \ge n + 1,$$

which is equivalent to

$$n - 1 \ge \left(1 + \frac{1}{n}\right)^n.$$

This is true for $n \ge 4$, since

$$n - 1 \ge 3 > \left(1 + \frac{1}{n}\right)^n.$$

The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. From the proof above, it follows that the following sharper inequality holds in the same conditions (*Gabriel Dospinescu* and *Calin Popa*):

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n \ge \frac{2n\sqrt[n]{n-1}}{n-1}(a_1 + a_2 + \cdots + a_n - n).$$

□

P 1.50. *Let a,b,c,d be positive real numbers such that $abcd=1$. If p and q are nonnegative real numbers such that $p+q=3$, then then*

$$\frac{1}{1+pa+qa^3}+\frac{1}{1+pb+qb^3}+\frac{1}{1+pc+qc^3}+\frac{1}{1+pd+qd^3}\ge 1.$$

(Vasile Cirtoaje, 2007)

Solution. Using the substitutions

$$a=e^x,\quad b=e^y,\quad c=e^z,\quad d=e^w,$$

we need to show that

$$f(x)+f(y)+f(z)+f(w)\ge 4f(s),\quad s=\frac{x+y+z+w}{4}=0,$$

where

$$f(u)=\frac{1}{1+pe^u+qe^{3u}},\quad u\in\mathbb{R}.$$

We will show that $f''(u)>0$ for $u\ge 0$, hence f is convex on $[0,\infty)$. Since

$$f''(u)=\frac{th(t)}{(1+pt+qt^3)^3},$$

where

$$h(t)=9q^2t^5+2pqt^3-9qt^2+p^2t-p,\quad t=e^u,\ t\ge 1,$$

we need to show that $h(t)>0$ for $t\ge 1$. Indeed, we have

$$h(t)\ge 9q^2t^3+2pqt^3-9qt^2+p^2t-pt=t[(9q^2+2pq)t^2-9qt+p^2-p]$$

and

$$(9q^2+2pq)t^2-9qt+p^2-p\ge(9q^2+2pq)(2t-1)-9qt+p^2-p$$
$$=q(18q+4p-9)t-9q^2-2pq+p^2-p\ge q(18q+4p-9)-9q^2-2pq+p^2-p$$
$$=p^2+2pq+9q^2-p-9q=p^2+2pq+9q^2-\frac{(p+9q)(p+q)}{3}$$
$$=\frac{2(p-q)^2+16q^2}{3}>0.$$

By HCF Theorem, it suffices to prove the original inequality for $b=c=d=t$ and $a=1/t^3$, where $t>0$; that is,

$$\frac{t^9}{t^9+pt^6+q}+\frac{3}{1+pt+qt^3}\ge 1,$$

$$\frac{3}{1+pt+qt^3} \ge \frac{pt^6+q}{t^9+pt^6+q},$$

$$(3-pq)t^9 - p^2t^7 + 2pt^6 - q^2t^3 - pqt + 2q \ge 0,$$

$$[(p+q)^2 - 3pq]t^9 - 3p^2t^7 + 2p(p+q)t^6 - 3q^2t^3 - 3pqt + 2q(p+q) \ge 0,$$

$$Ap^2 + Bq^2 \ge Cpq,$$

where

$$A = t^9 - 3t^7 + 2t^6 = t^6(t-1)^2(t+2) \ge 0,$$

$$B = t^9 - 3t^3 + 2 = (t^3-1)^2(t^3+2) \ge 0,$$

$$C = t^9 - 2t^6 + 3t - 2.$$

Since $A \ge 0$ and $B \ge 0$, it suffices to consider that $C \ge 0$ and to show that $4AB \ge C^2$. From

$$t^3 - 3t + 2 = (t-1)^2(t+2) \ge 0,$$

we get $3t - 2 \le t^3$. Therefore

$$C \le t^9 - 2t^6 + t^3 = t^3(t^3-1)^2,$$

hence

$$4AB - C^2 \ge 4AB - t^6(t^3-1)^4 = t^6(t-1)^2(t^3-1)^2[4(t+2)(t^3+2) - (t^2+t+1)^2]$$

$$= t^6(t-1)^2(t^3-1)^2(3t^4 + 6t^3 - 3t^2 + 6t + 15) \ge 0.$$

The proof is completed. The inequality holds for $a = b = c = d = 1$.

Remark. Similarly, we can prove the following generalization.

- *Let a, b, c, d be positive real numbers such that $abcd = 1$. If p and q are nonnegative real numbers such that $p + q \ge 3$, then then*

$$\frac{1}{1+pa+qa^3} + \frac{1}{1+pb+qb^3} + \frac{1}{1+pc+qc^3} + \frac{1}{1+pd+qd^3} \ge \frac{4}{1+p+q}.$$

□

P 1.51. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be positive real numbers such that $a_1a_2\ldots a_n = 1$. If $p, q \ge 0$ such that $p + q \ge n - 1$, then*

$$\frac{1}{1+pa_1+qa_1^2} + \frac{1}{1+pa_2+qa_2^2} + \cdots + \frac{1}{1+pa_n+qa_n^2} \ge \frac{n}{1+p+q}.$$

(Vasile Cirtoaje, 2007)

Solution. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

where

$$f(u) = \frac{1}{1 + pe^u + qe^{2u}}, \quad u \in \mathbb{R}.$$

For $u \geq 0$, we have

$$\begin{aligned} f''(u) &= \frac{e^u[4q^2e^{3u} + 3pqe^{2u} + (p^2 - 4q)e^u - p]}{(1 + pe^u + qe^{2u})^3} \\ &\geq \frac{e^{2u}[4q^2 + 3pq + (p^2 - 4q) - p]}{(1 + pe^u + qe^{2u})^3} \\ &= \frac{e^{2u}[(p + 2q)(p + q - 2) + 2q^2 + p]}{(1 + pe^u + qe^{2u})^3} > 0, \end{aligned}$$

therefore $f(u)$ is convex for $u \geq 0$. By HCF Theorem, it suffices to prove the original inequality for $a_2 = \cdots = a_n := t$ and $a_1 = 1/t^{n-1}$, where $t > 0$. Write this inequality as

$$\frac{t^{2n-2}}{t^{2n-2} + pt^{n-1} + q} + \frac{n-1}{1 + pt + qt^2} \geq \frac{n}{1 + p + q}.$$

Applying the Cauchy-Schwarz inequality, it suffices to prove that

$$\frac{(t^{n-1} + n - 1)^2}{(t^{2n-2} + pt^{n-1} + q) + (n-1)(1 + pt + qt^2)} \geq \frac{n}{1 + p + q},$$

which is equivalent to

$$pB + qC \geq A,$$

where

$$A = (n-1)(t^{n-1} - 1)^2 \geq 0,$$

$$B = (t^{n-1} - 1)^2 + nE = \frac{A}{n-1} + nE, \quad E = t^{n-1} + n - 2 - (n-1)t,$$

$$C = (t^{n-1} - 1)^2 + nF = \frac{A}{n-1} + nF, \quad F = 2t^{n-1} + n - 3 - (n-1)t^2.$$

By the AM-GM inequality applied to $n-1$ positive numbers, we have $E \geq 0$ and $F \geq 0$ for $n \geq 3$. Since $A \geq 0$ and $p + q \geq n - 1$, we have

$$pB + qC - A \geq pB + qC - \frac{(p+q)A}{n-1} = n(pE + qF) \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark 1. For $p=2k$ and $q=k^2$, we get the following result.

- *Let $a_1,a_2,\cdots,a_n$ $(n\ge 3)$ be positive real numbers such that $a_1a_2\cdots a_n=1$. If $k\ge\sqrt{n}-1$, then*

$$\frac{1}{(1+ka_1)^2}+\frac{1}{(1+ka_2)^2}+\cdots+\frac{1}{(1+ka_n)^2}\ge\frac{n}{(1+k)^2},$$

with equality for $a_1=a_2=\cdots=a_n=1$.

In addition, for $n=4$ and $k=1$, we get the known inequality (*Vasile Cirtoaje*, 1999):

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2}+\frac{1}{(1+d)^2}\ge 1,$$

where $a,b,c,d>0$ such that $abcd=1$.

Remark 2. For $p+q=n-1$, we get the beautiful inequality

$$\frac{1}{1+pa_1+qa_1^2}+\frac{1}{1+pa_2+qa_2^2}+\cdots+\frac{1}{1+pa_n+qa_n^2}\ge 1,\quad n\ge 3,$$

which is a generalization of the following inequalities:

$$\frac{1}{1+(n-1)a_1}+\frac{1}{1+(n-1)a_2}+\cdots+\frac{1}{1+(n-1)a_n}\ge 1,$$

$$\frac{1}{[1+(\sqrt{n}-1)a_1]^2}+\frac{1}{[1+(\sqrt{n}-1)a_1]^2}+\cdots+\frac{1}{[1+(\sqrt{n}-1)a_1]^2}\ge 1,$$

$$\frac{1}{2+(n-1)(a_1+a_1^2)}+\frac{1}{2+(n-1)(a_2+a_2^2)}+\cdots+\frac{1}{2+(n-1)(a_n+a_n^2)}\ge\frac{1}{2}.$$

Remark 3. Similarly, we can prove the following statement:

- *Let $a_1,a_2,\ldots,a_n$ $(n\ge 4)$ be positive real numbers such that $a_1a_2\ldots a_n=1$. If $p,q,r\ge 0$ such that $p+q+r\ge n-1$, then*

$$\sum_{i=1}^{n}\frac{1}{1+pa_i+qa_i^2+ra_i^3}\ge\frac{n}{1+p+q+r}.$$

For $n=4$ and $p+q+r=3$, we get the beautiful inequality

$$\sum_{i=1}^{4}\frac{1}{1+pa_i+qa_i^2+ra_i^3}\ge 1.$$

If $p = q = r = 1$, then we get the known inequality (*Vasile Cirtoaje*, 1999):

$$\sum_{i=1}^{4} \frac{1}{1 + a_i + a_i^2 + a_i^3} \geq 1.$$

Since $2a_i^2 \leq a_i + a_i^3$, the best inequality with respect to q if for $q = 0$; that is,

$$\sum_{i=1}^{4} \frac{1}{1 + pa_i + ra_i^3} \geq 1, \quad p + r = 3.$$

For $p = 1$ and $p = 2$, we get the following strong inequalities:

$$\sum_{i=1}^{4} \frac{1}{1 + a_i + 2a_i^3} \geq 1,$$

$$\sum_{i=1}^{4} \frac{1}{1 + 2a_i + a_i^3} \geq 1.$$

Actually, the following generalization holds.

• *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \ldots a_n = 1$, and let $k_1, k_2, \ldots, k_m \geq 0$ such that $k_1 + k_2 + \cdots + k_m \geq n - 1$. If $m \leq n - 1$, then*

$$\sum_{i=1}^{n} \frac{1}{1 + k_1 a_i + k_2 a_i^2 + \cdots + k_m a_i^m} \geq \frac{n}{1 + k_1 + k_2 + \cdots + k_m}. \qquad (*)$$

For $m = n - 1$ and $k_1 = k_2 = \cdots = k_m = 1$, (*) turns into the known beautiful inequality

$$\sum_{i=1}^{n} \frac{1}{1 + a_i + a_i^2 + \cdots + a_i^{n-1}} \geq 1.$$

Since

$$(m-1)a_i^k \leq (m-k)a_i + (k-1)a_i^m, \quad k = 2, 3, \ldots, m-1$$

(by the AM-GM inequality), the best inequality (*) with respect to $k_2, \cdots, k_{m-1}$ is for $k_2, \cdots, k_{m-1} = 0$; that is,

$$\sum_{i=1}^{n} \frac{1}{1 + k_1 a_i + k_m a_i^m} \geq \frac{n}{1 + k_1 + k_m}, \qquad k_1 + k_m \geq n - 1, \ \ 1 \leq m \leq n - 1.$$

If $k_1 + k_m = n - 1$ and $m = n - 1$, then

$$\sum_{i=1}^{n} \frac{1}{1 + k_1 a_i + k_{n-1} a_i^{n-1}} \geq 1.$$

For $k_1 = 1$ and $k_1 = n-2$, we get the following strong inequalities:

$$\sum_{i=1}^{n} \frac{1}{1+a_i+(n-2)a_i^{n-1}} \geq 1,$$

$$\sum_{i=1}^{n} \frac{1}{1+(n-2)a_i+a_i^{n-1}} \geq 1.$$

□

P 1.52. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\ldots a_n = 1$. If $k \geq n^2-1$, then*

$$\frac{1}{\sqrt{1+ka_1}} + \frac{1}{\sqrt{1+ka_2}} + \cdots + \frac{1}{\sqrt{1+ka_n}} \geq \frac{n}{\sqrt{1+k}}.$$

Solution. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

where

$$f(u) = \frac{1}{\sqrt{1+ke^u}}, \quad u \in \mathbb{R}.$$

For $u \geq 0$, we have

$$f''(u) = \frac{ke^u(ke^u-2)}{4(1+ke^u)^{5/2}} \geq \frac{ke^u(k-2)}{4(1+ke^u)^{5/2}} > 0.$$

Therefore, f is convex on $[0, \infty)$. By HCF Theorem and Remark 5, it suffices to prove the original inequality for $a_2 = \cdots = a_n := t$ and $a_1 = 1/t^{n-1}$, where $t \geq 1$. Write this inequality as $h(t) \geq 0$, where

$$h(t) = \sqrt{\frac{t^{n-1}}{t^{n-1}+k}} + \frac{n-1}{\sqrt{1+kt}} - \frac{n}{\sqrt{1+k}}.$$

The derivative

$$h'(t) = \frac{(n-1)kt^{(n-3)/2}}{2(t^{n-1}+k)^{3/2}} - \frac{(n-1)k}{2(kt+1)^{3/2}}$$

has the same sign as

$$h_1(t) = t^{n/3-1}(kt+1) - t^{n-1} - k.$$

Denoting $m = n/3$, $m \geq 2/3$, we see that

$$h_1(t) = kt^m + t^{m-1} - t^{3m-1} - k = k(t^m - 1) - t^{m-1}(t^{2m} - 1) = (t^m - 1)h_2(t),$$

where

$$h_2(t) = k - t^{m-1} - t^{2m-1}.$$

For $t > 1$, we have

$$\begin{aligned} h_2'(t) &= t^{m-2}[-m+1-(2m-1)t^m] < t^{m-2}[-m+1-(2m-1)] \\ &= -(3m-2)t^{m-2} \le 0, \end{aligned}$$

hence $h_2(t)$ is strictly decreasing for $t \ge 1$. Since $h_2(1) = k-2 > 0$ and $\lim_{t\to\infty} h_2(t) = -\infty$, there exists $t_1 > 1$ such that $h_2(t_1) = 0$, $h_2(t) > 0$ for $t \in (1, t_1)$, and $h_2(t) < 0$ for $t \in (t_1, \infty)$. Since h_2, h_1 and h' has the same sign for $t > 1$, $h(t)$ is strictly increasing for $t \in [1, t_1]$ and strictly decreasing for $t \in [t_1, \infty)$; this yields $h(t) \ge \min\{h(1), h(\infty)\}$. From $h(1) = 0$ and $h(\infty) = 1 - \dfrac{n}{\sqrt{1+k}} \ge 0$, it follows that $h(t) \ge 0$ for all $t \ge 1$. The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. The following generalization holds (*Vasile Cirtoaje*, 2005):

- *Let $a_1, a_2, \cdots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n = 1$. If k and m are positive numbers such that*

$$m \le n-1, \qquad k \ge n^{1/m} - 1,$$

then

$$\frac{1}{(1+ka_1)^m} + \frac{1}{(1+ka_2)^m} + \cdots + \frac{1}{(1+ka_n)^m} \ge \frac{n}{(1+k)^m},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$.

For $0 < m \le n-1$ and $k = n^{1/m} - 1$, we get the beautiful inequality

$$\frac{1}{(1+ka_1)^m} + \frac{1}{(1+ka_2)^m} + \cdots + \frac{1}{(1+ka_n)^m} \ge 1.$$

□

P 1.53. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 a_2 \ldots a_n = 1$, then*

$$\frac{1}{1+a_1+\cdots+a_1^{n-1}} + \frac{1}{1+a_2+\cdots+a_2^{n-1}} + \cdots + \frac{1}{1+a_n+\cdots+a_n^{n-1}} \ge 1.$$

(Vasile Cirtoaje, 2007)

Solution. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

where

$$f(u) = \frac{1}{1 + e^u + \cdots + e^{(n-1)u}}, \quad u \in \mathbb{R}.$$

We will show that f is convex for $u \ge 0$. Setting $t = e^u$, $t \ge 1$, the necessary and sufficient condition $f''(u) \ge 0$ for $u \ge 0$ is equivalent to

$$2A^2 \ge B(1 + C),$$

where

$$A = t + 2t^2 + \cdots + (n-1)t^{n-1},$$

$$B = t + 4t^2 + \cdots + (n-1)^2 t^{n-1},$$

$$C = t + t^2 + \cdots + t^{n-1}.$$

We will prove this inequality by induction on n. For $n = 2$, the inequality becomes $t(t-1) \ge 0$, which is clearly true for $t \ge 1$. Assume now that the inequality is true for n and prove it for $n+1$, $n \ge 2$. So, we need to show that $2A^2 \ge B(1 + C)$ involves

$$2(A + nt^n)^2 \ge (B + n^2 t^n)(1 + C + t^n),$$

which is equivalent to

$$2A^2 - B(1 + C) + t^n[n^2(t^n - 1) + D] \ge 0,$$

where

$$D = 2nA - B - n^2 C = \sum_{i=1}^{n-1} b_i t^i, \quad b_i = 3n^2 - (2n - i)^2.$$

Since $2A^2 - B(1 + C) \ge 0$, it suffices to show that $D \ge 0$. Since

$$b_1 < b_2 < \cdots < b_{n-1}, \quad t \le t^2 \le \cdots \le t^{n-1},$$

we may apply Chebyshev's inequality to get

$$D \ge \frac{1}{n}(b_1 + b_2 + \cdots + b_{n-1})(t + t^2 + \cdots + t^{n-1}).$$

Thus, it suffices to show that $b_1 + b_2 + \cdots + b_{n-1} \ge 0$. Indeed,

$$b_1 + b_2 + \cdots + b_{n-1} = \sum_{i=1}^{n-1}[3n^2 - (2n - i)^2] = \frac{n(n-1)(4n+1)}{6} > 0.$$

By HCF Theorem and Remark 5, it suffices to prove the original inequality for $a_2 = \cdots = a_n := t$ and $a_1 = 1/t^{n-1}$, where $t \ge 1$. Setting $k = n-1$, $k \ge 1$, we need to show that

$$\frac{t^{k^2}}{1+t^k+\cdots+t^{k^2}} + \frac{k}{1+t+\cdots+t^k} \ge 1.$$

For the nontrivial case $t > 1$, this inequality is equivalent to the following sequence of inequalities:

$$\frac{k}{1+t+\cdots+t^k} \ge \frac{1+t^k+\cdots+t^{(k-1)k}}{1+t^k+\cdots+t^{k^2}},$$

$$\frac{k(t-1)}{t^{k+1}-1} \ge \frac{t^{k^2}-1}{t^k-1}\cdot\frac{t^k-1}{t^{(k+1)k}-1},$$

$$\frac{k(t-1)}{t^{k+1}-1} \ge \frac{t^{k^2}-1}{t^{(k+1)k}-1},$$

$$k\frac{t^{k(k+1)}-1}{t^{k+1}-1} \ge \frac{t^{k^2}-1}{t-1},$$

$$k\left[1+t^{k+1}+t^{2(k+1)}+\cdots+t^{(k-1)(k+1)}\right] \ge 1+t+t^2+\cdots+t^{(k-1)(k+1)},$$

$$k\left[1\cdot 1+t\cdot t^k+\cdots+t^{k-1}\cdot t^{(k-1)k}\right] \ge \left(1+t+\cdots+t^{k-1}\right)\left[1+t^k+\cdots+t^{(k-1)k}\right].$$

Since $1 < t < \cdots < t^{k-1}$ and $1 < t^k < \cdots < t^{(k-1)k}$, the last inequality follows from Chebyshev's inequality. This completes the proof. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.54. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2\ldots a_n = 1$. If $p, q \ge 0$ such that $0 < p+q \le \dfrac{1}{n-1}$, then*

$$\frac{1}{1+pa_1+qa_1^2} + \frac{1}{1+pa_2+qa_2^2} + \cdots + \frac{1}{1+pa_n+qa_n^2} \le \frac{n}{1+p+q}.$$

(Vasile Cirtoaje, 2007)

Solution. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that

$$f(x_1)+f(x_2)+\cdots+f(x_n) \ge nf(s), \quad s = \frac{x_1+x_2+\cdots+x_n}{n} = 0,$$

where

$$f(u) = \frac{-1}{1+pe^u+qe^{2u}}, \quad u \in \mathbb{R}.$$

For $u \le 0$, we have

$$\begin{aligned} f''(u) &= \frac{e^u[-4q^2e^{3u}-3pqe^{2u}+(4q-p^2)e^u+p]}{(1+pe^u+qe^{2u})^3} \\ &\ge \frac{e^{2u}[-4q^2-3pq+(4q-p^2)+p]}{(1+pe^u+qe^{2u})^3} \\ &= \frac{e^{2u}[(p+4q)(1-p-q)+2pq]}{(1+pe^u+qe^{2u})^3} \ge 0, \end{aligned}$$

therefore $f(u)$ is convex for $u \le 0$. By HCF Theorem, it suffices to prove the original inequality for $a_2 = \cdots = a_n := t$ and $a_1 = 1/t^{n-1}$, where $t > 0$. Write this inequality as

$$\frac{t^{2n-2}}{t^{2n-2}+pt^{n-1}+q} + \frac{n-1}{1+pt+qt^2} \le \frac{n}{1+p+q},$$

$$p^2A + q^2B + pqC \le pD + qE,$$

where

$$A = t^{n-1}(t^n - nt + n - 1), \quad B = t^{2n} - nt^2 + n - 1,$$

$$C = t^{2n-1} + t^{2n} - nt^{n+1} + (n-1)t^{n-1} - nt + n - 1,$$

$$D = t^{n-1}[(n-1)t^n - nt^{n-1} + 1], \quad E = (n-1)t^{2n} - nt^{2n-2} + 1.$$

Applying the AM-GM inequality to n positive numbers yields $D \ge 0$ and $E \ge 0$. Then, since $p + q \le 1/(n-1)$ involves $pD + qE \ge (n-1)(p+q)(pD+qE)$, it suffices to show that

$$p^2A + q^2B + pqC \le (n-1)(p+q)(pD+qE).$$

Write this inequality as

$$p^2A_1 + q^2B_1 + pqC_1 \ge 0,$$

where

$$A_1 = (n-1)D - A = nt^n[(n-2)t^{n-1} - (n-1)t^{n-2} + 1],$$

$$B_1 = (n-1)E - B = nt^2[(n-2)t^{2n-2} - (n-1)t^{2n-4} + 1],$$

$$C_1 = (n-1)(D+E) - C = ny[(n-2)(t^{2n-1} + t^{2n-2}) - 2(n-1)t^{2n-3} + t^n + 1].$$

Applying the AM-GM inequality to $n-1$ nonnegative numbers yields $A_1 \ge 0$ and $B_1 \ge 0$. So, it suffices to show that $C_1 \ge 0$. Indeed, we have

$$(n-2)(t^{2n-1} + t^{2n-2}) - 2(n-1)t^{2n-3} + t^n + 1 = A_2 + B_2 + C_2,$$

where

$$A_2 = (n-2)t^{2n-1} - (n-1)t^{2n-3} + t \ge 0,$$

$$B_2 = (n-2)t^{2n-2} - (n-1)t^{2n-3} + t^{n-1} \ge 0,$$

$$C_2 = t^n - t^{n-1} - t + 1 = (t-1)(t^{n-1}-1) \geq 0.$$

The inequalities $A_2 \geq 0$ and $B_2 \geq 0$ follows by the AM-GM inequality applied to $n-1$ nonnegative numbers. This completes the proof. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark 1. For $p + q = \dfrac{1}{n-1}$, we get the inequality

$$\frac{1}{1+pa_1+qa_1^2} + \frac{1}{1+pa_2+qa_2^2} + \cdots + \frac{1}{1+pa_n+qa_n^2} \leq n-1,$$

which is a generalization of the following inequalities:

$$\frac{1}{n-1+a_1} + \frac{1}{n-1+a_2} + \cdots + \frac{1}{n-1+a_n} \leq 1,$$

$$\frac{1}{2n-2+a_1+a_1^2} + \frac{1}{2n-2+a_2+a_2^2} + \cdots + \frac{1}{2n-2+a_n+a_n^2} \leq \frac{1}{2}.$$

Remark 2. For $p = 2k$ and $q = k^2$, we get the following result:

- *Let $a_1, a_2, \cdots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n = 1$. If*

$$0 < k \leq \sqrt{\frac{n}{n-1}} - 1,$$

then

$$\frac{1}{(1+ka_1)^2} + \frac{1}{(1+ka_2)^2} + \cdots + \frac{1}{(1+ka_n)^2} \leq \frac{n}{(1+k)^2},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.55. *Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be positive real numbers such that $a_1 a_2 \ldots a_n = 1$. If*

$$0 < k \leq \left(\frac{n}{n-1}\right)^2 - 1,$$

then

$$\frac{1}{\sqrt{1+ka_1}} + \frac{1}{\sqrt{1+ka_2}} + \cdots + \frac{1}{\sqrt{1+ka_n}} \leq \frac{n}{\sqrt{1+k}}.$$

Solution. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \dots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

where

$$f(u) = \frac{-1}{\sqrt{1 + ke^u}}, \quad u \in \mathbb{R}.$$

For $u \le 0$, we have

$$f''(u) = \frac{ke^u(2 - ke^u)}{4(1 + ke^u)^{5/2}} \ge \frac{ke^u(2 - k)}{4(1 + ke^u)^{5/2}} > 0.$$

Therefore, f is convex on $(-\infty, 0]$. By HCF Theorem and Remark 5, it suffices to prove the original inequality for $a_2 = \cdots = a_n := t$ and $a_1 = 1/t^{n-1}$, where $0 < t \le 1$. Write this inequality as $h(t) \le 0$, where

$$h(t) = \sqrt{\frac{t^{n-1}}{t^{n-1} + k}} + \frac{n - 1}{\sqrt{1 + kt}} - \frac{n}{\sqrt{1 + k}}.$$

The derivative

$$h'(t) = \frac{(n - 1)kt^{(n-3)/2}}{2(t^{n-1} + k)^{3/2}} - \frac{(n - 1)k}{2(kt + 1)^{3/2}}$$

has the same sign as

$$h_1(t) = t^{n/3-1}(kt + 1) - t^{n-1} - k.$$

Denoting $m = n/3$, $m \ge 1$, we see that

$$h_1(t) = kt^m + t^{m-1} - t^{3m-1} - k = -k(1 - t^m) + t^{m-1}(1 - t^{2m}) = (1 - t^m)h_2(t),$$

where

$$h_2(t) = t^{m-1} + t^{2m-1} - k$$

is strictly increasing for $t \in (0, 1]$. There are two possible cases: $h_2(0) \ge 0$ and $h_2(0) < 0$.

Case 1: $h_2(0) \ge 0$. This case is possible only for $m = 1$ $(n = 3)$ and $k \le 1$, when $h_2(t) = t + 1 - k > 0$ for $t \in (0, 1]$. Also, we have $h_1(t) > 0$ and $h'(t) > 0$ for $t \in (0, 1)$. Therefore, h is strictly increasing on $[0, 1]$, hence $h(t) \le h(1) = 0$.

Case 2: $h_2(0) < 0$. This case is possible for either $m = 1$ $(n = 3)$ and $1 < k \le 5/4$, or $m > 1$ $(n \ge 4)$. Since $h_2(1) = 2 - k > 0$, there exists $t_1 \in (0, 1)$ such that $h_2(t_1) = 0$, $h_2(t) < 0$ for $t \in (0, t_1)$, and $h_2(t) > 0$ for $t \in (t_1, 1)$. Since h' has the same sign as h_2 on $(0, 1)$, it follows that h is strictly decreasing on $[0, t_1]$, and strictly increasing on $[t_1, 1]$. Therefore, $h(t) \le \min\{h(0), h(1)\}$. Since $h(0) = n - 1 - \dfrac{n}{\sqrt{1 + k}} \le 0$ and $h(1) = 0$, we have $h(t) \le 0$ for all $t \in (0,1]$. This completes the proof. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. The following generalization holds (*Vasile Cirtoaje*, 2005):

- *Let $a_1, a_2, \cdots, a_n$ $(n \ge 3)$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. If k and m are positive numbers such that*

$$m \ge \frac{1}{n-1}, \qquad k \le \left(\frac{n}{n-1}\right)^{1/m} - 1,$$

then

$$\frac{1}{(1+ka_1)^m} + \frac{1}{(1+ka_2)^m} + \cdots + \frac{1}{(1+ka_n)^m} \le \frac{n}{(1+k)^m},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$.

For $n \ge 3$, $m \ge \dfrac{1}{n-1}$ and $k = \left(\dfrac{n}{n-1}\right)^{1/m} - 1$, we get the beautiful inequality

$$\frac{1}{(1+ka_1)^m} + \frac{1}{(1+ka_2)^m} + \cdots + \frac{1}{(1+ka_n)^m} \le n-1.$$

□

P 1.56. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1a_2\ldots a_n = 1$, then*

$$a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} + n(n-2) \ge (n-1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

Solution. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

where

$$f(u) = e^{(n-1)u} - (n-1)e^{-u}, \quad u \in \mathbb{R}.$$

For $u \ge 0$, we have

$$f''(u) = (n-1)^2e^{(n-1)u} - (n-1)e^{-u} = (n-1)e^{-u}[(n-1)e^{nu} - 1] \ge 0;$$

therefore, $f(u)$ is convex on $[0, \infty)$. By HCF Theorem and Remark 2, it suffices to show that $H(x,y) \ge 0$ for $x, y \in \mathbb{R}$ such that $x + (n-1)y = 0$, where

$$H(x,y) = \frac{f'(x) - f'(y)}{x-y}.$$

From

$$f'(u) = (n-1)[e^{(n-1)u} + e^{-u}],$$

we get

$$\begin{aligned} H(x,y) &= \frac{(n-1)(e^x - e^y)}{x-y}\left[e^{(n-2)x} + e^{(n-3)x+y} + \cdots + e^{x+(n-3)y} + e^{(n-2)y} - e^{-x-y}\right] \\ &= \frac{(n-1)(e^x - e^y)}{x-y}\left[e^{(n-2)x} + e^{(n-3)x+y} + \cdots + e^{x+(n-3)y})\right]. \end{aligned}$$

Since $(e^x - e^y)/(x-y) > 0$, we have $H(x,y) > 0$. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.57. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1a_2 \ldots a_n = 1$. If $k \ge n$, then*

$$a_1^k + a_2^k + \cdots + a_n^k + kn \ge (k+1)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

(Vasile Cirtoaje, 2006)

Solution. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

where

$$f(u) = e^{ku} - (k+1)e^{-u}, \quad u \in \mathbb{R}.$$

For $u \ge 0$, we have

$$f''(u) = k^2e^{ku} - (k+1)e^{-u} = e^{-u}\left[k^2e^{(k+1)u} - k - 1\right] \ge e^{-u}(k^2 - k - 1) > 0;$$

therefore, f is convex on $[0, \infty)$. By HCF Theorem and Remark 5, it suffices to to prove the original inequality for $a_2 = \cdots = a_n := b \ge 1$ and $a_1 := a \le 1$, $ab^{n-1} = 1$; that is

$$a^k + (n-1)b^k - \frac{k+1}{a} - \frac{(k+1)(n-1)}{b} + kn \ge 0.$$

By the weighted AM-GM inequality, we have

$$a^k + (kn-k-1) \ge [1 + (kn-k-1)](a^k)^{\frac{1}{1+(kn-k-1)}} = \frac{k(n-1)}{b}.$$

Then, we still have to show that

$$(n-1)\left(b^k - \frac{1}{b}\right) - (k+1)\left(\frac{1}{a} - 1\right) \ge 0,$$

which is equivalent to $h(b) \ge 0$ for $b \ge 1$, where

$$h(b) = (n-1)(b^{k+1} - 1) - (k+1)(b^n - b).$$

Since

$$\begin{aligned}
\frac{h'(b)}{k+1} &= (n-1)b^k - nb^{n-1} + 1 \ge (n-1)b^n - nb^{n-1} + 1 \\
&= nb^{n-1}(b-1) - (b^n - 1) \\
&= (b-1)\left[(b^{n-1} - b^{n-2}) + (b^{n-1} - b^{n-3}) + \cdots + (b^{n-1} - 1)\right] \ge 0,
\end{aligned}$$

h is increasing on $[1,\infty)$, hence $h(b) \ge h(1) = 0$. The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.58. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 a_2 \ldots a_n = 1$, then*

$$\left(1 - \frac{1}{n}\right)^{a_1} + \left(1 - \frac{1}{n}\right)^{a_2} + \cdots + \left(1 - \frac{1}{n}\right)^{a_n} \le n - 1.$$

(Vasile Cîrtoaje, 2006)

Solution. Let

$$k = \frac{n}{n-1}, \quad k > 1,$$

and

$$m = \ln k, \quad 0 < m \le \ln 2 < 1.$$

Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

where

$$f(u) = -k^{-e^u}, \quad u \in \mathbb{R}.$$

From

$$f''(u) = me^u k^{-e^u}(1 - me^u),$$

it follows that $f''(u) > 0$ for $u \le 0$, since

$$1 - me^u \ge 1 - m \ge 1 - \ln 2 > 0.$$

Therefore, f is convex on $(-\infty, 0]$. By HCF Theorem and Remark 5, it suffices to prove the original inequality for $a_2 = \cdots = a_n := t$ and $a_1 = t^{-n+1}$, where $0 < t \le 1$. Write this inequality as

$$h(t) \le n - 1,$$

where

$$h(t) = k^{-t^{-n+1}} + (n-1)k^{-t}, \quad t \in (0,1].$$

We have

$$h'(t) = (n-1)mt^{-n}k^{-t^{-n+1}}h_1(t), \quad h_1(t) = 1 - t^n k^{-t^{-n+1}-t},$$

$$h_1'(t) = k^{-t^{-n+1}-t}h_2(t), \quad h_2(t) = [m(n-1+t^n) - nt^{n-1},$$

$$h_2'(t) = nt^{n-2}(mt-n+1) \le nt^{n-2}(m-n+1) \le nt^{n-2}(m-1) < 0,$$

hence h_2 is strictly decreasing on $[0,1]$. Since $h_2(0) = (n-1)m > 0$ and $h_2(1) = n(m-1) < 0$, there is $t_1 \in (0,1)$ such that $h_2(t_1) = 0$, $h_2(t) > 0$ for $t \in [0,t_1)$ and $h_2(t) < 0$ for $t \in (t_1,1]$. Therefore, h_1 is strictly increasing on $(0,t_1]$ and is strictly decreasing on $[t_1,1]$. Since $\lim_{t\to 0} h_1(u) = -\infty$ and $h_1(1) = 0$, there is $t_2 \in (0,t_1)$ such that $h_1(t_2) = 0$, $h_1(t) < 0$ for $t \in (0,t_2)$ and $h_1(t) > 0$ for $t \in (t_2,1)$. Thus, h is strictly decreasing on $(0,t_2]$ and is strictly increasing on $[t_2,1]$. Since $\lim_{t\to 0} h(u) = n-1$ and $h(1) = n-1$, we have $h(t) \le n-1$ for all $t \in (0,1]$. This completes the proof. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.59. *If a,b,c are positive real numbers such that $abc = 1$, then*

$$\frac{1}{1+\sqrt{1+3a}} + \frac{1}{1+\sqrt{1+3b}} + \frac{1}{1+\sqrt{1+3c}} \le 1.$$

(Vasile Cîrtoaje, 2008)

Solution. Using the substitution

$$a = e^{-x}, \quad b = e^{-y}, \quad c = e^{-z},$$

we need to show that

$$f(x) + f(y) + f(z) \ge 3f(s), \quad s = \frac{x+y+z}{3} = 0,$$

where

$$f(u) = \frac{-3}{1+\sqrt{1+3e^{-u}}} = e^u - \sqrt{e^{2u}+3e^u}, \quad u \in \mathbb{R}.$$

For $u \ge 0$, which involves $t = e^u \ge 1$, we have

$$f''(u) = t\left[1 - \frac{4t^2+18t+9}{4(t+3)\sqrt{t(t+3)}}\right] > 0$$

since

$$16t(t+3)^3 - (4t^2+18t+9)^2 = 9(4t^2+12t-9) > 0.$$

Therefore, f is half convex for $u \ge 0$. By HCF Theorem, it suffices to prove that $f(x) + 2f(y) \ge 3f(0)$, where $x, y \in \mathbb{R}$ such that $x + 2y = 0$. Substituting $a = e^x$ and $b = e^y$, we need to show that

$$a - \sqrt{a^2 + 3a} + 2(b - \sqrt{b^2 + 3b}\ ) \ge -3,$$

where $a, b > 0$ such that $ab^2 = 1$. Write this inequality as

$$2b^3 + 3b^2 + 1 \ge \sqrt{3b^2 + 1} + 2b^2\sqrt{b^2 + 3b}.$$

Squaring and dividing by b^2, the inequality becomes

$$9b^2 + 4b + 3 \ge 4\sqrt{(b^2 + 3b)(3b^2 + 1)}.$$

Since

$$2\sqrt{(b^2 + 3b)(3b^2 + 1)} \le (b^2 + 3b) + (3b^2 + 1) = 4b^2 + 3b + 1,$$

we get

$$9b^2 + 4b + 3 - 4\sqrt{(b^2 + 3b)(3b^2 + 1)} \ge 9b^2 + 4b + 3 - 2(4b^2 + 3b + 1) = (b - 1)^2 \ge 0.$$

The equality holds for $a = b = c = 1$.

Remark. Similarly, we can prove the following generalization.

- *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n = 1$. If*

$$0 < k \le \frac{4n}{(n-1)^2},$$

then

$$\frac{1}{1 + \sqrt{1 + ka_1}} \frac{1}{1 + \sqrt{1 + ka_2}} + \cdots + \frac{1}{1 + \sqrt{1 + ka_n}} \le \frac{n}{1 + \sqrt{1 + k}}.$$

□

P 1.60. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 a_2 \cdots a_n = 1$, then*

$$\frac{1}{1 + \sqrt{1 + 4n(n-1)a_1}} \frac{1}{1 + \sqrt{1 + 4n(n-1)a_2}} + \cdots + \frac{1}{1 + \sqrt{1 + 4n(n-1)a_n}} \ge \frac{1}{2}.$$

(Vasile Cîrtoaje, 2008)

Solution. Let $k = 4n(n-1)$, $k \ge 8$. Using the substitutions $a_i = e^{-x_i}$ for $i = 1, 2, \dots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s), \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

where

$$f(u) = \frac{k}{1 + \sqrt{1 + ke^{-u}}} = \sqrt{e^{2u} + ke^u} - e^u, \quad u \in \mathbb{R}.$$

For $u \le 0$, which involves $t = e^u \in (0, 1]$, we have

$$f''(u) = t\left[\frac{4t^2 + 6kt + k^2}{4(t+k)\sqrt{t(t+k)}} - 1\right] > 0$$

since

$$(4t^2 + 6kt + k^2)^2 - 16t(t+k)^3 = k^2(k^2 - 4kt - 4t^2) \ge k^2(k^2 - 4k - 4) > 0.$$

Therefore, f is convex on $(-\infty, 0]$. By HCF Theorem, it suffices to prove that $f(x) + (n-1)f(y) \ge nf(0)$, where $x, y \in \mathbb{R}$ such that $x + (n-1)y = 0$. Substituting $a = e^x$ and $b = e^y$, we need to show that

$$\sqrt{a^2 + ka} - a + (n-1)(\sqrt{b^2 + kb} - b) \ge n(\sqrt{1+k} - 1),$$

where $a, b > 0$ such that $ab^{n-1} = 1$. Write this inequality as

$$\sqrt{4n(n-1)b^{n-1} + 1} + (n-1)\sqrt{4n(n-1)b^{2n-1} + b^{2n}} \ge (n-1)b^n + 2n(n-1)b^{n-1} + 1.$$

By Minkowski's inequality, we have

$$\sqrt{4n(n-1)b^{n-1} + 1} + (n-1)\sqrt{4n(n-1)b^{2n-1} + b^{2n}} \ge$$

$$\ge \sqrt{4n(n-1)b^{n-1}[1 + (n-1)b^{n/2}]^2 + [1 + (n-1)b^n]^2}.$$

Thus, it suffices to show that

$$4n(n-1)b^{n-1}[1 + (n-1)b^{n/2}]^2 + [1 + (n-1)b^n]^2 \ge [(n-1)b^n + 2n(n-1)b^{n-1} + 1]^2,$$

which is equivalent to

$$4n(n-1)^2 b^{\frac{3n-2}{2}}\left[2 + (n-2)b^{\frac{n}{2}} - nb^{\frac{n-2}{2}}\right] \ge 0.$$

This inequality follows immediately by the AM-GM inequality. Thus, the proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.61. *If a, b, c are positive real numbers such that $abc = 1$, then*

$$\frac{a^6}{1+2a^5}+\frac{b^6}{1+2b^5}+\frac{c^6}{1+2c^5} \ge 1.$$

(Vasile Cîrtoaje, 2008)

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x)+f(y)+f(z) \ge 3f(s), \quad s = \frac{x+y+z}{3} = 0,$$

where

$$f(u) = \frac{e^{6u}}{1+2e^{5u}}, \quad u \in \mathbb{R}.$$

For $u \le 0$, which involves $w = e^u \in (0,1]$, we have

$$f''(u) = \frac{2w^6(2-w^5)(9-2w^5)}{(1+2w^5)^3} > 0.$$

Therefore, f is convex for $u \le 0$. By HCF Theorem, it suffices to prove the original inequality for $b = c := t$ and $a = 1/t^2$, where $t > 0$; that is,

$$\frac{1}{t^2(t^{10}+2)}+\frac{2t^6}{1+2t^5} \ge 1.$$

Since

$$1+2t^5 \le 1+t^4+t^6,$$

it suffices to show that

$$\frac{1}{x(x^5+2)}+\frac{2x^3}{1+x^2+x^3} \ge 1, \quad x = \sqrt{t}.$$

This inequality can be written as follows:

$$x^3(x^6-x^5-x^3+2x-1)+(x-1)^2 \ge 0,$$

$$x^3(x-1)^2(x^4+x^3+x^2-1)+(x-1)^2 \ge 0,$$

$$(x-1)^2[x^7+x^5+(x^6-x^3+1)] \ge 0.$$

Clearly, the last inequality is true. The equality holds for $a = b = c = 1$.

□

P 1.62. *If a, b, c are positive real numbers such that $abc = 1$, then*

$$\sqrt{25a^2+144}+\sqrt{25b^2+144}+\sqrt{25c^2+144} \le 5(a+b+c)+24.$$

(Vasile Cîrtoaje, 2008)

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x)+f(y)+f(z) \ge 3f(s), \quad s = \frac{x+y+z}{3} = 0,$$

where

$$f(u) = 5e^u - \sqrt{25e^{2u}+144}, \quad u \in \mathbb{R}.$$

We will show that f is convex for $u \le 0$. From

$$f''(u) = 5w\left[1 - \frac{5w(25w^2+288)}{(25w^2+144)^{3/2}}\right], \quad w = e^u \in (0,1],$$

we need to show that

$$(25w^2+144)^3 \ge 25w^2(25w^2+288)^2.$$

Setting $25w^2 = 144z$, $z \in (0, 25/144]$, we have

$$(25w^2+144)^3 - 25w^2(25w^2+288)^2 = 144^3(z+1)^3 - 144^3z(z+2)^2 = 144^3(1-z-z^2) > 0.$$

By HCF Theorem, it suffices to prove the original inequality for $a = t^2$ and $b = c = 1/t$, where $t > 0$; that is,

$$5t^3+24t+10 \ge \sqrt{25t^6+144t^2}+2\sqrt{25+144t^2}.$$

Squaring and dividing by $4t$ give

$$60t^3+25t^2-36t+120 \ge \sqrt{(25t^4+144)(144t^2+25)}.$$

Squaring again and dividing by 120, the inequality becomes

$$25t^5-36t^4+105t^3-112t^2-72t+90 \ge 0,$$

$$(t-1)^2(25t^3+14t^2+108t+90) \ge 0.$$

Since the last inequality is true, the proof is completed. The equality holds for $a = b = c = 1$.

□

P 1.63. *If a, b, c are positive real numbers such that $abc = 1$, then*

$$\sqrt{16a^2+9}+\sqrt{16b^2+9}+\sqrt{16c^2+9} \geq 4(a+b+c)+3.$$

(Vasile Cîrtoaje, 2008)

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z,$$

we need to show that

$$f(x)+f(y)+f(z) \geq 3f(s), \quad s = \frac{x+y+z}{3} = 0,$$

where

$$f(u) = \sqrt{16e^{2u}+9} - 4e^u, \quad u \in \mathbb{R}.$$

We will show that f is convex for $u \geq 0$. From

$$f''(u) = 4w\left[\frac{4w(16w^2+18)}{(16w^2+9)^{3/2}} - 1\right], \quad w = e^u \geq 1,$$

we need to show that

$$16w^2(16w^2+18)^2 \geq (16w^2+9)^3.$$

Setting $16w^2 = 9z$, $z \geq 16/9$, we have
Indeed,

$$16w^2(16w^2+18)^2 - (16w^2+9)^3 = 729z(z+2)^2 - 729(z+1)^3 = 729(z^2+z-1) > 0.$$

By HCF Theorem, it suffices to prove the original inequality for $a = t^2$ and $b = c = 1/t$, where $t > 0$; that is,

$$\sqrt{16a^6+9a^2} + 2\sqrt{16+9a^2} \geq 4t^3 + 3t + 8.$$

Squaring and dividing by $4a$ give

$$\sqrt{(16a^4+9)(9a^2+16)} \geq 6a^3 + 16a^2 - 9a + 12.$$

Squaring again and dividing by $12a$, the inequality becomes

$$9a^5 - 16a^4 + 9a^3 + 12a^2 - 32a + 18 \geq 0,$$

$$(a-1)^2(9a^3 + 2a^2 + 4a + 18) \geq 0.$$

Since the last inequality is true, the proof is completed. The equality holds for $a = b = c = 1$.

□

P 1.64. *If a,b,c,d are real numbers such that $a+b+c+d=4$, then*

$$\frac{a}{a^2-a+4}+\frac{b}{b^2-b+4}+\frac{c}{c^2-c+4}+\frac{d}{d^2-d+4}\le 1.$$

(Sqing, 2015)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)\ge 4f(s),\qquad s=\frac{a+b+c+d}{4}=1,$$

where

$$f(u)=\frac{-u}{u^2-u+4},\qquad u\in\mathbb{R}.$$

From

$$f'(u)=\frac{u^2-4}{(u^2-u+4)^2},\qquad f''(u)=\frac{2(-u^3+12u-4)}{(u^2-u+4)^3},$$

it follows that f is increasing on $(-\infty,-2]\cup[2,\infty)$, decreasing on $[-2,2]$ and convex on $[1,2]$. Define the function

$$f_0(u)=\begin{cases} f(u), & u\le 2\\ f(2), & u>2\end{cases}.$$

Since $f_0(u)\le f(u)$ for $u\in\mathbb{R}$ and $f_0(1)=f(1)$, it suffices to show that

$$f_0(a)+f_0(b)+f_0(c)+f_0(d)\ge 4f_0(s),\qquad s=\frac{a+b+c+d}{4}=1.$$

It is easy to check that f_0 is convex on $[1,\infty)$. Therefore, by HCF Theorem and Remark 5, we only need to show that $f_0(x)+3f_0(y)\ge 4f_0(1)$ for all $x,y\in\mathbb{R}$ such that $x\le 1\le y$ and $x+3y=4$. There are two cases to consider: $y\le 2$ and $y>2$.

Case 1: $y\le 2$. We need to show that $f(x)+3f(y)\ge 3f(1)$ for all $x,y\in\mathbb{R}$ such that $x+3y=4$. According to Remark 1, this is true if $h(x,y)\ge 0$ for $x+3y=4$. We have

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{u-4}{4(u^2-u+4)},$$

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{4(x+y)-xy}{4(x^2-x+4)(y^2-y+4)}=\frac{3(y-2)^2+4}{4(x^2-x+4)(y^2-y+4)}>0.$$

Case 2: $y>2$. From $y>2$ and $x+3y=4$, we get $x<-2$ and

$$f_0(x)+3f_0(y)-4f_0(1)=f(x)+3f(2)-4f(1)=\frac{-x}{x^2-x+4}>0.$$

The equality holds for $a=b=c=d=1$. □

P 1.65. *If a, b, c are nonnegative real numbers such that $a+b+c=12$, then*

$$(a^2+10)(b^2+10)(c^2+10) \ge 13310.$$

(Vasile Cîrtoaje, 2006)

Solution. Let

$$f(u)=\ln(u^2+10), \quad u \in [0,\infty).$$

From

$$f''(u)=\frac{2(10-u^2)}{(u^2+10)^2},$$

it follows that f is convex on $[0,\sqrt{10}]$ and concave on $[\sqrt{10},\infty)$. According to LCRCF-Theorem, the sum $f(a)+f(b)+f(c)$ is minimum when two of a, b, c are equal. Therefore, it suffices to prove the original inequality for $b=c$. So, we need to show that

$$(a^2+10)(b^2+10)^2 \ge 13310$$

for $a+2b=12$. Write this inequality as follows:

$$[(12-2b)^2+10](b^2+10)^2 \ge 13310,$$

$$(2b^2-24b+77)(b^4+20b^2+100) \ge 6655,$$

$$2b^6-24b^5+117b^4-480b^3+1740b^2-2400b+1045 \ge 0,$$

$$(b-1)^2(2b^4-20b^3+75b^2-310b+1045) \ge 0,$$

$$(b-1)^2[2b^2(b-5)^2+5(5b^2-62b+209)] \ge 0.$$

The last inequality is true since

$$5b^2-62b+209=5\left(b-\frac{31}{5}\right)^2+\frac{84}{5}>0.$$

The proof is completed. The equality holds for $a=10$ and $b=c=1$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization.

- *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_n=2n(n-1)$, then*

$$(a_1^2+k)(a_2^2+k)\cdots(a_n^2+k) \ge k(k+1)^n, \qquad k=(n-1)(2n-1).$$

The equality holds for $a_1=k$ and $a_2=\cdots=a_n=1$ (or any cyclic permutation).

□

P 1.66. *Let a,b,c be nonnegative real numbers. If*

$$k_0 \le k \le 3, \quad k_0 = \frac{\ln 2}{\ln 3 - \ln 2} \approx 1.71,$$

then

$$a^k(b+c) + b^k(c+a) + c^k(a+b) \le 2\left(\frac{a+b+c}{2}\right)^{k+1}.$$

Solution. Due to homogeneity, we may assume that $a+b+c=2$. Write the inequality as

$$f(a)+f(b)+f(c) \le 2,$$

where

$$f(u) = u^k(2-u), \quad u \in [0,\infty).$$

From

$$f''(u) = ku^{k-2}[2k-2-(k+1)u],$$

it follows that therefore, f is convex on $\left[0, \frac{2k-2}{k+1}\right]$ and concave on $\left[\frac{2k-2}{k+1}, 2\right]$. By LCRCF-Theorem, the sum $f(a)+f(b)+f(c)$ is maximum when $a=0$ or $0<a\le b=c$.

Case 1: $a=0$. We need to show that

$$bc(b^{k-1}+c^{k-1}) \le 2$$

for $b+c=2$. Since $0<(k-1)/2 \le 1$, Bernoulli's inequality gives

$$b^{k-1}+c^{k-1} = (b^2)^{(k-1)/2} + (c^2)^{(k-1)/2} \le 1 + \frac{k-1}{2}(b^2-1) + 1 + \frac{k-1}{2}(b^2-1)$$
$$= 3-k+\frac{k-1}{2}(b^2+c^2).$$

Thus, it suffices to show that

$$(3-k)bc + \frac{k-1}{2}bc(b^2+c^2) \le 2.$$

Since

$$bc \le \left(\frac{b+c}{2}\right)^2 = 1,$$

we only need to show that

$$bc(b^2+c^2) \le 2.$$

Indeed, we have

$$8[2-bc(b^2+c^2)] = (b+c)^4 - 8bc(b^2+c^2) = (b-c)^4 \ge 0.$$

Case 2: $0 < a \le b = c$. We only need to prove the original homogeneous inequality for $b = c = 1$ and $0 < a \le 1$; that is,

$$\left(1+\frac{a}{2}\right)^{k+1} - a^k - a - 1 \ge 0.$$

Since $\left(1+\frac{a}{2}\right)^{k+1}$ is increasing and a^k is decreasing when k increases, it suffices to prove that $g(a) \ge 0$ for $0 < a \le 1$, where

$$g(a) = \left(1+\frac{a}{2}\right)^{k_0+1} - a^{k_0} - a - 1.$$

We have

$$g'(a) = \frac{k_0+1}{2}\left(1+\frac{a}{2}\right)^{k_0} - k_0 a^{k_0-1} - 1,$$

$$\frac{1}{k_0}g''(a) = \frac{k_0+1}{4}\left(1+\frac{a}{2}\right)^{k_0} - \frac{k_0-1}{a^{2-k_0}}.$$

Since g'' is increasing on $(0,1]$, $\lim_{x\to 0} g''(a) = -\infty$ and

$$\frac{1}{k_0}g''(1) = \frac{k_0+1}{4}\left(\frac{3}{2}\right)^{k_0} - k_0 + 1 = \frac{k_0+1}{2} - k_0 + 1 = \frac{3-k_0}{2} > 0,$$

there exists $a_1 \in (0,1)$ such that $g''(a_1) = 0$, $g''(a) < 0$ for $a \in (0, a_1)$, and $g''(a) > 0$ for $a \in (a_1, 1]$. Therefore, g' is strictly decreasing on $[0, a_1]$ and strictly increasing on $[a_1, 1]$. Since $g'(0) = \frac{k_0-1}{2} > 0$ and $g'(1) = \frac{k_0+1}{2}\left[(3/2)^{k_0} - 2\right] = 0$, there exists $a_2 \in (0, a_1)$ such that $g'(a_2) = 0$, $g'(a) > 0$ for $a \in [0, a_2)$, and $g'(a) < 0$ for $a \in (a_2, 1)$. Thus, g is strictly increasing on $[0, a_2]$, and strictly decreasing on $[a_2, 1]$. Consequently,

$$g(a) \ge \min\{g(0), g(1)\},$$

and from

$$g(0) = 0, \quad g(1) = (3/2)^{k_0+1} - 3 = 0,$$

we get $g(a) \ge 0$.

This completes the proof. The equality holds for $a = 0$ and $b = c$ (or any cyclic permutation). If $k = k_0$, then the equality holds also for $a = b = c$.

□

P 1.67. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then*

$$(n+1)^2\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \ge 4(n+2)(a_1^2 + a_2^2 + \cdots + a_n^2) + n(n^2 - 3n - 6).$$

(Vasile Cîrtoaje, 2006)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge n(n^2-3n-6),$$

where

$$f(u)=\frac{(n+1)^2}{u}-4(n+2)u^2, \quad u\in(0,\infty).$$

From

$$f''(u)=\frac{2(n+1)^2}{u^3}-8(n+2),$$

it follows that f is strictly convex on (0,c] and strictly concave on $[c,\infty)$, where

$$c=\sqrt[3]{\frac{(n+1)^2}{4(n+2)}}.$$

According to LCRCF-Theorem and Remark 6, it suffices to consider the case

$$a_1=a_2=\cdots=a_{n-1}=x,\ \ 0<x\le 1, \quad a_n=n-(n-1)x,$$

when the inequality becomes as follows:

$$(n+1)^2\left(\frac{n-1}{x}+\frac{1}{a_n}\right)\ge 4(n+2)[(n-1)x^2+a_n^2)+n(n^2-3n-6),$$

$$n(n-1)(2x-1)^2[(n+2)(n-1)x^2-(n+2)(2n-1)x+(n+1)^2]\ge 0.$$

The last inequality is true since

$$(n+2)(n-1)x^2-(n+2)(2n-1)x+(n+1)^2=(n+2)(n-1)\left(x-\frac{2n-1}{2n-2}\right)^2+\frac{3(n-2)}{4(n-1)}\ge 0.$$

The equality holds for $a_1=a_2=\cdots=a_{n-1}=\frac{1}{2}$ and $a_n=\frac{n+1}{2}$ (or any cyclic permutation).

□

Chapter 2

Half Convex Function Method for Ordered Variables

2.1 Theoretical Basis

Half Convex Function Theorem for Ordered Variables (*Vasile Cirtoaje*, 2007). *Let $f(u)$ be a function defined on a real interval $\mathbb{I}$ and convex on $\mathbb{I}_{u\geq s}$ / $\mathbb{I}_{u\leq s}$, where $s\in\mathbb{I}$. The inequality*

$$f(a_1)+f(a_2)+\cdots+f(a_n)\geq nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

holds for all $a_1,a_2,\ldots,a_n\in\mathbb{I}$ such that $a_1+a_2+\cdots+a_n=ns$ and at least $n-m$ of $a_1,a_2,\ldots,a_n$ are smaller/greater than or equal to s if and only if

$$f(x)+mf(y)\geq(1+m)f(s)$$

for all $x,y\in\mathbb{I}$ such that $x+my=(1+m)s$.

Proof (for the *right convexity*). The necessity is obvious. Without loss of generality, assume that

$$a_1\leq a_2\leq\cdots\leq a_n,\quad a_1\leq\cdots\leq a_{n-m}\leq s.$$

Since at least $n-m$ of $a_1,a_2,\ldots,a_n$ are smaller than or equal to s , there is an integer $k\in\{n-m,\ldots,n-1\}$ such that

$$a_1\leq\cdots\leq a_k\leq s\leq a_{k+1}\leq\cdots\leq a_n.$$

Since f is convex on $\mathbb{I}_{u\geq s}$, we can apply Jensen's inequality to get

$$f(a_{k+1})+\cdots+f(a_n)\geq(n-k)f(z),\quad z=\frac{a_{k+1}+\cdots+a_n}{n-k}.$$

Thus, it suffices to show that

$$f(a_1)+\cdots+f(a_k)+(n-k)f(z)\ge nf(s).$$

Let $b_1,\ldots,b_k\in\mathbb{I}$ defined by

$$a_i+mb_i=(1+m)s,\quad i=1,\ldots,k.$$

We claim that

$$z\ge b_1\ge\cdots\ge b_k\ge s.$$

Indeed, we have

$$b_1\ge\cdots\ge b_k,$$

$$b_k-s=\frac{s-a_k}{m}\ge 0,$$

$$\begin{aligned}mb_1&=(1+m)s-a_1=(m-n+1)s+(a_2+\cdots+a_k)+(a_{k+1}+\cdots+a_n)\\&\le(m-n+1)s+(k-1)s+a_{k+1}+\cdots+a_n=(m-n+k)s+(a_{k+1}+\cdots+a_n)\\&=(m-n+k)s+(n-k)z=(m-n+k)(s-z)+mz\le mz.\end{aligned}$$

By hypothesis, we have

$$f(a_1)+mf(b_1)\ge(1+m)f(s),$$

$$\cdots$$

$$f(a_k)+mf(b_k)\ge(1+m)f(s),$$

hence

$$f(a_1)+\cdots+f(a_k)+m[f(b_1)+\cdots+f(b_k)]\ge k(1+m)f(s).$$

Consequently, it suffices to show that

$$k(1+m)f(s)-m[f(b_1)+\cdots+f(b_k)]+(n-k)f(z)\ge nf(s),$$

which is equivalent to

$$pf(z)+(k-p)f(s)\ge f(b_1)+\cdots+f(b_k),\quad p=\frac{n-k}{m}\le 1.$$

By Jensen's inequality, we have

$$pf(z)+(1-p)f(s)\ge f(w),\quad w=pz+(1-p)s\ge s.$$

Thus, we only need to show that

$$f(w)+(k-1)f(s)\ge f(b_1)+\cdots+f(b_k).$$

Since the decreasingly ordered vector $\vec{A_k} = (w, s, \ldots, s)$ majorizes the decreasingly ordered vector $\vec{B_k} = (b_1, b_2, \ldots, b_k)$, this inequality follows from Karamata's inequality for convex functions.

The proof for the *left convexity* is similar.

Remark 1. Let us denote

$$g(u) = \frac{f(u) - f(s)}{u - s}, \quad h(x, y) = \frac{g(x) - g(y)}{x - y}.$$

In many applications, it is useful to replace the hypothesis

$$f(x) + mf(y) \geq (1 + m)f(s)$$

in Half Convex Function Theorem for Ordered Variables (HCF-OV Theorem) by the equivalent condition

$$h(x, y) \geq 0 \ \textit{ for all } \ x, y \in \mathbb{I} \ \textit{ such that } \ x + my = (1 + m)s.$$

This equivalence is true since

$$\begin{aligned} f(x) + mf(y) - (1 + m)f(s) &= [f(x) - f(s)] + m[f(y) - f(s)] \\ &= (x - s)g(x) + m(y - s)g(y) \\ &= \frac{m}{1 + m}(x - y)[g(x) - g(y)] \\ &= \frac{m}{1 + m}(x - y)^2 h(x, y). \end{aligned}$$

Remark 2. Assume that f is differentiable on $\mathbb{I}$, and let

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$

Then, the desired inequality of Jensen's type in HCF-OV Theorem holds true by replacing the hypothesis

$$f(x) + mf(y) \geq (1 + m)f(s)$$

with the more restrictive condition

$$H(x, y) \geq 0 \ \textit{ for all } \ x, y \in \mathbb{I} \ \textit{ such that } \ x + my = (1 + m)s.$$

To prove this claim, we will show that the new condition implies $f(x) + mf(y) \geq (1 + m)f(s)$ for all $x, y \in \mathbb{I}$ such that $x + my = (1 + m)s$. Write this inequality as $f_1(x) \geq (1 + m)f(s)$, where

$$f_1(x) = f(x) + mf(y) = f(x) + mf\left(\frac{(1 + m)s - x}{m}\right).$$

From

$$f_1'(x) = f'(x) - f'\left(\frac{(1+m)s - x}{m}\right) = f'(x) - f'(y) = \frac{1+m}{m}(x-s)H(x,y),$$

it follows that f_1 is decreasing for $x \le s$ and increasing for $x \ge s$; therefore,

$$f_1(x) \ge f_1(s) = (1+m)f(s).$$

Remark 3. HCF-OV Theorem is also valid in the case when $\mathbb{I} = [a,b] \setminus \{u_0\}$ or $\mathbb{I} = (a,b) \setminus \{u_0\}$, where a, b, u_0 are real numbers such that $a < u_0 < b$. Clearly, two cases are possible:

(1) $u_0 < s$ - when f is right convex, i.e. convex on $\mathbb{I}_{u \ge s}$;

(2) $u_0 > s$ - when f is left convex, i.e. convex on $\mathbb{I}_{u \le s}$.

Remark 4. In HCF-OV Theorem for the *right convexity* (when HCF-OV Theorem is called RHCF-OV Theorem), it suffices to consider

$$x \le s \le y.$$

This claim is true because in the proof of HCF-OV Theorem, the hypothesis

$$f(x) + mf(y) \ge (1+m)f(s)$$

is used to get the inequalities

$$f(a_i) + mf(b_i) \ge (1+m)f(s), \quad i = 1, 2, \cdots, k,$$

where $a_i \le s \le b_i$.

Similarly, in HCF-OV Theorem for the *left convexity* (when HCF-OV Theorem is called LHCF-OV Theorem), it suffices to consider

$$x \ge s \ge y.$$

Remark 5. HCF-OV Theorem is a generalization of HCF-Theorem, because the last theorem can be obtained from the first theorem for $m = n - 1$.

Remark 6. If

$$a_1 \ge \cdots \ge a_m \ge s \ge a_{m+1} \ge \cdots \ge a_n,$$

then at least $n - m$ of $a_1, a_2, \ldots, a_n$ are smaller than or equal to s. Also, if

$$a_1 \ge \cdots \ge a_{n-m} \ge s \ge a_{n-m+1} \ge \cdots \ge a_n,$$

then at least $n - m$ of $a_1, a_2, \ldots, a_n$ are greater than or equal to s. Thus, from HCF-OV Theorem and Remark 4, we get the following corollaries.

RHCF-OV Corollary. *Let $f(u)$ be a function defined on a real interval $\mathbb{I}$ and convex on $\mathbb{I}_{u\geq s}$, where $s \in \mathbb{I}$. The inequality*

$$f(a_1)+f(a_2)+\cdots+f(a_n) \geq nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

holds for all $a_1,a_2,\ldots,a_n \in \mathbb{I}$ satisfying

$$a_1 \geq \cdots \geq a_m \geq s \geq a_{m+1} \geq \cdots \geq a_n, \qquad a_1+a_2+\cdots+a_n = ns,$$

if and only if

$$f(x)+mf(y) \geq (1+m)f(s)$$

for all $x,y \in \mathbb{I}$ such that $x \leq s \leq y$ and $x+my=(1+m)s$.

LHCF-OV Corollary. *Let $f(u)$ be a function defined on a real interval $\mathbb{I}$ and convex on $\mathbb{I}_{u\leq s}$, where $s \in \mathbb{I}$. The inequality*

$$f(a_1)+f(a_2)+\cdots+f(a_n) \geq nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

holds for all $a_1,a_2,\ldots,a_n \in \mathbb{I}$ satisfying

$$a_1 \leq \cdots \leq a_m \leq s \leq a_{m+1} \leq \cdots \leq a_n, \qquad a_1+a_2+\cdots+a_n = ns,$$

if and only if

$$f(x)+mf(y) \geq (1+m)f(s)$$

for all $x,y \in \mathbb{I}$ such that $x \geq s \geq y$ and $x+my=(1+m)s$.

Remark 7. The inequality in RHCF-OV Corollary becomes an equality for

$$a_1=a_2=\cdots=a_n=s$$

and also for

$$a_1=\cdots=a_m=y, \quad a_{m+1}=\cdots=a_{n-1}=s, \quad a_n=x \quad (x<y),$$

where $x,y \in \mathbb{I}$ satisfy the equations

$$x+my=(1+m)s, \quad f(x)+mf(y)=(1+m)f(s).$$

For $x \neq y$, these equations are equivalent to

$$x+my=(1+m)s, \quad h(x,y)=0.$$

Remark 8. The inequality in LHCF-OV Corollary becomes an equality for

$$a_1=a_2=\cdots=a_n=s$$

and also for

$$a_1 = x, \quad a_2 = \cdots = a_{n-m} = s, \quad a_{n-m+1} = \cdots = a_n = y \quad (x > y),$$

where $x, y \in \mathbb{I}$ satisfy the equations

$$x + my = (1+m)s, \quad f(x) + mf(y) = (1+m)f(s).$$

For $x \neq y$, these equations are equivalent to

$$x + my = (1+m)s, \quad h(x,y) = 0.$$

2.2 Applications

2.1. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$.

(a) If at least $n - m$ of $a_1, a_2, \ldots, a_n$ are smaller than or equal to 1, then

$$m(a_1^3 + a_2^3 + \cdots + a_n^3 - n) \ge (2m+1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n);$$

(b) If at least $n - m$ of $a_1, a_2, \ldots, a_n$ are greater than or equal to 1, then

$$a_1^3 + a_2^3 + \cdots + a_n^3 - n \le (m+2)(a_1^2 + a_2^2 + \cdots + a_n^2 - n).$$

2.2. If a, b, c, d are real numbers such that

$$a \ge b \ge 1 \ge c \ge d, \qquad a + b + c + d = 4,$$

then

$$(3a^2 - 2)(a-1)^2 + (3b^2 - 2)(b-1)^2 + (3c^2 - 2)(c-1)^2 + (3d^2 - 2)(d-1)^2 \ge 0.$$

2.3. If $a_1, a_2, \ldots, a_{2n} \ge \dfrac{-2n-1}{n-1}$ such that

$$a_1 \ge \cdots \ge a_n \ge 1 \ge a_{n+1} \ge \cdots \ge a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n,$$

then

$$a_1^3 + a_2^3 + \cdots + a_{2n}^3 \ge 2n.$$

2.4. Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 + a_2 + \cdots + a_n = n$.

(a) If $a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n$, then

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \ge 6(a_1^2 + a_2^2 + \cdots + a_n^2 - n);$$

(b) If $a_1 \ge a_2 \ge 1 \ge a_3 \ge \cdots \ge a_n$, then

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \ge \frac{14}{3}(a_1^2 + a_2^2 + \cdots + a_n^2 - n);$$

(c) If $a_1 \ge \cdots \ge a_{n-2} \ge 1 \ge a_{n-1} \ge a_n$, then

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \ge \frac{2(n^2 - 3n + 3)}{n^2 - 5n + 7}\,(a_1^2 + a_2^2 + \cdots + a_n^2 - n).$$

2.5. Let a,b,c,d,e be nonnegative real numbers such that $a+b+c+d+e=5$.

(a) If $a\ge b\ge 1\ge c\ge d\ge e$, then

$$21(a^2+b^2+c^2+d^2+e^2)\ge a^4+b^4+c^4+d^4+e^4+100;$$

(b) If $a\ge b\ge c\ge 1\ge d\ge e$, then

$$13(a^2+b^2+c^2+d^2+e^2)\ge a^4+b^4+c^4+d^4+e^4+60.$$

2.6. Let $a_1,a_2,\dots,a_n$ be nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$.

(a) If $a_1\ge\cdots\ge a_{n-1}\ge 1\ge a_n$, then

$$7(a_1^3+a_2^3+\cdots+a_n^3)\ge 3(a_1^4+a_2^4+\cdots+a_n^4)+4n;$$

(b) If $a_1\ge\cdots\ge a_{n-2}\ge 1\ge a_{n-1}\ge a_n$, then

$$13(a_1^3+a_2^3+\cdots+a_n^3)\ge 4(a_1^4+a_2^4+\cdots+a_n^4)+9n.$$

2.7. If $a_1,a_2,\dots,a_n$ are positive real numbers such that

$$a_1\ge\cdots\ge a_{n-m}\ge 1\ge a_{n-m+1}\ge\cdots\ge a_n,\qquad a_1+a_2+\cdots+a_n=n,$$

then

$$(m+1)^2\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-n\right)\ge 4m(a_1^2+a_2^2+\cdots+a_n^2-n).$$

2.8. If $a_1,a_2,\dots,a_n$ are positive real numbers such that

$$a_1\le\cdots\le a_{n-m}\le 1\le a_{n-m+1}\le\cdots\le a_n,\qquad \frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}=n,$$

then

$$a_1^2+a_2^2+\cdots+a_n^2-n\ge 2\left(1+\frac{\sqrt{m}}{1+m}\right)(a_1+a_2+\cdots+a_n-n).$$

2.9. Let $a_1,a_2,\dots,a_n$ be nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$.

(a) If $a_1\ge 1\ge a_2\ge\cdots\ge a_n$, then

$$\frac{1}{a_1^2+2}+\frac{1}{a_2^2+2}+\cdots+\frac{1}{a_n^2+2}\ge\frac{n}{3};$$

(b) If $a_1\ge a_2\ge 1\ge a_3\ge\cdots\ge a_n$, then

$$\frac{1}{2a_1^2+3}+\frac{1}{2a_2^2+3}+\cdots+\frac{1}{2a_n^2+3}\ge\frac{n}{5}.$$

2.10. If $a_1, a_2, \ldots, a_{2n}$ are nonnegative real numbers such that

$$a_1 \ge \cdots \ge a_n \ge 1 \ge a_{n+1} \ge \cdots \ge a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n,$$

then

$$\frac{1}{na_1^2 + n^2 + n + 1} + \frac{1}{na_2^2 + n^2 + n + 1} + \cdots + \frac{1}{na_{2n}^2 + n^2 + n + 1} \le \frac{2n}{(n+1)^2}.$$

2.11. If a, b, c, d, e, f are nonnegative real numbers such that

$$a \ge b \ge c \ge 1 \ge d \ge e \ge f, \quad a + b + c + d + e + f = 6,$$

then

$$\frac{3a+4}{3a^2+4} + \frac{3b+4}{3b^2+4} + \frac{3c+4}{3c^2+4} + \frac{3d+4}{3d^2+4} + \frac{3e+4}{3e^2+4} + \frac{3f+4}{3f^2+4} \le 6.$$

2.12. If a, b, c, d, e, f are nonnegative real numbers such that

$$a \ge b \ge 1 \ge c \ge d \ge e \ge f, \quad a + b + c + d + e + f = 6,$$

then

$$\frac{a^2-1}{(2a+7)^2} + \frac{b^2-1}{(2b+7)^2} + \frac{c^2-1}{(2c+7)^2} + \frac{d^2-1}{(2d+7)^2} + \frac{e^2-1}{(2e+7)^2} + \frac{f^2-1}{(2f+7)^2} \ge 0.$$

2.13. If a, b, c, d, e, f are nonnegative real numbers such that

$$a \ge b \ge c \ge d \ge 1 \ge e \ge f, \quad a + b + c + d + e + f = 6,$$

then

$$\frac{a^2-1}{(2a+5)^2} + \frac{b^2-1}{(2b+5)^2} + \frac{c^2-1}{(2c+5)^2} + \frac{d^2-1}{(2d+5)^2} + \frac{e^2-1}{(2e+5)^2} + \frac{f^2-1}{(2f+5)^2} \le 0.$$

2.14. If a, b, c, d are nonnegative real numbers such that

$$a \ge b \ge 1 \ge c \ge d, \quad a + b + c + d = 4,$$

then

$$\frac{1}{2a^3+5} + \frac{1}{2b^3+5} + \frac{1}{2c^3+5} + \frac{1}{2d^3+5} \le \frac{4}{7}.$$

2.15. If $a_1, a_2, \ldots, a_8$ are nonnegative real numbers such that

$$a_1 \geq a_2 \geq a_3 \geq a_4 \geq 1 \geq a_5 \geq a_6 \geq a_7 \geq a_8, \quad a_1 + a_2 + \cdots + a_8 = 8,$$

then

$$(a_1^2 + 1)(a_2^2 + 1) \cdots (a_8^2 + 1) \geq (a_1 + 1)(a_2 + 1) \cdots (a_8 + 1).$$

2.16. If a, b, c, d are positive real numbers such that

$$a \geq b \geq 1 \geq c \geq d, \quad abcd = 1,$$

then

$$a^2 + b^2 + c^2 + d^2 - 4 \geq 18\left(a + b + c + d - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} - \frac{1}{d}\right).$$

2.17. If a, b, c, d are positive real numbers such that

$$a \geq b \geq 1 \geq c \geq d, \quad abcd = 1,$$

then

$$\sqrt{a^2 - a + 1} + \sqrt{b^2 - b + 1} + \sqrt{c^2 - c + 1} + \sqrt{d^2 - d + 1} \geq a + b + c + d.$$

2.18. If a, b, c, d are positive real numbers such that

$$a \geq 1 \geq b \geq c \geq d, \quad abcd = 1,$$

then

$$\frac{1}{a^3 + 3a + 2} + \frac{1}{b^3 + 3b + 2} + \frac{1}{c^3 + 3c + 2} + \frac{1}{d^3 + 3d + 2} \geq \frac{2}{3}.$$

2.19. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that

$$a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n, \quad a_1 a_2 \cdots a_n = 1.$$

If $k \geq 1$, then

$$\frac{1}{1 + ka_1} + \frac{1}{1 + ka_2} + \cdots + \frac{1}{1 + ka_n} \geq \frac{n}{1 + k}.$$

2.20. If $a_1, a_2, \ldots, a_9$ are positive real numbers such that

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_9, \qquad a_1 a_2 \cdots a_9 = 1,$$

then

$$\frac{1}{(a_1+2)^2} + \frac{1}{(a_2+2)^2} + \cdots + \frac{1}{(a_9+2)^2} \ge 1.$$

2.21. If $a_1, a_2, \ldots, a_n$ $(n \ge 4)$ are positive real numbers such that

$$a_1 \ge a_2 \ge a_3 \ge 1 \ge a_4 \ge \cdots \ge a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{(a_1+1)^2} + \frac{1}{(a_2+1)^2} + \cdots + \frac{1}{(a_n+1)^2} \ge \frac{n}{4}.$$

2.22. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{(a_1+3)^2} + \frac{1}{(a_2+3)^2} + \cdots + \frac{1}{(a_n+3)^2} \le \frac{n}{16}.$$

2.23. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1 a_2 \cdots a_n = 1.$$

If $p, q \ge 0$ such that $p + q \le 1$, then

$$\frac{1}{1+pa_1+qa_1^2} + \frac{1}{1+pa_2+qa_2^2} + \cdots + \frac{1}{1+pa_n+qa_n^2} \le \frac{n}{1+p+q}.$$

2.24. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$a_1 \le a_2 \le 1 \le a_3 \le \cdots \le a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{(a_1+5)^2} + \frac{1}{(a_2+5)^2} + \cdots + \frac{1}{(a_n+5)^2} \le \frac{n}{36}.$$

2.25. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{\sqrt{1+3a_1}} + \frac{1}{\sqrt{1+3a_2}} + \cdots + \frac{1}{\sqrt{1+3a_n}} \ge \frac{n}{2}.$$

2.26. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{\sqrt{1+2a_1}} + \frac{1}{\sqrt{1+2a_2}} + \cdots + \frac{1}{\sqrt{1+2a_n}} \le \frac{n}{\sqrt{3}}.$$

2.3 Solutions

P 2.1. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$.*

(a) If at least $n - m$ of $a_1, a_2, \ldots, a_n$ are smaller than or equal to 1, then

$$m(a_1^3 + a_2^3 + \cdots + a_n^3 - n) \ge (2m+1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n);$$

(b) If at least $n - m$ of $a_1, a_2, \ldots, a_n$ are greater than or equal to 1, then

$$a_1^3 + a_2^3 + \cdots + a_n^3 - n \le (m+2)(a_1^2 + a_2^2 + \cdots + a_n^2 - n).$$

(Vasile Cirtoaje, 2007)

Solution. (a) Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = mu^3 - (2m+1)u^2, \quad u \in [0, n].$$

From $f''(u) = 2(3mu - 2m - 1)$, it follows that f is convex on $[1, n]$. By HCF-OV Theorem (or RHCF-OV Corollary) and Remark 1, it suffices to show that $h(x, y) \ge 0$ for all $x, y \ge 0$ such that $x + my = m + 1$. We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = m(u^2 + u + 1) - (2m+1)(u+1) = mu^2 - (m+1)u - m - 1,$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = m(x + y) - m - 1 = (m-1)x \ge 0.$$

From $x + my = m + 1$ and $h(x, y) = 0$, we get $x = 0$, $y = (m+1)/m$. Therefore, in accordance with Remark 7, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \cdots = a_m = \frac{m+1}{m}, \quad a_{m+1} = \cdots = a_{n-1} = 1, \quad a_n = 0$$

(or any thereof permutation).

(b) Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = (m+2)u^2 - u^3, \quad u \in [0, n].$$

From $f''(u) = 2(m+2-3u)$, it follows that f is convex on $[0,1]$. By HCF-OV Theorem (or LRHCF-OV Corollary) and Remark 1, it suffices to show that $h(x,y) \ge 0$ for all $x, y \ge 0$ such that $x + my = m+1$. We have

$$g(u) = \frac{f(u)-f(1)}{u-1} = (m+2)(u+1) - (u^2+u+1) = -u^2 + (m+1)u + m + 1,$$

$$h(x,y) = \frac{g(x)-g(y)}{x-y} = -(x+y) + m + 1 = (m-1)y \ge 0.$$

From $x+my = m+1$ and $h(x,y) = 0$, we get $x = m+1$, $y = 0$. Therefore, in accordance with Remark 8, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = m+1, \quad a_2 = \cdots = a_{n-m} = 1, \quad a_{n-m+1} = \cdots = a_n = 0$$

(or any thereof permutation).

Remark 1. For $m = n-1$, we get the following results:

- *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then*

$$(n-1)(a_1^3 + a_2^3 + \cdots + a_n^3 - n) \ge (2n-1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n),$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = 0$ and $a_2 = a_3 = \cdots = a_n = \frac{n}{n-1}$ (or any cyclic permutation).

- *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then*

$$a_1^3 + a_2^3 + \cdots + a_n^3 - n \le (n+1)(a_1^2 + a_2^2 + \cdots + a_n^2 - n),$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = n$ and $a_2 = a_3 = \cdots = a_n = 0$ (or any cyclic permutation).

Remark 2. For $m = 1$, we get the following results:

- *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that*

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \quad a_1 + a_2 + \cdots + a_n = n,$$

then

$$a_1^3 + a_2^3 + \cdots + a_n^3 + 2n \ge 3(a_1^2 + a_2^2 + \cdots + a_n^2),$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = 2$, $a_2 = \cdots = a_{n-1} = 1$, $a_n = 0$.

- *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that*

$$a_1 \ge \cdots \ge a_{n-1} \ge 1 \ge a_n, \quad a_1 + a_2 + \cdots + a_n = n,$$

then

$$a_1^3 + a_2^3 + \cdots + a_n^3 + 2n \le 3(a_1^2 + a_2^2 + \cdots + a_n^2),$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$*, and also for* $a_1 = 2$*,* $a_2 = \cdots = a_{n-1} = 1$*,* $a_n = 0$*.*

Remark 3. Replacing n with $2n$ and choosing $m = n$, we get the following results:

- *If* $a_1, a_2, \ldots, a_{2n}$ *are nonnegative real numbers such that*

$$a_1 \ge \cdots \ge a_n \ge 1 \ge a_{n+1} \ge \cdots \ge a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n,$$

then

$$n(a_1^3 + a_2^3 + \cdots + a_{2n}^3 - 2n) \ge (2n+1)(a_1^2 + a_2^2 + \cdots + a_{2n}^2 - 2n),$$

with equality for $a_1 = a_2 = \cdots = a_{2n} = 1$*, and also for*

$$a_1 = \cdots = a_n = \frac{n+1}{n}, \quad a_{n+2} = \cdots = a_{2n-1} = 1, \quad a_{2n} = 0.$$

- *If* $a_1, a_2, \ldots, a_{2n}$ *are nonnegative real numbers such that*

$$a_1 \ge \cdots \ge a_n \ge 1 \ge a_{n+1} \ge \cdots \ge a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n,$$

then

$$a_1^3 + a_2^3 + \cdots + a_{2n}^3 - 2n \le (n+2)(a_1^2 + a_2^2 + \cdots + a_{2n}^2 - 2n),$$

with equality for $a_1 = a_2 = \cdots = a_{2n} = 1$*, and also for*

$$a_1 = n+1, \quad a_2 = \cdots = a_n = 1, \quad a_{n+1} = \cdots = a_{2n} = 0.$$

□

P 2.2. *If* a, b, c, d *are real numbers such that*

$$a \ge b \ge 1 \ge c \ge d, \qquad a + b + c + d = 4,$$

then

$$(3a^2 - 2)(a-1)^2 + (3b^2 - 2)(b-1)^2 + (3c^2 - 2)(c-1)^2 + (3d^2 - 2)(d-1)^2 \ge 0.$$

(Vasile Cirtoaje, 2007)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)\ge 4f(s),\quad s=\frac{a+b+c+d}{4}=1,$$

where

$$f(u)=(3u^2-2)(u-1)^2,\quad u\in\mathbb{R}.$$

From

$$f''(u)=2(18u^2-18u+1),$$

it follows that $f''(u)>0$ for $u\ge 1$, hence f is convex for $u\ge 1$. Therefore, we may apply HCF-OV Theorem or RHCF-OV Corollary for $n=4$ and $m=2$. By these, it suffices to show that $f(x)+2f(y)\ge 3f(1)$ for all real x,y such that $x+2y=3$. Using Remark 1, we only need to show that $h(x,y)\ge 0$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y},\quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

We have

$$g(u)=3(u^3+u^2+u+1)-6(u^2+u+1)+u+1=3u^3-3u^2-2u-2,$$

$$h(x,y)=3(x^2+xy+y^2)-3(x+y)-2=(3y-4)^2\ge 0.$$

From $x+2y=3$ and $h(x,y)=0$, we get $x=1/3$, $y=4/3$. Therefore, in accordance with Remark 7, the equality holds for $a=b=c=d=1$, and also for $a=b=4/3$, $c=1$ and $d=1/3$.

Remark. Similarly, we can prove the following generalization:

- *Let $a_1,a_2,\dots,a_{2n}$ be real numbers such that*

$$a_1\ge\cdots\ge a_n\ge 1\ge a_{n+1}\ge\cdots\ge a_{2n},\quad a_1+a_2+\cdots+a_{2n}=2n.$$

If $k=\dfrac{n}{n^2-n+1}$, then

$$(a_1^2-k)(a_1-1)^2+(a_2^2-k)(a_2-1)^2+\cdots+(a_{2n}^2-k)(a_{2n}-1)^2\ge 0,$$

with equality for $a_1=a_2=\cdots=a_{2n}=1$, and also for

$$a_1=\cdots=a_n=\frac{n^2}{n^2-n+1},\quad a_{n+1}=\cdots=a_{2n-1}=1,\quad a_{2n}=\frac{1}{n^2-n+1}.$$

□

P 2.3. *If* $a_1, a_2, \ldots, a_{2n} \ge \dfrac{-2n-1}{n-1}$ *such that*

$$a_1 \ge \cdots \ge a_n \ge 1 \ge a_{n+1} \ge \cdots \ge a_{2n}, \quad a_1 + a_2 + \cdots + a_{2n} = 2n,$$

then

$$a_1^3 + a_2^3 + \cdots + a_{2n}^3 \ge 2n.$$

(Vasile Cirtoaje, 2007)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_{2n}) \ge 2nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_{2n}}{2n} = 1,$$

where

$$f(u) = u^3, \quad u \ge \frac{-2n-1}{n-1}.$$

From $f''(u) = 6u$, it follows that f is convex for $u \ge 1$. Therefore, we may apply HCF-OV Theorem or RHCF-OV Corollary for $2n$ numbers and $m = n$. By Remark 1, it suffices to show that $h(x,y) \ge 0$ for all $x, y \ge \dfrac{-2n-1}{n-1}$ such that $x + ny = n + 1$. We have

$$g(u) = \frac{f(u) - f(1)}{u-1} = u^2 + u + 1,$$

$$h(x,y) = \frac{g(x) - g(y)}{x - y} = x + y + 1 = \frac{(n-1)x + 2n + 1}{n-1} \ge 0.$$

From $x + ny = n + 1$ and $h(x,y) = 0$, we get $x = \dfrac{-2n-1}{n-1}$ and $y = \dfrac{n+2}{n-1}$. Therefore, in accordance with Remark 7, the equality holds for $a_1 = a_2 = \cdots = a_{2n} = 1$, and also for

$$a_1 = \cdots = a_n = \frac{n+2}{n-1}, \quad a_{n+1} = \cdots = a_{2n-1} = 1, \quad a_{2n} = \frac{-2n-1}{n-1}.$$

□

P 2.4. *Let* $a_1, a_2, \ldots, a_n$ *be real numbers such that* $a_1 + a_2 + \cdots + a_n = n$.

(a) If $a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n$, *then*

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \ge 6(a_1^2 + a_2^2 + \cdots + a_n^2 - n);$$

(b) If $a_1 \ge a_2 \ge 1 \ge a_3 \ge \cdots \ge a_n$, *then*

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \ge \frac{14}{3}(a_1^2 + a_2^2 + \cdots + a_n^2 - n);$$

(c) If $a_1 \ge \cdots \ge a_{n-2} \ge 1 \ge a_{n-1} \ge a_n$, *then*

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \ge \frac{2(n^2-3n+3)}{n^2-5n+7}\,(a_1^2 + a_2^2 + \cdots + a_n^2 - n).$$

(Vasile Cirtoaje, 2009)

Solution. Consider the inequality

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \ge k(a_1^2 + a_2^2 + \cdots + a_n^2 - n), \quad k \le 6,$$

and write it as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = u^4 - ku^2, \quad u \ge 0.$$

From $f''(u) = 2(6u^2 - k)$, it follows that f is convex for $u \ge 1$. Therefore, we may apply HCF-OV Theorem or RHCF-OV Corollary for $m = 1$, $m = 2$ and $m = n-2$, respectively. By Remark 1, it suffices to show that $h(x, y) \ge 0$ for all $x, y \ge 0$ such that $x + my = m+1$. We have

$$g(u) = \frac{f(u) - f(1)}{u-1} = u^3 + u^2 + u + 1 - k(u+1),$$

$$h(x, y) = \frac{g(x) - g(y)}{x - y} = x^2 + xy + y^2 + x + y - k + 1.$$

(a) We have $k = 6$, $m = 1$ and $x + y = 2$, which involve

$$h(x, y) = 1 - xy = \frac{1}{4}(x-y)^2 \ge 0.$$

From $x + y = 2$ and $h(x, y) = 0$, we get $x = y = 1$. Therefore, in accordance with Remark 7, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

(b) We have $k = 14/3$, $m = 2$ and $x + 2y = 3$, which involve

$$h(x, y) = \frac{1}{3}(3y-5)^2 \ge 0.$$

From $x + 2y = 3$ and $h(x, y) = 0$, we get $x = -1/3$ and $y = 5/3$. Therefore, in accordance with Remark 7, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = a_2 = \frac{5}{3}, \quad a_3 = \cdots = a_{n-1} = 1, \quad a_n = \frac{-1}{3}.$$

(c) We have $k = \dfrac{2(n^2-3n+3)}{n^2-5n+7}$, $m = n-2$ and $x+(n-2)y = n-1$, which involve

$$h(x,y) = \frac{1}{n^2-5n+7}[(n^2-5n+7)y - n^2 + 3n - 1]^2 \ge 0.$$

From $x+(n-2)y = n-1$ and $h(x,y) = 0$, we get $x = \dfrac{-n^2+5n-4}{n^2-5n+7}$ and $\dfrac{n^2-3n+1}{n^2-5n+7}$. Therefore, in accordance with Remark 7, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \cdots = a_{n-2} = \frac{n^2-3n+1}{n^2-5n+7}, \quad a_{n-1} = 1, \quad a_n = \frac{-n^2+5n-4}{n^2-5n+7}.$$

□

P 2.5. *Let a,b,c,d,e be nonnegative real numbers such that $a+b+c+d+e = 5$.*

(a) If $a \ge b \ge 1 \ge c \ge d \ge e$, then

$$21(a^2+b^2+c^2+d^2+e^2) \ge a^4+b^4+c^4+d^4+e^4+100;$$

(b) If $a \ge b \ge c \ge 1 \ge d \ge e$, then

$$13(a^2+b^2+c^2+d^2+e^2) \ge a^4+b^4+c^4+d^4+e^4+60.$$

(Vasile Cirtoaje, 2009)

Solution. Consider the inequality

$$k(a^2+b^2+c^2+d^2+e^2-5) \ge a^4+b^4+c^4+d^4+e^4-5, \quad k \ge 6,$$

and write it as

$$f(a)+f(b)+f(c)+f(d)+f(e) \ge 5f(s), \quad s = \frac{a+b+c+d+e}{5} = 1,$$

where

$$f(u) = ku^2 - u^4, \quad u \ge 0.$$

From $f''(u) = 2(k-6u^2)$, it follows that f is convex on $[0,1]$. Therefore, we may apply HCF-OV Theorem or LHCF-OV Corollary for $m = 3$ and $m = 2$, respectively. By Remark 1, it suffices to show that $h(x,y) \ge 0$ for all $x,y \ge 0$ such that $x+my = m+1$. We have

$$g(u) = \frac{f(u)-f(1)}{u-1} = k(u+1)-(u^3+u^2+u+1),$$

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=k-(x^2+xy+y^2+x+y+1).$$

(a) We have $k=21$, $m=3$ and $x+3y=4$, which involve $y\le 4/3$ and

$$h(x,y)=20-(x^2+xy+y^2+x+y)=y(22-7y)\ge 0.$$

From $x+3y=4$ and $h(x,y)=0$, we get $x=4$ and $y=0$. Therefore, in accordance with Remark 8, the equality holds for $a=b=c=d=e=1$ and also for $a=4$, $b=1$, $c=d=e=0$.

(b) We have $k=13$, $m=2$ and $x+2y=3$, which involve $y\le 3/2$ and

$$h(x,y)=12-(x^2+xy+y^2+x+y)=y(10-3y)\ge 0.$$

From $x+2y=3$ and $h(x,y)=0$, we get $x=3$ and $y=0$. Therefore, in accordance with Remark 8, the equality holds for $a=b=c=d=e=1$ and also for $a=3$, $b=c=1$, $d=e=0$.

□

P 2.6. *Let $a_1,a_2,\dots,a_n$ be nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$.*

(a) If $a_1\ge\cdots\ge a_{n-1}\ge 1\ge a_n$, then

$$7(a_1^3+a_2^3+\cdots+a_n^3)\ge 3(a_1^4+a_2^4+\cdots+a_n^4)+4n;$$

(b) If $a_1\ge\cdots\ge a_{n-2}\ge 1\ge a_{n-1}\ge a_n$, then

$$13(a_1^3+a_2^3+\cdots+a_n^3)\ge 4(a_1^4+a_2^4+\cdots+a_n^4)+9n.$$

(Vasile Cirtoaje, 2009)

Solution. Consider the inequality

$$k(a_1^3+a_2^3+\cdots+a_n^3-n)\ge a_1^4+a_2^4+\cdots+a_n^4-n,\quad k>2,$$

and write it as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s),\quad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=ku^3-u^4,\quad u\ge 0.$$

From $f''(u)=6u(k-2u^2)$, it follows that f is convex on $[0,1]$. Therefore, we may apply HCF-OV Theorem or LHCF-OV Corollary for $m=1$ and $m=2$, respectively. By

Remark 1, it suffices to show that $h(x,y) \ge 0$ for all $x, y \ge 0$ such that $x + my = m+1$. We have

$$g(u) = \frac{f(u)-f(1)}{u-1} = k(u^2+u+1)-(u^3+u^2+u+1),$$

$$h(x,y) = \frac{g(x)-g(y)}{x-y} = -(x^2+xy+y^2)+(k-1)(x+y+1).$$

(a) We have $k = 7/3$, $m = 1$ and $x + y = 2$, which involve

$$h(x,y) = xy \ge 0.$$

From $x + y = 2$ and $h(x,y) = 0$, we get $x = 2$ and $y = 0$. Therefore, in accordance with Remark 8, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 2, \quad a_2 = \cdots = a_{n-1} = 1, \quad a_n = 0.$$

(b) We have $k = 13/4$, $m = 2$ and $x + 2y = 3$, which involve

$$h(x,y) = 3y(9-4y) = 3y(3+2x) \ge 0.$$

From $x + 2y = 3$ and $h(x,y) = 0$, we get $x = 3$ and $y = 0$. Therefore, in accordance with Remark 8, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = 3, \quad a_2 = \cdots = a_{n-2} = 1, \quad a_{n-1} = a_n = 0.$$

□

P 2.7. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that*

$$a_1 \ge \cdots \ge a_{n-m} \ge 1 \ge a_{n-m+1} \ge \cdots \ge a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$(m+1)^2\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-n\right) \ge 4m(a_1^2+a_2^2+\cdots+a_n^2-n).$$

(Vasile Cirtoaje, 2007)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n) \ge nf(s), \quad s = \frac{a_1+a_2+\cdots+a_n}{n} = 1,$$

where

$$f(u) = \frac{(m+1)^2}{u} - 4mu^2, \quad u \in (0,n).$$

For $u \in (0,1]$, we have

$$f''(u) = \frac{2(m+1)^2}{u^3} - 8m \geq 2(m+1)^2 - 8m = 2(m-1)^2 \geq 0.$$

Since f is convex on $(0,1]$, we may apply HCF-OV Theorem or LHCF-OV Corollary. By Remark 1, it suffices to show that $h(x,y) \geq 0$ for all $x, y > 0$ such that $x + my = m+1$. We have

$$g(u) = \frac{f(u)-f(1)}{u-1} = \frac{-(m+1)^2}{u} - 4m(u+1),$$

$$h(x,y) = \frac{(m+1)^2}{xy} - 4m = \frac{[m+1-2my]^2}{xy} \geq 0.$$

From $x + my = m+1$ and $h(x,y) = 0$, we get $x = \dfrac{1+m}{2}$ and $y = \dfrac{1+m}{2m}$. Therefore, in accordance with Remark 8, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = \frac{1+m}{2}, \quad a_2 = a_3 = \cdots = a_{n-m} = 1, \quad a_{n-m+1} = \cdots = a_n = \frac{1+m}{2m}.$$

Remark 1. For $m = 1$, we get the following elegant statement:

- *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that*

$$a_1 \geq \cdots \geq a_{n-1} \geq 1 \geq a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \geq a_1^2 + a_2^2 + \cdots + a_n^2,$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$

Remark 2. Replacing n with $2n$ and choosing $m = n$, we get the statement:

- *If $a_1, a_2, \ldots, a_{2n}$ are positive real numbers such that*

$$a_1 \geq \cdots \geq a_n \geq 1 \geq a_{n+1} \geq \cdots \geq a_{2n}, \qquad a_1 + a_2 + \cdots + a_{2n} = 2n,$$

then

$$(n+1)^2\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{2n}} - 2n\right) \geq 4n(a_1^2 + a_2^2 + \cdots + a_{2n}^2 - 2n),$$

with equality for $a_1 = a_2 = \cdots = a_{2n} = 1$, and also for

$$a_1 = \frac{1+n}{2}, \quad a_2 = a_3 = \cdots = a_n = 1, \quad a_{n+1} = \cdots = a_{2n} = \frac{1+n}{2n}.$$

□

P 2.8. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that*

$$a_1 \le \cdots \le a_{n-m} \le 1 \le a_{n-m+1} \le \cdots \le a_n, \qquad \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = n,$$

then

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n \ge 2\left(1 + \frac{\sqrt{m}}{1+m}\right)(a_1 + a_2 + \cdots + a_n - n).$$

(Vasile Cirtoaje, 2007)

Solution. Replacing each a_i by $1/a_i$, we need to prove that

$$a_1 \ge \cdots \ge a_{n-m} \ge 1 \ge a_{n-m+1} \ge \cdots \ge a_n, \qquad a_1 + a_2 + \cdots + a_n = n$$

involves

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{1}{u^2} - 2\left(1 + \frac{\sqrt{m}}{1+m}\right)\frac{1}{u}, \quad u \in (0, n).$$

For $u \in (0, 1]$, we have

$$f''(u) = \frac{6 - 4\left(1 + \frac{\sqrt{m}}{1+m}\right)u}{u^4} \ge \frac{6 - 4\left(1 + \frac{\sqrt{m}}{m+1}\right)}{u^4} = \frac{2(\sqrt{m}-1)^2}{(1+m)u^4} \ge 0.$$

Thus, f is convex on $(0, 1]$. By HCF-OV Theorem (or LHCF-OV Corollary) and Remark 1, it suffices to show that $h(x, y) \ge 0$ for $x, y > 0$ such that $x + my = 1 + m$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

We have

$$g(u) = \frac{-1}{u^2} + \left(1 + \frac{2\sqrt{m}}{1+m}\right)\frac{1}{u}$$

and

$$h(x, y) = \frac{1}{xy}\left(\frac{1}{x} + \frac{1}{y} - 1 - \frac{2\sqrt{m}}{1+m}\right).$$

We only need to show that

$$\frac{1}{x} + \frac{1}{y} \ge 1 + \frac{2\sqrt{m}}{1+m}.$$

Indeed, using the Cauchy-Schwarz inequality, we get

$$\frac{1}{x} + \frac{1}{y} \ge \frac{(1+\sqrt{m})^2}{x + my} = \frac{(1+\sqrt{m})^2}{1+m} = 1 + \frac{2\sqrt{m}}{1+m}.$$

From $x+my=1+m$ and $h(x,y)=0$, we get $x=\dfrac{1+m}{1+\sqrt{m}}$ and $y=\dfrac{1+m}{m+\sqrt{m}}$. In accordance with Remark 8, we have $f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(1)$ for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=\frac{1+m}{1+\sqrt{m}},\quad a_2=a_3=\cdots=a_{n-m}=1,\quad a_{n-m+1}=\cdots=a_n=\frac{1+m}{m+\sqrt{m}}.$$

Therefore, the original inequality becomes an equality for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=\frac{1+\sqrt{m}}{1+m},\quad a_2=a_3=\cdots=a_{n-m}=1,\quad a_{n-m+1}=\cdots=a_n=\frac{m+\sqrt{m}}{1+m}.$$

Remark. Replacing n with $2n$ and choosing $m=n$, we get the statement:

- *If $a_1,a_2,\dots,a_{2n}$ are positive real numbers such that*

$$a_1\le\cdots\le a_n\le 1\le a_{n+1}\le\cdots\le a_{2n},\qquad \frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_{2n}}=2n,$$

then

$$a_1^2+a_2^2+\cdots+a_{2n}^2-2n\ge 2\left(1+\frac{\sqrt{n}}{1+n}\right)(a_1+a_2+\cdots+a_{2n}-2n).$$

with equality for $a_1=a_2=\cdots=a_{2n}=1$, and also for

$$a_1=\frac{1+\sqrt{n}}{1+n},\quad a_2=a_3=\cdots=a_n=1,\quad a_{n+1}=\cdots=a_{2n}=\frac{n+\sqrt{n}}{1+n}.$$

□

P 2.9. *Let $a_1,a_2,\dots,a_n$ be nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$.*

(a) If $a_1\ge 1\ge a_2\ge\cdots\ge a_n$, then

$$\frac{1}{a_1^2+2}+\frac{1}{a_2^2+2}+\cdots+\frac{1}{a_n^2+2}\ge\frac{n}{3};$$

(b) If $a_1\ge a_2\ge 1\ge a_3\ge\cdots\ge a_n$, then

$$\frac{1}{2a_1^2+3}+\frac{1}{2a_2^2+3}+\cdots+\frac{1}{2a_n^2+3}\ge\frac{n}{5}.$$

(Vasile Cirtoaje, 2007)

Solution. Consider the inequality

$$\frac{1}{a_1^2+k}+\frac{1}{a_2^2+k}+\cdots+\frac{1}{a_n^2+k}\ge\frac{n}{1+k},\quad k\in[0,3];$$

and write it as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s),\quad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=\frac{1}{u^2+k},\quad u\in[0,n].$$

For $u\in[1,n]$, we have

$$f''(u)=\frac{2(3u^2-k)}{(u^2+k)^3}\ge\frac{2(3-k)}{(u^2+k)^3}\ge 0,$$

hence f is convex on $[1,n]$. Therefore, we may apply HCF-OV Theorem or RHCF-OV Corollary for $m=1$ and $m=2$, respectively. By Remark 1, it suffices to show that $h(x,y)\ge 0$ for all $x,y\ge 0$ such that $x+my=m+1$. Since

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{-u-1}{(1+k)(u^2+k)},$$

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{xy+x+y-k}{(1+k)(x^2+k)(y^2+k)},$$

we only need to show that

$$xy+x+y-k\ge 0.$$

(a) We have $k=2$, $m=1$ and $x+y=2$, which involve

$$xy+x+y-k=xy\ge 0.$$

From $x+y=2$ and $h(x,y)=0$, we get $x=0$ and $y=2$. Therefore, in accordance with Remark 7, the equality holds for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=2,\quad a_2=\cdots=a_{n-1}=1,\quad a_n=0.$$

(b) We have $k=3/2$, $m=2$ and $x+2y=3$, which involve

$$xy+x+y-k=\frac{x(4-x)}{2}=\frac{x(1+2y)}{2}\ge 0.$$

From $x+2y=3$ and $h(x,y)=0$, we get $x=0$ and $y=3/2$. Therefore, in accordance with Remark 7, the equality holds for $a_1=a_2=\cdots=a_n=1$, and also for

$$a_1=a_2=3/2,\quad a_3=\cdots=a_{n-1}=1,\quad a_n=0.$$

□

P 2.10. *If $a_1,a_2,\dots,a_{2n}$ are nonnegative real numbers such that*

$$a_1\ge\cdots\ge a_n\ge 1\ge a_{n+1}\ge\cdots\ge a_{2n},\quad a_1+a_2+\cdots+a_{2n}=2n,$$

then

$$\frac{1}{na_1^2+n^2+n+1}+\frac{1}{na_2^2+n^2+n+1}+\cdots+\frac{1}{na_{2n}^2+n^2+n+1}\le\frac{2n}{(n+1)^2}.$$

(Vasile Cirtoaje, 2007)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_{2n})\ge 2nf(s),\quad s=\frac{a_1+a_2+\cdots+a_{2n}}{2n}=1,$$

where

$$f(u)=\frac{-1}{nu^2+n^2+n+1},\quad u\in[0,2n].$$

For $u\in[0,1]$, we have

$$f''(u)=\frac{2nu(n^2+n+1-3nu^2)}{(nu^2+n^2+n+1)^3}\ge\frac{2nu(n^2+n+1-3n)}{(nu^2+n^2+n+1)^3}\ge 0,$$

hence f is convex on $[0,1]$. Therefore, we may apply HCF-OV Theorem or LHCF-OV Corollary for $m=n$. By Remark 1, it suffices to show that $h(x,y)\ge 0$ for all $x,y\ge 0$ such that $x+ny=n+1$. Since

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{n(u+1)}{(n+1)^2(nu^2+n^2+n+1)},$$

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{n(n^2+n+1-nx-ny-nxy)}{(n+1)^2(nx^2+n^2+n+1)(ny^2+n^2+n+1)},$$

we only need to show that

$$n^2+n+1-n(x+y+xy)\ge 0.$$

Indeed,

$$n^2+n+1-n(x+y+xy)=(ny-1)^2\ge 0.$$

From $x+ny=n+1$ and $h(x,y)=0$, we get $x=n$ and $y=1/n$. Therefore, in accordance with Remark 7, the equality holds for $a_1=a_2=\cdots=a_{2n}=1$, and also for

$$a_1=n,\quad a_2=\cdots=a_n=1,\quad a_{n+1}=\cdots=a_n=1/n.$$

□

P 2.11. *If a,b,c,d,e,f are nonnegative real numbers such that*

$$a \ge b \ge c \ge 1 \ge d \ge e \ge f, \qquad a+b+c+d+e+f=6,$$

then

$$\frac{3a+4}{3a^2+4}+\frac{3b+4}{3b^2+4}+\frac{3c+4}{3c^2+4}+\frac{3d+4}{3d^2+4}+\frac{3e+4}{3e^2+4}+\frac{3f+4}{3f^2+4} \le 6.$$

(Vasile Cirtoaje, 2009)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)+f(e)+f(f) \ge 6f(s), \qquad s=\frac{a+b+c+d+e+f}{6}=1,$$

where

$$f(u)=\frac{-3u-4}{3u^2+4}, \qquad u \in [0,6].$$

For $u \in [0,1]$, we have

$$f''(u)=\frac{6(16-9u^3)+216u(1-u)}{(3u^2+4)^3} \ge 0,$$

hence f is convex on $[0,1]$. Therefore, we may apply HCF-OV Theorem or LHCF-OV Corollary for $m=3$. By Remark 1, it suffices to show that $h(x,y) \ge 0$ for all $x,y \ge 0$ such that $x+3y=4$. We have

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{3u}{3u^2+4},$$

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{3(4-3xy)}{(3x^2+4)(3y^2+4)}=\frac{3(x-2)^2}{(3x^2+4)(3y^2+4)} \ge 0.$$

From $x+3y=4$ and $h(x,y)=0$, we get $x=2$ and $y=2/3$. Therefore, in accordance with Remark 8, the equality holds only for $a=b=c=d=e=f=1$, and also for $a=2$, $b=c=1$, $d=e=f=2/3$.

□

P 2.12. *If a,b,c,d,e,f are nonnegative real numbers such that*

$$a \ge b \ge 1 \ge c \ge d \ge e \ge f, \qquad a+b+c+d+e+f=6,$$

then

$$\frac{a^2-1}{(2a+7)^2}+\frac{b^2-1}{(2b+7)^2}+\frac{c^2-1}{(2c+7)^2}+\frac{d^2-1}{(2d+7)^2}+\frac{e^2-1}{(2e+7)^2}+\frac{f^2-1}{(2f+7)^2} \ge 0.$$

(Vasile Cirtoaje, 2009)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)+f(e)+f(f)\ge 6f(s),\quad s=\frac{a+b+c+d+e+f}{6}=1,$$

where

$$f(u)=\frac{u^2-1}{(2u+7)^2},\quad u\in[0,6].$$

For $u\in[0,1]$, we have

$$f''(u)=\frac{2(37-28u)}{(2u+7)^4}\ge 0,$$

hence f is convex on $[0,1]$. Therefore, we may apply HCF-OV Theorem or LHCF-OV Corollary for $m=4$. By Remark 1, it suffices to show that $h(x,y)\ge 0$ for all $x,y\ge 0$ such that $x+4y=5$. We have

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{u+1}{(2u+7)^2},$$

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{21-4x-4y-4xy}{(2x+7)^2(2y+7)^2}=\frac{(x-4)^2}{(2x+7)^2(2y+7)^2}\ge 0.$$

From $x+4y=5$ and $h(x,y)=0$, we get $x=4$ and $y=1/4$. Therefore, in accordance with Remark 8, the equality holds only for $a=b=c=d=e=f=1$, and also for $a=4$, $b=1$, $c=d=e=f=1/4$.

□

P 2.13. *If a,b,c,d,e,f are nonnegative real numbers such that*

$$a\ge b\ge c\ge d\ge 1\ge e\ge f,\quad a+b+c+d+e+f=6,$$

then

$$\frac{a^2-1}{(2a+5)^2}+\frac{b^2-1}{(2b+5)^2}+\frac{c^2-1}{(2c+5)^2}+\frac{d^2-1}{(2d+5)^2}+\frac{e^2-1}{(2e+5)^2}+\frac{f^2-1}{(2f+5)^2}\le 0.$$

(Vasile Cirtoaje, 2009)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)+f(e)+f(f)\ge 6f(s),\quad s=\frac{a+b+c+d+e+f}{6}=1,$$

where

$$f(u)=\frac{1-u^2}{(2u+5)^2},\quad u\in[0,6].$$

For $u \geq 1$, we have

$$f''(u) = \frac{2(20u-13)}{(2u+5)^4} \geq 0,$$

hence f is convex on $[1,6]$. Therefore, we may apply HCF-OV Theorem or RHCF-OV Corollary for $m = 4$. By Remark 1, it suffices to show that $h(x,y) \geq 0$ for all $x,y \geq 0$ such that $x + 4y = 5$. We have

$$g(u) = \frac{f(u)-f(1)}{u-1} = \frac{-u-1}{(2u+5)^2},$$

$$h(x,y) = \frac{g(x)-g(y)}{x-y} = \frac{4xy+4x+4y-5}{(2x+5)^2(2y+5)^2} = \frac{x(8-x)}{(2x+5)^2(2y+5)^2} \geq 0.$$

From $x + 4y = 5$ and $h(x,y) = 0$, we get $x = 0$ and $y = 5/4$. Therefore, in accordance with Remark 7, the equality holds only for $a = b = c = d = e = f = 1$, and also for $a = b = c = d = 5/4$, $e = 1$, $f = 0$.

□

P 2.14. *If a,b,c,d are nonnegative real numbers such that*

$$a \geq b \geq 1 \geq c \geq d, \qquad a+b+c+d = 4,$$

then

$$\frac{1}{2a^3+5} + \frac{1}{2b^3+5} + \frac{1}{2c^3+5} + \frac{1}{2d^3+5} \leq \frac{4}{7}.$$

(Vasile Cirtoaje, 2009)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d) \geq 4f(s), \qquad s = \frac{a+b+c+d}{4} = 1,$$

where

$$f(u) = \frac{-1}{2u^3+5}, \quad u \in [0,4].$$

From

$$f''(u) = \frac{12u(5-4u^3)}{(2u^3+5)^3},$$

it follows that $f''(u) \geq 0$ for $u \in [0,1]$, hence f is convex on $[0,1]$. Therefore, we may apply HCF-OV Theorem or LHCF-OV Corollary for $n = 4$ and $m = 2$. Thus, it suffices to show that $f(x)+2f(y) \geq 3f(1)$ for all $x,y \geq 0$ such that $x+2y = 3$. Using Remark 1, we only need to show that $h(x,y) \geq 0$. We have

$$g(u) = \frac{f(u)-f(1)}{u-1} = \frac{2(u^2+u+1)}{7(2u^3+5)},$$

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{2E}{7(2x^3+5)(2y^3+5)},$$

where

$$E=-2x^2y^2-2xy(x+y)-2(x^2+xy+y^2)+5(x+y)+5.$$

Since

$$E=(1-2y)^2(2+3y-2y^2)=(1-2y)^2(2+xy)\geq 0,$$

the proof is completed. From $x+2y=3$ and $h(x,y)=0$, we get $x=2$, $y=1/2$. Therefore, in accordance with Remark 8, the equality holds for $a=b=c=d=1$, and also for $a=2$, $b=1$, $c=d=1/2$.

Remark. Similarly, we can prove the following generalization:

- *If $a_1,a_2,\ldots,a_{2n}$ are nonnegative real numbers such that*

$$a_1\geq\cdots\geq a_n\geq 1\geq a_{n+1}\geq\cdots\geq a_{2n},\quad a_1+a_2+\cdots+a_{2n}=2n.$$

then

$$\frac{1}{a_1^3+n+\frac{1}{n}}+\frac{1}{a_2^3+n+\frac{1}{n}}+\cdots+\frac{1}{a_{2n}^3+n+\frac{1}{n}}\geq\frac{2n^2}{n^2+n+1},$$

with equality for $a_1=a_2=\cdots=a_{2n}=1$, and also for

$$a_1=n,\quad a_2=\cdots=a_n=1,\quad a_{n+1}=\cdots=a_{2n}=1/n.$$

□

P 2.15. *If $a_1,a_2,\ldots,a_8$ are nonnegative real numbers such that*

$$a_1\geq a_2\geq a_3\geq a_4\geq 1\geq a_5\geq a_6\geq a_7\geq a_8,\quad a_1+a_2+\cdots+a_8=8,$$

then

$$(a_1^2+1)(a_2^2+1)\cdots(a_8^2+1)\geq(a_1+1)(a_2+1)\cdots(a_8+1).$$

(Vasile Cirtoaje, 2008)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_8)\geq 8f(s),\quad s=\frac{a_1+a_2+\cdots+a_8}{8}=1,$$

where

$$f(u)=\ln(u^2+1)-\ln(u+1),\quad u\in[0,8].$$

For $u \in [0,1]$, we have

$$f''(u) = \frac{2(1-u^2)}{(u^2+1)^2} + \frac{1}{(u+1)^2} = \frac{(u^2-u^4)+4u(1-u^2)+u^2+3}{(u^2+1)^2(u+1)^2} > 0.$$

Therefore, f is convex on $[0,1]$. According to HCF-OV Theorem or LHCF-OV Corollary applied for $n = 8$ and $m = 4$, it suffices to show that $f(x)+4f(y) \geq 5f(1)$ for all $x, y \geq 0$ such that $x+4y = 5$. Using Remark 2, we only need to show that $H(x,y) \geq 0$ for $x, y \geq 0$ such that $x+4y = 5$, where

$$H(x,y) = \frac{f'(x)-f'(y)}{x-y} = \frac{2(1-xy)}{(x^2+1)(y^2+1)} + \frac{1}{(x+1)(y+1)}.$$

Since $2(x^2+1) \geq (x+1)^2$ and $2(y^2+1) \geq (y+1)^2$, it suffices to prove that $8(1-xy)+(x+1)(y+1) \geq 0$. Indeed,

$$8(1-xy)+(x+1)(y+1) = 28x^2-38x+14 = 28(x-19/28)^2+31/28 > 0.$$

The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_8$.

□

P 2.16. *If a, b, c, d are positive real numbers such that*

$$a \geq b \geq 1 \geq c \geq d, \qquad abcd = 1,$$

then

$$a^2+b^2+c^2+d^2-4 \geq 18\left(a+b+c+d-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}-\frac{1}{d}\right).$$

(Vasile Cirtoaje, 2008)

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z, \quad d = e^w,$$

we need to show that

$$f(x)+f(y)+f(z)+f(w) \geq 4f(s),$$

where

$$x \geq y \geq 0 \geq z \geq w, \qquad s = \frac{x+y+z+w}{4} = 0,$$

$$f(u) = e^{2u}-1-18(e^u-e^{-u}), \quad u \in \mathbb{R}.$$

For $u \leq 0$, we have

$$f''(u) = 4e^{2u}+18(e^{-u}-e^u) > 0,$$

hence f is convex on $(-\infty, 0]$. By HCF-OV Theorem or LHCF-OV Corollary applied for $n=4$ and $m=2$, it suffices to show that $f(x)+2f(y) \ge 3f(0)$ for all real x,y such that $x+2y=0$; that is, to show that

$$a^2+2b^2-3-18\left(a+2b-\frac{1}{a}-\frac{2}{b}\right) \ge 0$$

for all $a,b>0$ such that $ab^2=1$. This inequality is equivalent to

$$\frac{(b^2-1)^2(2b^2+1)}{b^4}+\frac{18(b-1)^3(b+1)}{b^2} \ge 0,$$

$$\frac{(b-1)^2(2b-1)^2(b+1)(5b+1)}{b^4} \ge 0.$$

The proof is completed. The equality holds for $a=b=c=d=1$, and also for $a=4$, $b=1$, $c=d=1/2$.

□

P 2.17. *If a,b,c,d are positive real numbers such that*

$$a \ge b \ge 1 \ge c \ge d, \qquad abcd=1,$$

then

$$\sqrt{a^2-a+1}+\sqrt{b^2-b+1}+\sqrt{c^2-c+1}+\sqrt{d^2-d+1} \ge a+b+c+d.$$

(Vasile Cirtoaje, 2008)

Solution. Using the substitution

$$a=e^x, \quad b=e^y, \quad c=e^z, \quad d=e^w,$$

we need to show that

$$f(x)+f(y)+f(z)+f(w) \ge 4f(s),$$

where

$$x \ge y \ge 0 \ge z \ge w, \qquad s=\frac{x+y+z+w}{4}=0,$$

$$f(u)=\sqrt{e^{2u}-e^u+1}-e^u, \quad u \in \mathbb{R}.$$

We claim that f is convex for $u \ge 0$. Since

$$e^{-u}f''(u)=\frac{4e^{3u}-6e^{2u}+9e^u-2}{4(e^{2u}-e^u+1)^{3/2}}-1,$$

we need to show that

$$(4t^3-6t^2+9t-2)^2 \ge 16(t^2-t+1)^3,$$

where $t=e^u \ge 1$. Indeed,

$$(4t^3-6t^2+9t-2)^2-16(t^2-t+1)^3 = 12t^3(t-1)+9t^2+12(t-1) > 0.$$

By HCF-OV Theorem or RHCF-OV Corollary applied for $n=4$ and $m=2$, it suffices to show that $f(x)+2f(y) \ge 3f(0)$ for all real x,y such that $x+2y=0$; that is, to show that

$$\sqrt{a^2-a+1}+2\sqrt{b^2-b+1} \ge a+2b$$

for all $a,b>0$ such that $ab^2=1$. This inequality is equivalent to

$$\frac{\sqrt{b^4-b^2+1}}{b^2}+2\sqrt{b^2-b+1} \ge \frac{1}{b^2}+2b,$$

$$\frac{b^2-1}{\sqrt{b^4-b^2+1}+1}+\frac{2(1-b)}{\sqrt{b^2-b+1}+b} \ge 0.$$

Since

$$\frac{b^2-1}{\sqrt{b^4-b^2+1}} \ge \frac{b^2-1}{b^2+1},$$

it suffices to show that

$$\frac{b^2-1}{b^2+1}+\frac{2(1-b)}{\sqrt{b^2-b+1}+b} \ge 0,$$

which is equivalent to

$$(b-1)\left[\frac{b+1}{b^2+1}-\frac{2}{\sqrt{b^2-b+1}+b}\right] \ge 0,$$

$$(b-1)\left[(b+1)\sqrt{b^2-b+1}-b^2+b-2\right] \ge 0,$$

$$\frac{(b-1)^2(3b^2-2b+3)}{(b+1)\sqrt{b^2-b+1}+b^2-b+2} \ge 0.$$

Clearly, the last inequality is true. The equality holds for $a=b=c=d=1$.

□

P 2.18. *If a,b,c,d are positive real numbers such that*

$$a \ge 1 \ge b \ge c \ge d, \qquad abcd=1,$$

then

$$\frac{1}{a^3+3a+2}+\frac{1}{b^3+3b+2}+\frac{1}{c^3+3c+2}+\frac{1}{d^3+3d+2} \ge \frac{2}{3}.$$

(Vasile Cirtoaje, 2007)

Solution. Using the substitution

$$a = e^x, \quad b = e^y, \quad c = e^z, \quad d = e^w,$$

we need to show that

$$f(x) + f(y) + f(z) + f(w) \ge 4f(s),$$

where

$$x \ge y \ge 0 \ge z \ge w, \qquad s = \frac{x + y + z + w}{4} = 0,$$

$$f(u) = \frac{1}{e^{3u} + 3e^u + 2}, \quad u \in \mathbb{R}.$$

We claim that f is convex for $u \ge 0$. Indeed, denoting $t = e^u$, $t \ge 1$, we have

$$\begin{aligned} f''(u) &= \frac{3t(3t^5 + 2t^3 - 6t^2 + 3t - 2)}{(t^3 + 3t + 2)^3} \\ &= \frac{3t(t-1)(3t^4 + 3t^3 + 5t^2 - t + 2)}{(t^3 + 3t + 2)^3} \ge 0. \end{aligned}$$

By HCF-OV Theorem or RHCF-OV Corollary applied for $n = 4$ and $m = 1$, it suffices to show that $f(x)+f(y) \ge 2f(0)$ for all real x, y such that $x + y = 0$; that is, to show that

$$\frac{1}{a^3 + 3a + 2} + \frac{1}{b^3 + 3b + 2} \ge \frac{1}{3}$$

for all $a, b > 0$ such that $ab = 1$. This inequality is equivalent to

$$(a-1)^4(a^2 + a + 1) \ge 0,$$

which is clearly true. The equality holds for $a = b = c = d = 1$.

□

P 2.19. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that*

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1 a_2 \cdots a_n = 1.$$

If $k \ge 1$, then

$$\frac{1}{1 + ka_1} + \frac{1}{1 + ka_2} + \cdots + \frac{1}{1 + ka_n} \ge \frac{n}{1 + k}.$$

(Vasile Cirtoaje, 2007)

Solution. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \dots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s),$$

where

$$x_1 \ge 0 \ge x_2 \ge \cdots \ge x_n, \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

$$f(u) = \frac{1}{1 + ke^u}, \quad u \in \mathbb{R}.$$

For $u \ge 0$, we have

$$f''(u) = \frac{ke^u(ke^u - 1)}{(1 + ke^u)^3} \ge 0,$$

therefore $f(u)$ is convex for $u \ge 0$. By HCF-OV Theorem or RHCF-OV Corollary applied for $m = 1$, it suffices to show that $f(x) + f(y) \ge 2f(0)$ for all real x, y such that $x + y = 0$; that is, to show that

$$\frac{1}{1 + ka} + \frac{1}{1 + kb} \ge \frac{2}{1 + k}$$

for all $a, b > 0$ such that $ab = 1$. This is true since

$$\frac{1}{1 + ka} + \frac{1}{1 + kb} - \frac{2}{1 + k} = \frac{k(k - 1)(a - 1)^2}{(1 + ka)(a + k)} \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 2.20. *If $a_1, a_2, \dots, a_9$ are positive real numbers such that*

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_9, \quad a_1 a_2 \cdots a_9 = 1,$$

then

$$\frac{1}{(a_1 + 2)^2} + \frac{1}{(a_2 + 2)^2} + \cdots + \frac{1}{(a_9 + 2)^2} \ge 1.$$

(Vasile Cirtoaje, 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \quad i = 1, 2, \dots, 9,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \cdots + f(x_9) \ge 9f(s),$$

where

$$x_1 \ge 0 \ge x_2 \ge \cdots \ge x_9, \qquad s = \frac{x_1 + x_2 + \cdots + x_9}{9} = 0,$$

$$f(u) = \frac{1}{(e^u + 2)^2}, \quad u \in \mathbb{R}.$$

For $u \in [0, \infty)$, we have

$$f''(u) = \frac{4e^u(e^u - 1)}{(e^u + 2)^4} \ge 0,$$

hence f is convex on $[0, \infty)$. According to HCF-OV Theorem or RHCF-OV Corollary (case $m = 1$), it suffices to show that $f(x) + f(y) \ge 2f(0)$ for all real x, y such that $x + y = 0$; that is, to show that

$$\frac{1}{(a+2)^2} + \frac{1}{(b+2)^2} \ge \frac{2}{9}$$

for all $a, b > 0$ such that $ab = 1$. Write this inequality as

$$\frac{b^2}{(2b+1)^2} + \frac{1}{(b+2)^2} \ge \frac{2}{9},$$

which is equivalent to the obvious inequality

$$(b-1)^4 \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_9 = 1$.

Remark. Similarly, we can prove the following generalization.

- *Let $a_1, a_2, \cdots, a_n$ be positive real numbers such that*

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1 a_2 \cdots a_n = 1.$$

If $p, q \ge 0$ such that

$$p + q \ge 1 + \frac{2pq}{p + 4q},$$

then

$$\frac{1}{1 + pa_1 + qa_1^2} + \frac{1}{1 + pa_2 + qa_2^2} + \cdots + \frac{1}{1 + pa_n + qa_n^2} \ge \frac{n}{1 + p + q}.$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 2.21. *If $a_1, a_2, \ldots, a_n$ $(n \ge 4)$ are positive real numbers such that*

$$a_1 \ge a_2 \ge a_3 \ge 1 \ge a_4 \ge \cdots \ge a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{(a_1+1)^2} + \frac{1}{(a_2+1)^2} + \cdots + \frac{1}{(a_n+1)^2} \ge \frac{n}{4}.$$

(Vasile Cirtoaje, 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \qquad i = 1, 2, \ldots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s),$$

where

$$x_1 \ge x_2 \ge x_3 \ge 0 \ge x_4 \ge \cdots \ge x_n, \qquad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

$$f(u) = \frac{1}{(e^u+1)^2}, \qquad u \in \mathbb{R}.$$

For $u \in [0, \infty)$, we have

$$f''(u) = \frac{2e^u(2e^u - 1)}{(e^u+1)^4} > 0,$$

hence f is convex on $[0, \infty)$. According to HCF-OV Theorem or RHCF-OV Corollary (case $m = 3$), it suffices to show that $f(x) + 3f(y) \ge 4f(0)$ for all real x, y such that $x + 3y = 0$; that is, to show that

$$\frac{1}{(a+1)^2} + \frac{3}{(b+1)^2} \ge 1$$

for all $a, b > 0$ such that $ab^3 = 1$. The inequality is equivalent to

$$\frac{b^6}{(b^3+1)^2} + \frac{3}{(b+1)^2} \ge 1.$$

Using the Cauchy-Schwarz inequality, it suffices to show that

$$\frac{(b^3+3)^2}{(b^3+1)^2 + 3(b+1)^2} \ge 1,$$

which is equivalent to the obvious inequality

$$(b-1)^2(4b+5) \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 2.22. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that*

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1a_2\cdots a_n = 1,$$

then

$$\frac{1}{(a_1+3)^2}+\frac{1}{(a_2+3)^2}+\cdots+\frac{1}{(a_n+3)^2} \le \frac{n}{16}.$$

(Vasile Cirtoaje, 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \qquad i = 1,2,\ldots,n,$$

we can write the inequality as

$$f(x_1)+f(x_2)+\cdots+f(x_n) \ge nf(s),$$

where

$$x_1 \le 0 \le x_2 \le \cdots \le x_n, \qquad s = \frac{x_1+x_2+\cdots+x_n}{n} = 0,$$

$$f(u) = \frac{-1}{(e^u+3)^2}, \qquad u \in \mathbb{R}.$$

For $u \in (-\infty, 0]$, we have

$$f''(u) = \frac{2e^u(3-2e^u)}{(e^u+3)^4} > 0,$$

hence f is convex on $(-\infty, 0]$. According to HCF-OV Theorem or LHCF-OV Corollary (case $m = 1$), it suffices to show that $f(x)+f(y) \ge 2f(0)$ for all real x, y such that $x + y = 0$; that is, to show that

$$\frac{1}{(a+3)^2}+\frac{1}{(b+3)^2} \le \frac{1}{8}$$

for all $a, b > 0$ such that $ab = 1$. Write this inequality as

$$\frac{b^2}{(3b+1)^2}+\frac{1}{(b+3)^2} \le \frac{1}{8},$$

which is equivalent to the obvious inequality

$$(b^2-1)^2 + 12b(b-1)^2 \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. Similarly, we can prove the following generalization:

- *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that*

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1a_2\cdots a_n = 1.$$

If $k \ge 1+\sqrt{2}$, then

$$\frac{1}{(a_1+k)^2}+\frac{1}{(a_2+k)^2}+\cdots+\frac{1}{(a_n+k)^2} \le \frac{n}{(1+k)^2},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 2.23. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that*

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1a_2\cdots a_n = 1.$$

If $p, q \ge 0$ such that $p+q \le 1$, then

$$\frac{1}{1+pa_1+qa_1^2}+\frac{1}{1+pa_2+qa_2^2}+\cdots+\frac{1}{1+pa_n+qa_n^2} \le \frac{n}{1+p+q}.$$

(Vasile Cirtoaje, 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \qquad i = 1, 2, \ldots, n,$$

we can write the inequality as

$$f(x_1)+f(x_2)+\cdots+f(x_n) \ge nf(s),$$

where

$$x_1 \le 0 \le x_2 \le \cdots \le x_n, \qquad s = \frac{x_1+x_2+\cdots+x_n}{n} = 0,$$

$$f(u) = \frac{-1}{1+pe^u+qe^{2u}}, \qquad u \in \mathbb{R}.$$

For $u \le 0$, we have

$$\begin{aligned} f''(u) &= \frac{e^u[-4q^2e^{3u}-3pqe^{2u}+(4q-p^2)e^u+p]}{(1+pe^u+qe^{2u})^3} \\ &\ge \frac{e^{2u}[-4q^2-3pq+(4q-p^2)+p]}{(1+pe^u+qe^{2u})^3} \\ &= \frac{e^{2u}[(p+4q)(1-p-q)+2pq]}{(1+pe^u+qe^{2u})^3} \ge 0, \end{aligned}$$

therefore $f(u)$ is convex for $u \le 0$. According to HCF-OV Theorem or LHCF-OV Corollary (case $m=1$), it suffices to show that $f(x)+f(y) \ge 2f(0)$ for all real x,y such that $x+y=0$; that is, to show that

$$\frac{1}{1+pa+qa^2}+\frac{1}{1+pb+qb^2} \le \frac{2}{1+p+q}$$

for all $a,b>0$ such that $ab=1$. Write this inequality as

$$(a-1)^2[q(1-p-q)a^2+(p+2q-p^2-pq-2q^2)a+q(1-p-q)] \ge 0,$$

which is true because

$$p+2q-p^2-pq-2q^2 \ge (p+2q)(p+q)-p^2-pq-2q^2 = 2pq \ge 0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 2.24. *If $a_1,a_2,\ldots,a_n$ are positive real numbers such that*

$$a_1 \le a_2 \le 1 \le a_3 \le \cdots \le a_n, \qquad a_1a_2\cdots a_n=1,$$

then

$$\frac{1}{(a_1+5)^2}+\frac{1}{(a_2+5)^2}+\cdots+\frac{1}{(a_n+5)^2} \le \frac{n}{36}.$$

(Vasile Cirtoaje, 2007)

Solution. Using the substitution

$$a_i=e^{x_i}, \qquad i=1,2,\ldots,n,$$

we can write the inequality as

$$f(x_1)+f(x_2)+\cdots+f(x_n) \ge nf(s),$$

where

$$x_1 \le x_2 \le 0 \le x_3 \le \cdots \le x_n, \quad s=\frac{x_1+x_2+\cdots+x_n}{n}=0,$$

$$f(u)=\frac{-1}{(e^u+5)^2}, \quad u \in \mathbb{R}.$$

For $u \in (-\infty,0]$, we have

$$f''(u)=\frac{2e^u(5-2e^u)}{(e^u+5)^4}>0,$$

hence f is convex on $(-\infty, 0]$. According to HCF-OV Theorem or LHCF-OV Corollary (case $m = 2$), it suffices to show that $f(x) + 2f(y) \geq 3f(0)$ for all real x, y such that $x + 2y = 0$; that is, to show that

$$\frac{1}{(a+5)^2} + \frac{2}{(b+5)^2} \leq \frac{1}{12}$$

for all $a, b > 0$ such that $ab^2 = 1$. Since

$$\frac{1}{(a+5)^2} = \frac{b^4}{(5b^2+1)^2} \leq \frac{b^4}{(4b^2+2b)^2} = \frac{b^2}{4(2b+1)^2},$$

it suffices to show that

$$\frac{b^2}{4(2b+1)^2} + \frac{2}{(b+5)^2} \leq \frac{1}{12},$$

which is equivalent to the obvious inequality

$$(b-1)^2(b^2+16b+1) \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. Similarly, we can prove the following generalization:

- *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that*

$$a_1 \leq a_2 \leq 1 \leq a_3 \leq \cdots \leq a_n, \qquad a_1 a_2 \cdots a_n = 1.$$

If $k \geq 2 + \sqrt{6}$, then

$$\frac{1}{(a_1+k)^2} + \frac{1}{(a_2+k)^2} + \cdots + \frac{1}{(a_n+k)^2} \leq \frac{n}{(1+k)^2},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 2.25. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that*

$$a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{\sqrt{1+3a_1}} + \frac{1}{\sqrt{1+3a_2}} + \cdots + \frac{1}{\sqrt{1+3a_n}} \geq \frac{n}{2}.$$

(Vasile Cirtoaje, 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \quad i = 1, 2, \ldots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s),$$

where

$$x_1 \ge 0 \ge x_2 \ge \cdots \ge x_n, \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

$$f(u) = \frac{1}{\sqrt{1+3e^u}}, \quad u \in \mathbb{R}.$$

For $u \ge 0$, we have

$$f''(u) = \frac{e^u(3e^u - 2)}{2(1+3e^u)^{5/2}} > 0,$$

hence f is convex on $[0, \infty)$. According to HCF-OV Theorem or RHCF-OV Corollary (case $m = 1$), it suffices to show that $f(x) + f(y) \ge 2f(0)$ for all real x, y such that $x + y = 0$; that is, to show that

$$\frac{1}{\sqrt{1+3a}} + \frac{1}{\sqrt{1+3b}} \ge 1$$

for all $a, b > 0$ such that $ab = 1$. Write this inequality as

$$\frac{1}{\sqrt{1+3a}} + \sqrt{\frac{a}{a+3}} \ge 1.$$

Substituting $\dfrac{1}{\sqrt{1+3a}} = t$, $0 < t < 1$, the inequality becomes

$$\sqrt{\frac{1-t^2}{8t^2+1}} \ge 1 - t.$$

By squaring, we get

$$t(1-t)(2t-1)^2 \ge 0,$$

which is true. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 2.26. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that*

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \quad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{\sqrt{1+2a_1}} + \frac{1}{\sqrt{1+2a_2}} + \cdots + \frac{1}{\sqrt{1+2a_n}} \le \frac{n}{\sqrt{3}}.$$

(Vasile Cirtoaje, 2007)

Solution. Using the substitution

$$a_i = e^{x_i}, \qquad i = 1, 2, \ldots, n,$$

we can write the inequality as

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge n f(s),$$

where

$$x_1 \le 0 \le x_2 \le \cdots \le x_n, \quad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

$$f(u) = \frac{-1}{\sqrt{1 + 2e^u}}, \quad u \in \mathbb{R}.$$

For $u \le 0$, we have

$$f''(u) = \frac{e^u(1 - e^u)}{(1 + 2e^u)^{5/2}} > 0,$$

hence f is convex on $(-\infty, 0]$. According to HCF-OV Theorem or LHCF-OV Corollary (case $m = 1$), it suffices to show that $f(x) + f(y) \ge 2f(0)$ for all real x, y such that $x + y = 0$; that is, to show that

$$\sqrt{\frac{3}{1 + 2a}} + \sqrt{\frac{3}{1 + 2b}} \le 2$$

for all $a, b > 0$ such that $ab = 1$. By the Cauchy-Schwarz inequality, we get

$$\sqrt{\frac{3}{1 + 2a}} + \sqrt{\frac{3}{1 + 2b}} \le \sqrt{\left(\frac{3}{1 + 2a} + 1\right)\left(1 + \frac{3}{1 + 2b}\right)} = 2.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

Chapter 3

Partially Convex Function Method

3.1 Theoretical Basis

Right Partially Convex Function Theorem (*Vasile Cirtoaje*, 2012). *Let f be a function defined on a real interval $\mathbb{I}$ and convex on $[s,s_0]$, where $s,s_0 \in \mathbb{I}$, $s < s_0$. In addition, $f(u)$ is decreasing on $\mathbb{I}_{u\le s_0}$ and*

$$\min_{u\in\mathbb{I}} f(u) = f(s_0).$$

The inequality

$$f(a_1)+f(a_2)+\cdots+f(a_n) \ge nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

holds for all $a_1,a_2,\cdots,a_n \in \mathbb{I}$ satisfying $a_1+a_2+\cdots+a_n = ns$ if and only if

$$f(x)+(n-1)f(y) \ge nf(s)$$

for all $x,y \in \mathbb{I}$ such that $x \le s \le y$ and $x+(n-1)y = ns$.

Proof. The necessity is obvious. By Lemma below, to prove the sufficiency, it suffices to consider that $a_1,a_2,\ldots,a_n \in \mathbb{J}$, where

$$\mathbb{J} = \mathbb{I}_{u\le s_0}.$$

Because $f(u)$ is convex on $\mathbb{J}_{u\ge s}$, the desired inequality follows from HCF Theorem applied to the interval $\mathbb{J}$.

Lemma. *Let f be a function defined on a real interval $\mathbb{I}$ and convex on $[s,s_0]$, where $s,s_0 \in \mathbb{I}$, $s < s_0$. In addition, $f(u)$ is decreasing on $\mathbb{I}_{u\le s_0}$ and*

$$\min_{u\in\mathbb{I}} f(u) = f(s_0).$$

If the inequality

$$f(a_1)+f(a_2)+\cdots+f(a_n) \ge nf(s)$$

holds for all $a_1,a_2,\cdots,a_n \in \mathbb{I}_{u\le s_0}$ *such that* $a_1+a_2+\cdots+a_n = ns$*, then it holds for all* $a_1,a_2,\ldots,a_n \in \mathbb{I}$ *such that* $a_1+a_2+\cdots+a_n = ns$.

Proof. For $i=1,2,\ldots,n$, define the numbers $b_i \in \mathbb{I}_{u\le s_0}$ as follows

$$b_i = \begin{cases} a_i, & a_i \le s_0 \\ s_0, & a_i > s_0. \end{cases}$$

We have $b_i \le a_i$ and $f(b_i) \le f(a_i)$ for $i=1,2,\cdots,n$. Therefore,

$$b_1+b_2+\cdots+b_n \le a_1+a_2+\cdots+a_n = ns$$

and

$$f(b_1)+f(b_2)+\cdots+f(b_n) \le f(a_1)+f(a_2)+\cdots+f(a_n).$$

Thus, it suffices to show that

$$f(b_1)+f(b_2)+\cdots+f(b_n) \ge nf(s)$$

for all $b_1,b_2,\ldots,b_n \in \mathbb{I}_{u\le s_0}$ such that $b_1+b_2+\cdots+b_n \le ns$. By hypothesis, this inequality is true for $b_1,b_2,\ldots,b_n \in \mathbb{I}_{u\le s_0}$ and $b_1+b_2+\cdots+b_n = ns$. Since $f(u)$ is decreasing on $\mathbb{I}_{u\le s_0}$, the more we have $f(b_1)+f(b_2)+\cdots+f(b_n) \ge nf(s)$ for $b_1,b_2,\ldots,b_n \in \mathbb{I}_{u\le s_0}$ and $b_1+b_2+\cdots+b_n \le ns$.

Similarly, we can prove Left Partially Convex Function Theorem (*Vasile Cirtoaje*, 2012).

Left Partially Convex Function Theorem. *Let f be a function defined on a real interval $\mathbb{I}$ and convex on $[s_0,s]$, where $s_0,s \in \mathbb{I}$, $s_0 < s$. In addition, $f(u)$ is increasing on $\mathbb{I}_{u\ge s_0}$ and satisfies*

$$\min_{u\in\mathbb{I}} f(u) = f(s_0).$$

The inequality

$$f(a_1)+f(a_2)+\cdots+f(a_n) \ge nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

holds for all $a_1,a_2,\cdots,a_n \in \mathbb{I}$ *satisfying* $a_1+a_2+\cdots+a_n = ns$ *if and only if*

$$f(x)+(n-1)f(y) \ge nf(s)$$

for all $x,y \in \mathbb{I}$ *such that* $x \ge s \ge y$ *and* $x+(n-1)y = ns$.

Remark 1. Let us denote

$$g(u) = \frac{f(u)-f(s)}{u-s}, \quad h(x,y) = \frac{g(x)-g(y)}{x-y}.$$

As shown in Remark 1 from 1.1, we may replace the hypothesis condition in Right Partially Convex Function Theorem (RPCF Theorem) and Left Partially Convex Function Theorem (LPCF Theorem), namely

$$f(x)+(n-1)f(y)\geq nf(s),$$

by the condition

$$h(x,y)\geq 0 \ \textit{ for all } \ x,y\in\mathbb{I} \ \textit{ such that } \ x+(n-1)y=ns.$$

Remark 2. Assume that f is differentiable on $\mathbb{I}$, and let

$$H(x,y)=\frac{f'(x)-f'(y)}{x-y}.$$

As shown in Remark 2 from 1.1, the inequalities in RPCF Theorem and LPCF Theorem hold true by replacing the hypothesis

$$f(x)+(n-1)f(y)\geq nf(s)$$

with the more restrictive condition

$$H(x,y)\geq 0 \ \text{ for all } \ x,y\in\mathbb{I} \ \text{ such that } \ x+(n-1)y=ns.$$

Remark 3. The inequality in RPCF Theorem or LPCF Theorem becomes an equality for

$$a_1=a_2=\cdots=a_n=s$$

and also for

$$a_1=x, \quad a_2=\cdots=a_n=y$$

(or any cyclic permutation), where $x,y\in\mathbb{I}$ satisfy the equations

$$x+(n-1)y=ns, \quad f(x)+(n-1)f(y)=nf(s).$$

For $x\neq y$, these equations are equivalent to

$$x+(n-1)y=ns, \quad h(x,y)=0.$$

Remark 4. RPCF Theorem is also valid in the case when $\mathbb{I}=[a,b]\setminus\{u_0\}$ or $\mathbb{I}=(a,b)\setminus\{u_0\}$, where a,b,u_0 are real numbers such that

$$a<s<s_0<u_0<b.$$

Similarly, LPCF Theorem is also valid in the case when $\mathbb{I}=[a,b]\setminus\{u_0\}$ or $\mathbb{I}=(a,b)\setminus\{u_0\}$, where a,b,u_0 are real numbers such that

$$a<u_0<s_0<s<b.$$

Remark 5. The desired inequality in RPCF Theorem holds true by replacing the condition

$$f \textit{ is decreasing on } \mathbb{I}_{u \le s_0}$$

with the sufficient condition

$$ns - (n-1)s_0 \le \inf \mathbb{I}.$$

To prove this claim, we define the function

$$f_0(u) = \begin{cases} f(u), & u \le s_0,\ u \in \mathbb{I} \\ f(s_0), & u \ge s_0,\ u \in \mathbb{I}, \end{cases}$$

which is convex for $u \in \mathbb{I}$, $u \ge s$. Taking into account that $f_0(s) = f(s)$ and $f_0(u) \le f(u)$ for all $u \in \mathbb{I}$, it suffices to prove that

$$f_0(x_1) + f_0(x_2) + \cdots + f_0(x_n) \ge nf_0(s)$$

for all $x_1, x_2, \cdots, x_n \in \mathbb{I}$ satisfying $x_1 + x_2 + \cdots + x_n = ns$. According to HCF Theorem and Remark 5 from Section 1, we only need to show that

$$f_0(x) + (n-1)f_0(y) \ge nf_0(s) \tag{*}$$

for all $x, y \in \mathbb{I}$ such that $x \le s \le y$ and $x+(n-1)y = ns$. The case $y > s_0$ is not possible because

$$x = ns - (n-1)y < ns - (n-1)s_0 \le \inf \mathbb{I}$$

involves $x \notin \mathbb{I}$. For the possible case $y \le s_0$, (*) becomes $f(x) + (n-1)f(y) \ge nf(s)$, which holds (by hypothesis) for all $x, y \in \mathbb{I}$ such that $x + (n-1)y = ns$.

Similarly, we can show that the desired inequality in LPCF Theorem holds true by replacing the condition

$$f \textit{ is increasing on } \mathbb{I}_{u \ge s_0}$$

with the sufficient condition

$$ns - (n-1)s_0 \ge \sup \mathbb{I}.$$

3.2 Applications

3.1. If a, b, c are real numbers such that $a+b+c=3$, then

$$\frac{16a-5}{32a^2+1}+\frac{16b-5}{32b^2+1}+\frac{16c-5}{32c^2+1}\le 1.$$

3.2. If a, b, c, d are real numbers such that $a+b+c+d=4$, then

$$\frac{18a-5}{12a^2+1}+\frac{18b-5}{12b^2+1}+\frac{18c-5}{12c^2+1}+\frac{18d-5}{12d^2+1}\le 4.$$

3.3. If a, b, c, d, e, f are real numbers such that $a+b+c+d+e+f=6$, then

$$\frac{5a-1}{5a^2+1}+\frac{5b-1}{5b^2+1}+\frac{5c-1}{5c^2+1}+\frac{5d-1}{5d^2+1}+\frac{5e-1}{5e^2+1}+\frac{5f-1}{5f^2+1}\le 4.$$

3.4. If $a_1, a_2, \cdots, a_n$ $(n\ge 3)$ are real numbers such that $a_1+a_2+\cdots+a_n=n$, then

$$\frac{n(n+1)-2a_1}{n^2+(n-2)a_1^2}+\frac{n(n+1)-2a_2}{n^2+(n-2)a_2^2}+\cdots+\frac{n(n+1)-2a_n}{n^2+(n-2)a_n^2}\le n.$$

3.5. If a, b, c, d are real numbers such that $a+b+c+d=4$, then

$$\frac{a(a-1)}{3a^2+4}+\frac{b(b-1)}{3b^2+4}+\frac{c(c-1)}{3c^2+4}+\frac{d(d-1)}{3d^2+4}\ge 0.$$

3.6. Let $a_1, a_2, \cdots, a_n \ne -k$ be real numbers such that $a_1+a_2+\cdots+a_n=n$, where

$$k\ge\frac{n}{2\sqrt{n-1}}.$$

Then,

$$\frac{a_1(a_1-1)}{(a_1+k)^2}+\frac{a_2(a_2-1)}{(a_2+k)^2}+\cdots+\frac{a_n(a_n-1)}{(a_n+k)^2}\ge 0.$$

3.7. Let $a_1, a_2, \ldots, a_n \neq -k$ be real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If

$$k \geq 1 + \frac{n}{\sqrt{n-1}},$$

then

$$\frac{a_1^2 - 1}{(a_1 + k)^2} + \frac{a_2^2 - 1}{(a_2 + k)^2} + \cdots + \frac{a_n^2 - 1}{(a_n + k)^2} \geq 0.$$

3.8. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n \geq n$. If $k > 1$, then

$$\frac{a_1}{a_1^k + a_2 + \cdots + a_n} + \frac{a_2}{a_1 + a_2^k + \cdots + a_n} + \cdots + \frac{a_n}{a_1 + a_2 + \cdots + a_n^k} \leq 1.$$

3.9. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n \geq n$. If $k > 1$, then

$$\frac{1}{a_1^k + a_2 + \cdots + a_n} + \frac{1}{a_1 + a_2^k + \cdots + a_n} + \cdots + \frac{1}{a_1 + a_2 + \cdots + a_n^k} \leq 1.$$

3.10. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$, then

$$\frac{a_1 - 1}{a_1^2 + a_2 + \cdots + a_n} + \frac{a_2 - 1}{a_1 + a_2^2 + \cdots + a_n} + \cdots + \frac{a_n - 1}{a_1 + a_2 + \cdots + a_n^2} \leq 0.$$

3.11. Let $a_1, a_2, \cdots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $0 < k \leq \frac{n}{n-1}$, then

$$a_1^{k/a_1} + a_2^{k/a_2} + \cdots + a_n^{k/a_n} \leq n.$$

3.12. If a, b, c, d, e are nonzero real numbers such that $a + b + c + d + e = 5$, then

$$\left(7 - \frac{5}{a}\right)^2 + \left(7 - \frac{5}{b}\right)^2 + \left(7 - \frac{5}{c}\right)^2 + \left(7 - \frac{5}{d}\right)^2 + \left(7 - \frac{5}{e}\right)^2 \geq 20.$$

3.13. If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then

$$(1 - a + a^4)(1 - b + b^4)(1 - c + c^4) \geq 1.$$

3.14. If a, b, c, d are nonnegative real numbers such that $a+b+c+d=4$, then

$$(1-a+a^3)(1-b+b^3)(1-c+c^3)(1-d+d^3) \geq 1.$$

3.15. If $a_1, a_2, \ldots, a_{10}$ are real numbers such that $a_1+a_2+\cdots+a_{10}=10$, then

$$(1-a_1+a_1^2)(1-a_2+a_2^2)\cdots(1-a_{10}+a_{10}^2) \geq 1.$$

3.16. If a, b, c, d, e are nonzero real numbers such that $a+b+c+d+e=5$, then

$$5\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}+\frac{1}{e^2}\right)+45 \geq 14\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right).$$

3.17. If a, b, c are positive real numbers such that $abc=1$, then

$$\frac{7-6a}{2+a^2}+\frac{7-6b}{2+b^2}+\frac{7-6c}{2+c^2} \geq 1.$$

3.18. If a, b, c are positive real numbers such that $abc=1$, then

$$\frac{1}{a+5bc}+\frac{1}{b+5ca}+\frac{1}{c+5ab} \leq \frac{1}{2}.$$

3.19. If a, b, c are positive real numbers such that $abc=1$, then

$$\frac{1}{4-3a+4a^2}+\frac{1}{4-3b+4b^2}+\frac{1}{4-3c+4c^2} \leq \frac{3}{5}.$$

3.20. If a, b, c are positive real numbers such that $abc=1$, then

$$\frac{1}{(3a+1)(3a^2-5a+3)}+\frac{1}{(3b+1)(3b^2-5b+3)}+\frac{1}{(3c+1)(3c^2-5c+3)} \leq \frac{3}{4}.$$

3.21. Let $a_1, a_2, \cdots, a_n$ $(n \geq 3)$ be positive real numbers such that $a_1a_2\cdots a_n=1$. If $p, q \geq 0$ such that $p+4q \geq n-1$, then

$$\frac{1-a_1}{1+pa_1+qa_1^2}+\frac{1-a_2}{1+pa_2+qa_2^2}+\cdots+\frac{1-a_n}{1+pa_n+qa_n^2} \geq 0.$$

3.22. If a, b, c are positive real numbers such that $abc = 1$, then

$$\frac{1-a}{17+4a+6a^2}+\frac{1-b}{17+4b+6b^2}+\frac{1-c}{17+4c+6c^2}\ge 0.$$

3.23. If $a_1, a_2, \cdots, a_8$ are positive real numbers such that $a_1a_2\cdots a_8 = 1$, then

$$\frac{1-a_1}{(1+a_1)^2}+\frac{1-a_2}{(1+a_2)^2}+\cdots+\frac{1-a_8}{(1+a_8)^2}\ge 0.$$

3.24. Let a, b, c be positive real numbers such that $abc = 1$. If $k \in \left[\frac{-13}{3\sqrt{3}}, \frac{13}{3\sqrt{3}}\right]$, then

$$\frac{a+k}{a^2+1}+\frac{b+k}{b^2+1}+\frac{c+k}{c^2+1}\le\frac{3(1+k)}{2}.$$

3.25. If a, b, c are positive real numbers and $0 < k \le 2+2\sqrt{2}$, then

$$\frac{a^3}{ka^2+bc}+\frac{b^3}{kb^2+ca}+\frac{c^3}{kc^2+ab}\ge\frac{a+b+c}{k+1}.$$

3.26. If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are real numbers such that

$$a_1, a_2, \ldots, a_n \ge \frac{-1}{n-2}, \quad a_1+a_2+\cdots+a_n = n,$$

then

$$\frac{a_1^2}{a_1^2-a_1+1}+\frac{a_2^2}{a_2^2-a_2+1}+\cdots+\frac{a_n^2}{a_n^2-a_n+1}\le n.$$

3.27. If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are nonzero real numbers such that

$$a_1, a_2, \ldots, a_n \ge \frac{-n}{n-2}, \quad a_1+a_2+\cdots+a_n = n,$$

then

$$\frac{1}{a_1^2}+\frac{1}{a_2^2}+\cdots+\frac{1}{a_n^2}\ge\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}.$$

3.28. If $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ are real numbers such that

$$a_1, a_2, \ldots, a_n \geq \frac{-(3n-2)}{n-2}, \quad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{1-a_1}{(1+a_1)^2} + \frac{1-a_2}{(1+a_2)^2} + \cdots + \frac{1-a_n}{(1+a_n)^2} \geq 0.$$

3.29. Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$. If $n \geq 3$ and $k \geq 2 - \frac{2}{n}$, then

$$\frac{1-a_1}{(1-ka_1)^2} + \frac{1-a_2}{(1-ka_2)^2} + \cdots + \frac{1-a_n}{(1-ka_n)^2} \geq 0.$$

3.3 Solutions

P 3.1. *If a, b, c are real numbers such that $a+b+c=3$, then*

$$\frac{16a-5}{32a^2+1}+\frac{16b-5}{32b^2+1}+\frac{16c-5}{32c^2+1}\le 1.$$

(Vasile Cîrtoaje, 2012)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)\ge 3f(s),\quad s=\frac{a+b+c}{3}=1,$$

where

$$f(u)=\frac{5-16u}{32u^2+1},\quad u\in\mathbb{R}.$$

From

$$f'(u)=\frac{16(32u^2-20u-1)}{(32u^2+1)^2},$$

it follows that f is increasing on $\left(-\infty,\frac{5-\sqrt{33}}{16}\right]\cup[s_0,\infty)$ and decreasing on $\left[\frac{5-\sqrt{33}}{16},s_0\right]$, where $s_0=\frac{5+\sqrt{33}}{16}\approx 0.6715$. Since

$$\lim_{u\to-\infty}f(u)=0$$

and $f(s_0)<f(5/16)=0$, it follows that

$$\min_{u\in\mathbb{R}}f(u)=f(s_0).$$

For $u\in[5/16,1]$, we have

$$\frac{1}{64}f''(u)=\frac{-512u^3+480u^2+48u-5}{(32u^2+1)^3}$$
$$=\frac{512u^2(1-u)+32u(1-u)+(16u-5)}{(32u^2+1)^3}>0.$$

Therefore, f is convex on $[1/2,1]$, hence on $[s_0,1]$. According to LPCF Theorem, we only need to show that $f(x)+2f(y)\ge 3f(1)$ for all real x,y such that $x+2y=3$. Using Remark 1, it suffices to prove that $h(x,y)\ge 0$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y},\quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

Indeed, we have

$$g(u)=\frac{32(2u-1)}{3(32u^2+1)},$$

$$h(x,y)=\frac{64(1+16x+16y-32xy)}{3(32x^2+1)(32y^2+1)}=\frac{64(4x-5)^2}{3(32x^2+1)(32y^2+1)}\geq 0.$$

Thus, the proof is completed. From $x+2y=3$ and $h(x,y)=0$, we get $x=5/4$ and, $y=7/8$. Therefore, in accordance with Remark 3, the equality holds for $a=b=c=1$, and also for $a=5/4$ and $b=c=7/8$ (or any cyclic permutation).

□

P 3.2. *If a,b,c,d are real numbers such that $a+b+c+d=4$, then*

$$\frac{18a-5}{12a^2+1}+\frac{18b-5}{12b^2+1}+\frac{18c-5}{12c^2+1}+\frac{18d-5}{12d^2+1}\leq 4.$$

(Vasile Cîrtoaje, 2012)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)\geq 4f(s),\quad s=\frac{a+b+c+d}{4}=1,$$

where

$$f(u)=\frac{5-18u}{12u^2+1},\quad u\in\mathbb{R}.$$

From

$$f'(u)=\frac{6(36u^2-20u-3)}{(12u^2+1)^2},$$

it follows that f is increasing on $\left(-\infty,\frac{5-\sqrt{52}}{18}\right]\cup[s_0,\infty)$ and decreasing on $\left[\frac{5-\sqrt{52}}{18},s_0\right]$, where $s_0=\frac{5+\sqrt{52}}{18}\approx 0.678$. Since

$$\lim_{u\to-\infty}f(u)=0$$

and $f(s_0)<f(5/18)=0$, it follows that

$$\min_{u\in\mathbb{R}}f(u)=f(s_0).$$

For $u\in[5/18,1]$, we have

$$\frac{1}{24}f''(u)=\frac{-216u^3+180u^2+54u-5}{(12u^2+1)^3}$$

$$=\frac{216u^2(1-u)+36u(1-u)+(18u-5)}{(32u^2+1)^3}>0.$$

Therefore, f is convex on $[1/2,1]$, hence on $[s_0,1]$. According to LPCF Theorem and Remark 1, we only need to show that $h(x,y)\geq 0$ for $x,y\in\mathbb{R}$ such that $x+3y=4$. We have

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{6(2u-1)}{12u^2+1},$$

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{12(1+6x+6y-12xy)}{(12x^2+1)(12y^2+1)}=\frac{12(2x-3)^2}{(12x^2+1)(12y^2+1)}\geq 0.$$

Thus, the proof is completed. From $x+3y=4$ and $h(x,y)=0$, we get $x=3/2$ and, $y=5/6$. Therefore, in accordance with Remark 3, the equality holds for $a=b=c=d=1$, and also for $a=3/2$ and $b=c=d=5/6$ (or any cyclic permutation).

□

P 3.3. *If a,b,c,d,e,f are real numbers such that $a+b+c+d+e+f=6$, then*

$$\frac{5a-1}{5a^2+1}+\frac{5b-1}{5b^2+1}+\frac{5c-1}{5c^2+1}+\frac{5d-1}{5d^2+1}+\frac{5e-1}{5e^2+1}+\frac{5f-1}{5f^2+1}\leq 4.$$

(Vasile Cîrtoaje, 2012)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)+f(e)+f(f)\geq 4f(s),\quad s=\frac{a+b+c+d+e+f}{6}=1,$$

where

$$f(u)=\frac{1-5u}{5u^2+1},\quad u\in\mathbb{R}.$$

From

$$f'(u)=\frac{5(5u^2-2u-1)}{(5u^2+1)^2},$$

it follows that f is increasing on $\left(-\infty,\frac{1-\sqrt{6}}{5}\right]\cup[s_0,\infty)$ and decreasing on $\left[\frac{1-\sqrt{6}}{5},s_0\right]$, where $s_0=\frac{1+\sqrt{6}}{5}\approx 0.69$. Since

$$\lim_{u\to-\infty}f(u)=0$$

and $f(s_0)<f(1/5)=0$, it follows that

$$\min_{u\in\mathbb{R}}f(u)=f(s_0).$$

For $u \in [1/5,1]$, we have

$$\frac{1}{10}f''(u) = \frac{-25u^3+15u^2+15u-1}{(5u^2+1)^3}$$
$$= \frac{25u^2(1-u)+10u(1-u)+(5u-1)}{(3u^2+4)^3} > 0.$$

Therefore, f is convex on $[1/5,1]$, hence on $[s_0,1]$. According to LPCF Theorem and Remark 1, we only need to show that $h(x,y) \ge 0$ for $x,y \in \mathbb{R}$ such that $x+5y=6$. We have

$$g(u) = \frac{f(u)-f(1)}{u-1} = \frac{5(2u-1)}{3(5u^2+1)},$$

$$h(x,y) = \frac{g(x)-g(y)}{x-y} = \frac{5(2+5x+5y-10xy)}{3(5x^2+1)(5y^2+1)} = \frac{10(x-2)^2}{3(5x^2+1)(5y^2+1)} \ge 0.$$

Thus, the proof is completed. In accord with Remark 3, the equality holds for $a=b=c=d=e=f=1$, and also for $a=2$ and $b=c=d=e=f=4/5$ (or any cyclic permutation).

□

P 3.4. *If $a_1,a_2,\cdots,a_n$ $(n \ge 3)$ are real numbers such that $a_1+a_2+\cdots+a_n=n$, then*

$$\frac{n(n+1)-2a_1}{n^2+(n-2)a_1^2}+\frac{n(n+1)-2a_2}{n^2+(n-2)a_2^2}+\cdots+\frac{n(n+1)-2a_n}{n^2+(n-2)a_n^2} \le n.$$

(Vasile Cîrtoaje, 2008)

Solution. The desired inequality is true for $a_1 > \frac{n(n+1)}{2}$ since

$$\frac{n(n+1)-2a_1}{n^2+(n-2)a_1^2} < 0$$

and

$$\frac{n(n+1)-2a_i}{n^2+(n-2)a_i^2} < \frac{n}{n-1}, \quad i=2,3,\cdots,n.$$

The last inequality is equivalent to

$$n(n-2)a_i^2+2(n-1)a_i+n > 0,$$

and is true because

$$n(n-2)a_i^2+2(n-1)a_i+n \ge (n-1)a_i^2+2(n-1)a_i+n > (n-1)(a_i+1)^2 \ge 0.$$

Consider further that $a_1, a_2, \cdots, a_n \le \dfrac{n(n+1)}{2}$ and rewrite the desired inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{2u - n(n+1)}{(n-2)u^2 + n^2}, \quad u \in \mathbb{I} = \left(-\infty, \frac{n(n+1)}{2}\right].$$

We have

$$\frac{f'(u)}{2(n-2)} = \frac{n^2 + n(n+1)u - u^2}{[(n-2)u^2 + n^2]^2}$$

and

$$\frac{f''(u)}{2(n-2)} = \frac{f_1(u)}{[(n-2)u^2 + n^2]^3},$$

where

$$f_1(u) = 2(n-2)u^3 - 3n(n+1)(n-2)u^2 - 2n^2(2n-3)u + n^3(n+1).$$

From the expression of f', it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $\left[s_0, \dfrac{n(n+1)}{2}\right]$, where

$$s_0 = \frac{n}{2}(n + 1 - \sqrt{n^2 + 2n + 5}\,) \in (-1, 0);$$

therefore,

$$\min_{u \in \mathbb{I}} f(u) = f(s_0).$$

On the other hand, for $-1 \le u \le 1$, we have

$$\begin{aligned} f_1(u) &> -2(n-2) - 3n(n+1)(n-2) - 2n^2(2n-3) + n^3(n+1) \\ &= n^2(n-3)^2 + 4(n+1) > 0, \end{aligned}$$

and hence $f''(u) > 0$. Since $[s_0, 1] \subset [-1, 1]$, f is convex on $[s_0, 1]$. By LPCF Theorem and Remark 1, we only need to show that $h(x, y) \ge 0$ for $x, y \in \mathbb{R}$ and $x + (n-1)y = n$, where

$$h(x, y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

Indeed, we have

$$g(u) = \frac{(n-2)u + n}{(n-2)u^2 + n^2}$$

and

$$\frac{h(x,y)}{n-2} = \frac{n^2-n(x+y)-(n-2)xy}{[(n-2)x^2+n^2][(n-2)y^2+n^2]}$$
$$= \frac{(n-1)(n-2)y^2}{[(n-2)x^2+n^2][(n-2)y^2+n^2]} \geq 0.$$

The proof is completed. In accord with Remark 3, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = n$ and $a_2 = \cdots = a_n = 0$ (or any cyclic permutation).

□

P 3.5. *If a,b,c,d are real numbers such that $a+b+c+d=4$, then*

$$\frac{a(a-1)}{3a^2+4} + \frac{b(b-1)}{3b^2+4} + \frac{c(c-1)}{3c^2+4} + \frac{d(d-1)}{3d^2+4} \geq 0.$$

(Vasile Cîrtoaje, 2012)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d) \geq 4f(s), \quad s = \frac{a+b+c+d}{4} = 1,$$

where

$$f(u) = \frac{u^2-u}{3u^2+4}, \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{3u^2+8u-4}{(3u^2+4)^2},$$

it follows that f is increasing on $\left(-\infty, \frac{-4-2\sqrt{7}}{3}\right] \cup [s_0,\infty)$ and decreasing on $\left[\frac{-4-2\sqrt{7}}{3}, s_0\right]$, where $s_0 = \frac{-4+2\sqrt{7}}{3} \approx 0.43$. Since

$$\lim_{u\to-\infty} f(u) = 1/3$$

and $f(s_0) < f(0) = 0$, it follows that

$$\min_{u\in\mathbb{R}} f(u) = f(s_0).$$

For $u \in [0,1]$, we have

$$\frac{1}{2}f''(u) = \frac{-9u^3-36u^2+36u+14}{(3u^2+4)^3}$$
$$= \frac{9u^2(1-u)+45u(1-u)+(16-9u)}{(3u^2+4)^3} > 0.$$

Therefore, f is convex on $[0,1]$, hence on $[s_0,1]$. According to LPCF Theorem and Remark 1, we only need to show that $h(x,y) \ge 0$ for $x,y \in \mathbb{R}$ such that $x+3y=4$. We have

$$g(u) = \frac{f(u)-f(1)}{u-1} = \frac{u)}{3u^2+4},$$

$$h(x,y) = \frac{g(x)-g(y)}{x-y} = \frac{4-3xy}{(3x^2+4)(3y^2+4)} = \frac{(x-2)^2}{(12x^2+1)(12y^2+1)} \ge 0.$$

The proof is completed. From $x+3y=4$ and $h(x,y)=0$, we get $x=2$ and, $y=2/3$. In accord with Remark 3, the equality holds for $a=b=c=d=1$, and also for $a=2$ and $b=c=d=2/3$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization.

- *If $a_1,a_2,\dots,a_n$ are real numbers such that $a_1+a_2+\cdots+a_n=n$, then*

$$\frac{a_1(a_1-1)}{4(n-1)a_1^2+n^2} + \frac{a_2(a_2-1)}{4(n-1)a_2^2+n^2} + \cdots + \frac{a_n(a_n-1)}{4(n-1)a_n^2+n^2} \ge 0,$$

with equality for $a_1=a_2=\cdots=a_n=1$, and also for $a_1=\dfrac{n}{2}$ and $a_2=a_3=\cdots=a_n=\dfrac{n}{2(n-1)}$ (or any cyclic permutation).

□

P 3.6. *Let $a_1,a_2,\cdots,a_n \ne -k$ be real numbers such that $a_1+a_2+\cdots+a_n=n$, where*

$$k \ge \frac{n}{2\sqrt{n-1}}.$$

Then,

$$\frac{a_1(a_1-1)}{(a_1+k)^2} + \frac{a_2(a_2-1)}{(a_2+k)^2} + \cdots + \frac{a_n(a_n-1)}{(a_n+k)^2} \ge 0.$$

(Vasile Cîrtoaje, 2008)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n) \ge nf(s), \quad s = \frac{a_1+a_2+\cdots+a_n}{n} = 1,$$

where

$$f(u) = \frac{u(u-1)}{(u+k)^2}, \quad u \in \mathbb{I} = \mathbb{R} \setminus \{-k\}.$$

From

$$f'(u) = \frac{(2k+1)u-k}{(u+k)^3},$$

it follows that f is increasing on $(-\infty,-k)\cup[s_0,\infty)$ and decreasing on $(-k,s_0]$, where

$$s_0=\frac{k}{2k+1}<1.$$

Since

$$\lim_{u\to-\infty} f(u)=1$$

and $f(s_0)<f(1)=0$, we have

$$\min_{u\in\mathbb{I}} f(u)=f(s_0).$$

From

$$\frac{1}{2}f''(u)=\frac{k(k+2)-(2k+1)u}{(u+k)^4},$$

it follows that f is convex on $\left[0,\frac{k(k+2)}{2k+1}\right]$, hence on $[s_0,1]$. According to LPCF Theorem, Remark 4 and Remark 1, it suffices to show that $h(x,y)\ge 0$ for all $x,y\in\mathbb{I}$ which satisfy $x+(n-1)y=n$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y},\qquad g(u)=\frac{f(u)-f(1)}{u-1}.$$

Indeed, we have

$$g(u)=\frac{u}{(u+k)^2}$$

and

$$h(x,y)=\frac{k^2-xy}{(x+k)^2(y+k)^2}\ge 0,$$

because

$$k^2-xy\ge\frac{n^2}{4(n-1)}-xy=\frac{[2(n-1)y-n]^2}{4(n-1)}\ge 0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$. If $k=\frac{n}{2\sqrt{n-1}}$, then the equality holds also for $a_1=\frac{n}{2}$ and $a_2=\cdots=a_n=\frac{n}{2(n-1)}$ (or any cyclic permutation).

□

P 3.7. *Let $a_1,a_2,\ldots,a_n\ne -k$ be real numbers such that $a_1+a_2+\cdots+a_n=n$. If*

$$k\ge 1+\frac{n}{\sqrt{n-1}},$$

then

$$\frac{a_1^2-1}{(a_1+k)^2}+\frac{a_2^2-1}{(a_2+k)^2}+\cdots+\frac{a_n^2-1}{(a_n+k)^2}\ge 0.$$

(Vasile Cîrtoaje, 2008)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s), \quad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=\frac{u^2-1}{(u+k)^2}, \quad u\in\mathbb{I}=\mathbb{R}\setminus\{-k\}.$$

From

$$f'(u)=\frac{2(ku+1)}{(u+k)^3},$$

it follows that f is increasing on $(-\infty,-k)\cup[s_0,\infty)$ and decreasing on $(-k,s_0]$, where

$$s_0=\frac{-1}{k}\in(-1,0).$$

Since

$$\lim_{u\to-\infty}f(u)=1$$

and $f(s_0)<f(-1)=0$, we have

$$\min_{u\in\mathbb{I}}f(u)=f(s_0).$$

From

$$f''(u)=\frac{2(k^2-3-2ku)}{(u+k)^4},$$

it follows that f is convex on $[-1,1]$, hence on $[s_0,1]$, since

$$k^2-3-2ku\ge k^2-3-2k=(k+1)(k-3)\ge 0.$$

According to LPCF Theorem, Remark 4 and Remark 1, it suffices to show that $h(x,y)\ge 0$ for $x,y\in\mathbb{I}$ which satisfy $x+(n-1)y=n$. We have

$$g(u)=\frac{f(u)-f(1)}{u-1}=\frac{u+1}{(u+k)^2},$$

$$h(x,y)=\frac{g(x)-g(y)}{x-y}=\frac{(k-1)^2-1-x-y-xy}{(x+k)^2(y+k)^2}\ge 0,$$

since

$$(k-1)^2-1-x-y-xy\ge\frac{n^2}{n-1}-1-x-y-xy=\frac{[(n-1)y-1]^2}{n-1}\ge 0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$. If $k=1+\dfrac{n}{\sqrt{n-1}}$, then the equality holds also for $a_1=n-1$ and $a_2=\cdots=a_n=\dfrac{1}{n-1}$ (or any cyclic permutation).

□

P 3.8. *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n \ge n$. If $k > 1$, then*

$$\frac{a_1}{a_1^k + a_2 + \cdots + a_n} + \frac{a_2}{a_1 + a_2^k + \cdots + a_n} + \cdots + \frac{a_n}{a_1 + a_2 + \cdots + a_n^k} \le 1.$$

(Vasile Cîrtoaje, 2006)

Solution. According to the proof of P 2.106 from Volume 2, it suffices to consider the case where $a_1 + a_2 + \cdots + a_n = n$ and $k > n + 1$. Write the desired inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{-u}{u^k - u + n}, \quad u \in \mathbb{I} = [0, n].$$

We have

$$f'(u) = \frac{(k-1)u^k - n}{(u^k - u + n)^2}$$

and

$$f''(u) = \frac{f_1(u)}{(u^k - u + n)^3},$$

where

$$f_1(u) = k(k-1)u^{k-1}(u^k - u + n) - 2(ku^{k-1} - 1)[(k-1)u^k - n].$$

From the expression of f', it follows that f is decreasing on $[0, s_0]$ and increasing on $[s_0, n]$, where

$$s_0 = \left(\frac{n}{k-1}\right)^{1/k} < 1.$$

On the other hand, for $u \in [s_0, 1]$, we have

$$(k-1)u^k - n \ge (k-1)s_0^k - n = 0,$$

and hence

$$\begin{aligned} f_1(u) &\ge k(k-1)u^{k-1}(u^k - u + n) - 2ku^{k-1}[(k-1)u^k - n] \\ &= ku^{k-1}[-(k-1)(u^k + u) + n(k+1)] \\ &\ge ku^{k-1}[-2(k-1) + 2(k+1)] = 4ku^{k-1} > 0. \end{aligned}$$

Thus, $f''(u) > 0$, and hence f is convex on $[s_0, 1]$. By LPCF Theorem, we need to show that $f(x) + (n-1)f(y) \ge nf(1)$ for all positive x, y which satisfy $x \ge 1 \ge y \ge 0$ and $x + (n-1)y = n$. Consider the nontrivial case where $x > 1 > y \ge 0$ and write the inequality $f(x) + (n-1)f(y) \ge nf(1)$ as follows:

$$\frac{x}{x^k - x - n} + \frac{(n-1)y}{y^k - y + n} \le 1,$$

$$x^k - x + n \geq \frac{x(y^k - y + n)}{y^k - ny + n},$$

$$x^k - x \geq \frac{(n-1)y(y - y^k)}{y^k - ny + n}.$$

Since $y < 1$, it suffices to show that

$$x^k - x \geq \frac{(n-1)(y - y^k)}{y^k - y + 1},$$

which is equivalent to

$$h(x) \geq \frac{y - y^k}{(1-y)(y^k - y + 1)},$$

where

$$h(x) = \frac{x^k - x}{x - 1}.$$

By the weighted AM-GM inequality, we have

$$h'(x) = \frac{(k-1)x^k + 1 - kx^{k-1}}{(x-1)^2} > 0,$$

and hence h is strictly increasing. Since $x - 1 = (n-1)(1-y) \geq 1 - y$, we get

$$h(x) \geq h(2-y) = \frac{(2-y)^k - 2 + y}{1 - y}.$$

Thus, it suffices to show that

$$(2-y)^k - 2 + y \geq \frac{y - y^k}{y^k - y + 1}.$$

Putting $1 - y = t$, $0 < t \leq 1$, we write this inequality as

$$(2-y)^k - 1 + y \geq \frac{1}{y^k - y + 1},$$

$$(1+t)^k - t \geq \frac{1}{(1-t)^k + t},$$

$$(1-t^2)^k + t(1+t)^k \geq 1 + t^2 + t(1-t)^k.$$

By Bernoulli's inequality,

$$(1-t^2)^k + t(1+t)^k > 1 - kt^2 + t(1+kt) = 1 + t.$$

So, we only need to show that $t(1-t) \geq t(1-t)^k$, which is clearly true. The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 3.9. *Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n \ge n$. If $k > 1$, then*

$$\frac{1}{a_1^k + a_2 + \cdots + a_n} + \frac{1}{a_1 + a_2^k + \cdots + a_n} + \cdots + \frac{1}{a_1 + a_2 + \cdots + a_n^k} \le 1.$$

(Vasile Cîrtoaje, 2006)

Solution. Clearly, it suffices to consider the case $a_1+a_2+\cdots+a_n = n$, when the desired inequality can be written as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{-1}{u^k - u + n}, \quad u \in \mathbb{I} = [0, n].$$

From

$$f'(u) = \frac{ku^{k-1} - 1}{(u^k - u + n)^2},$$

it follows that f is decreasing on $[0, s_0]$ and increasing on $[s_0, n]$, where

$$s_0 = k^{\frac{1}{1-k}} < 1.$$

We will show that f is convex on $[s_0, 1]$. For $u \in [s_0, 1]$, we have

$$f''(u) = \frac{-k(k+1)u^{2k-2} + k(k+3)u^{k-1} + nk(k-1)u^{k-2} - 2}{(u^k - u + n)^3} > \frac{g(u)}{(u^k - u + n)^3},$$

where

$$g(u) = -k(k+1)u^{2k-2} + k(k+3)u^{k-1} - 2.$$

Denoting

$$t = ku^{k-1},$$

we have $1 \le t \le k$ and

$$kg(u) = -(k+1)t^2 + k(k+3)t - 2k = (k+1)(t-1)(k-t) + (k-1)(t+k) > 0.$$

By LPCF Theorem, it suffices to show that

$$\frac{1}{x^k - x + n} + \frac{n-1}{y^k - y + n} \le 1$$

for $x \ge 1 \ge y \ge 0$ and $x + (n-1)y = n$. Since this inequality is trivial for $x = y = 1$, assume next that $x > 1 > y \ge 0$, and write the desired inequality as follows:

$$x^k - x + n \ge \frac{y^k - y + n}{y^k - y + 1},$$

$$x^k - x \geq \frac{(n-1)(y-y^k)}{y^k - y + 1},$$

$$\frac{x^k - x}{x-1} \geq \frac{y-y^k}{(1-y)(y^k-y+1)}.$$

Let $h(x) = \dfrac{x^k - x}{x-1}$, $x > 1$. By the weighted AM-GM inequality, we have

$$h'(x) = \frac{(k-1)x^k + 1 - kx^{k-1}}{(x-1)^2} > 0.$$

Therefore, h is increasing. Since $x - 1 = (n-1)(1-y) \geq 1 - y$, hence $x \geq 2 - y > 1$, we get

$$h(x) \geq h(2-y) = \frac{(2-y)^k + y - 2}{1-y}.$$

Thus, it suffices to show that

$$(2-y)^k + y - 2 \geq \frac{y - y^k}{y^k - y + 1},$$

which is equivalent to

$$(2-y)^k + y - 1 \geq \frac{1}{y^k - y + 1}.$$

Substituting $t = 1 - y$, $0 < t \leq 1$, the inequality becomes

$$(1+t)^k - t \geq \frac{1}{(1-t)^k + t},$$

$$(1-t^2)^k + t(1+t)^k \geq 1 + t^2 + t(1-t)^k.$$

By Bernoulli's inequality,

$$(1-t^2)^k + t(1+t)^k \geq 1 - kt^2 + t(1+kt) = 1 + t.$$

Thus, it suffices to show that $t(1-t) \geq t(1-t)^k$, which is clearly true. The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. Using this result, we can get the following statement.

- *Let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers such that $x_1 + x_2 + \cdots + x_n \geq n$. If $k > 1$, then*

$$\frac{x_1^k - x_1}{x_1^k + x_2 + \cdots + x_n} + \frac{x_2^k - x_2}{x_1 + x_2^k + \cdots + x_n} + \cdots + \frac{x_n^k - x_n}{x_1 + x_2 + \cdots + x_n^k} \geq 0.$$

This inequality is equivalent to

$$\frac{1}{x_1^k+x_2+\cdots+x_n}+\frac{1}{x_1+x_2^k+\cdots+x_n}+\cdots+\frac{1}{x_1+x_2+\cdots+x_n^k}\le\frac{n}{x_1+x_2+\cdots+x_n}.$$

Using the substitutions

$$s=\frac{x_1+x_2+\cdots+x_n}{n},\quad s\ge 1,$$

and

$$a_i=\frac{x_i}{s},\quad i=1,2,\ldots,n,$$

which yields $a_1+a_2+\cdots+a_n=n$, the desired inequality becomes

$$\sum\frac{1}{s^{k-1}a_1^k+a_2+\cdots+a_n}\le 1.$$

Since $s^{k-1}\ge 1$, it suffices to show that

$$\sum\frac{1}{a_1^k+a_2+\cdots+a_n}\le 1,$$

which is just the inequality from P 3.9.

Since $x_1x_2\cdots x_n\ge 1$ involves $x_1+x_2+\cdots+x_n\ge n$, the inequality is also true under the more restrictive condition $x_1x_2\cdots x_n\ge 1$. For $n=3$ and $k=5/2$, we get the inequality from IMO-2005:

- *If x,y,z are nonnegative real numbers such that $xyz\ge 1$, then*

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{x^2+y^5+z^2}+\frac{z^5-z^2}{x^2+y^2+z^5}\ge 0.$$

□

P 3.10. *If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$, then*

$$\frac{a_1-1}{a_1^2+a_2+\cdots+a_n}+\frac{a_2-1}{a_1+a_2^2+\cdots+a_n}+\cdots+\frac{a_n-1}{a_1+a_2+\cdots+a_n^2}\le 0.$$

(Vasile Cîrtoaje, 2006)

Solution. For $n=2$, the inequality is equivalent to $(a_1-a_2)^2\ge 0$. Consider further that $n\ge 3$ and write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s),\quad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u) = \frac{1-u}{u^2-u+n}, \quad u \in \mathbb{I} = [0,n].$$

From

$$f'(u) = \frac{u^2-2u-n+1}{(u^2-u+n)^2},$$

it follows that f is decreasing on $[0,s_0]$ and increasing on $[s_0,n]$, where

$$s_0 = 1+\sqrt{n}, \quad s_0 \in (1,n).$$

We will show that f is convex on $[1,s_0]$. For $u \in [1,s_0]$, we have

$$f''(u) = \frac{2g(u)}{(u^2-u+n)^3}, \quad g(u) = -u^3+3u^2+3(n-1)u-2n+1.$$

Since

$$g'(u) = -3(u^2-2u-n+1) \ge 0,$$

g is increasing on $[1,s_0]$, hence $g(u) \ge g(1) = n > 0$ and $f''(u) > 0$. By RPCF Theorem, it suffices to show that

$$\frac{1-x}{x^2-x+n} + \frac{(n-1)(1-y)}{y^2-y+n} \ge 0$$

for $0 \le x \le 1 \le y$ and $x+(n-1)y = n$. Since $(n-1)(1-y) = x-1$, we have

$$\begin{aligned}\frac{1-x}{x^2-x+n} + \frac{(n-1)(1-y)}{y^2-y+n} &= (x-1)\left(-\frac{1}{x^2-x+n} + \frac{1}{y^2-y+n}\right)\\ &= \frac{(x-1)(x-y)(x+y-1)}{(x^2-x+n)(y^2-y+n)}\\ &= \frac{n(x-1)^2(x+y-1)}{(n-1)(x^2-x+n)(y^2-y+n)} \ge 0.\end{aligned}$$

The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 3.11. *Let $a_1, a_2, \cdots, a_n$ be positive real numbers such that $a_1+a_2+\cdots+a_n = n$. If $0 < k \le \dfrac{n}{n-1}$, then*

$$a_1^{k/a_1} + a_2^{k/a_2} + \cdots + a_n^{k/a_n} \le n.$$

(Vasile Cîrtoaje, 2006)

Solution. According to the power mean inequality, we have

$$\left(\frac{a_1^{p/a_1}+a_2^{p/a_2}+\cdots+a_n^{p/a_n}}{n}\right)^{1/p} \ge \left(\frac{a_1^{q/a_1}+a_2^{q/a_2}+\cdots+a_n^{q/a_n}}{n}\right)^{1/q}$$

for all $p \ge q > 0$. Thus, it suffices to prove the desired inequality for

$$k=\frac{n}{n-1}, \quad 1<k\le 2.$$

Rewrite the desired inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s), \quad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=-u^{k/u}, \quad u\in\mathbb{I}=(0,n).$$

We have

$$f'(u)=ku^{\frac{k}{u}-2}(\ln u-1),$$

$$f''(u)=ku^{\frac{k}{u}-4}[u+(1-\ln u)(2u-k+k\ln u)].$$

For $n=2$, when $k=2$ and $\mathbb{I}=(0,2)$, f is convex for $u\in\mathbb{I}$, $u\ge 1$, because

$$1-\ln u>0, \quad 2u-k+k\ln u=2u-2+2\ln u\ge 2u-2\ge 0.$$

Therefore, we may apply HCF Theorem. Consider now that $n\ge 3$. From the expression of f', it follows that f is decreasing on $(0,s_0]$ and increasing on $[s_0,n)$, where $s_0=e$. In addition, we claim that f is convex on $[1,s_0]$. Indeed, since

$$1-\ln u\ge 0, \quad 2u-k+k\ln u\ge 2-k>0,$$

we have $f''>0$ for $u\in[1,s_0]$. Therefore, by HCF Theorem (for $n=2$) and RPCF Theorem (for $n\ge 3$), we only need to show that

$$x^{k/x}+(n-1)y^{k/y}\le n$$

for $0<x\le 1\le y$ and $x+(n-1)y=n$. We have $k/x\ge k>1$. Also, from

$$\frac{k}{y}=\frac{n}{(n-1)y}>\frac{n}{x+(n-1)y}=1, \quad \frac{k}{y}\le\frac{2}{y}\le 2,$$

we get

$$0<\frac{k}{y}-1\le 1.$$

Therefore, by Bernoulli's inequality, we have

$$x^{k/x}+(n-1)y^{k/y}-n=\frac{1}{\left(\frac{1}{x}\right)^{k/x}}+(n-1)y\cdot y^{k/y-1}-n$$

$$\le \frac{1}{1+\frac{k}{x}\left(\frac{1}{x}-1\right)}+(n-1)y\left[1+\left(\frac{k}{y}-1\right)(y-1)\right]-n$$

$$=\frac{x^2}{x^2-kx+k}-(k-1)x^2-(2-k)x=\frac{-(x-1)^2[(k-1)x+k(2-k)]}{x^2-kx+k}\le 0.$$

The proof is completed. The equality holds for $a_1=a_2=\cdots=a_n=1$

□

P 3.12. *If a,b,c,d,e are nonzero real numbers such that $a+b+c+d+e=5$, then*

$$\left(7-\frac{5}{a}\right)^2+\left(7-\frac{5}{b}\right)^2+\left(7-\frac{5}{c}\right)^2+\left(7-\frac{5}{d}\right)^2+\left(7-\frac{5}{e}\right)^2\ge 20.$$

(Vasile Cîrtoaje, 2012)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)+f(e)\ge 5f(s),\quad s=\frac{a+b+c+d+e}{5}=1,$$

where

$$f(u)=\left(7-\frac{5}{u}\right)^2,\quad u\in\mathbb{I}=\mathbb{R}\setminus\{0\}.$$

From

$$f'(u)=\frac{10(7u-5)}{u^3},$$

it follows that f is increasing on $(-\infty,0)\cup[s_0,\infty)$ and decreasing on $(0,s_0]$, where $s_0=5/7$. Since

$$\lim_{u\to-\infty}f(u)=49$$

and $f(s_0)=0$, we have

$$\min_{u\in\mathbb{I}}f(u)=f(s_0).$$

Also, f is convex on $[s_0,s]=[5/7,1]$ because

$$f''(u)=\frac{10(15-14u)}{u^4}>0.$$

According to LPCF Theorem and Remark 4, we only need to show that that $f(x)+4f(y)\geq 5f(1)$ for all nonzero real x,y such that $x+4y=5$. Using Remark 1, it suffices to prove that $h(x,y)\geq 0$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y},\quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

From

$$g(u)=5\left(\frac{9}{u}-\frac{5}{u^2}\right),$$

$$h(x,y)=\frac{5(5x+5y-9xy)}{x^2y^2}\geq 0$$

and

$$5x+5y-9xy=(6y-5)^2\geq 0,$$

it follows that $h(x,y)\geq 0$. The proof is completed. In accordance with Remark 3, the equality holds for $a=b=c=d=e=1$, and also for $a=5/3$ and $b=c=d=e=5/6$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization.

- *Let $a_1,a_2,\ldots,a_n$ be nonzero real numbers such that $a_1+a_2+\cdots+a_n=n$. If $k=\dfrac{n}{n+\sqrt{n-1}}$, then*

$$\left(1-\frac{k}{a_1}\right)^2+\left(1-\frac{k}{a_2}\right)^2+\cdots+\left(1-\frac{k}{a_n}\right)^2\geq n(1-k)^2,$$

with equality for $a_1=a_2=\cdots=a_n=1$, and also for $a_1=\dfrac{n}{1+\sqrt{n-1}}$ and $a_2=a_3=\cdots=a_n=\dfrac{n}{n-1+\sqrt{n-1}}$ (or any cyclic permutation).

□

P 3.13. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$(1-a+a^4)(1-b+b^4)(1-c+c^4)\geq 1.$$

Solution. Write the inequality as

$$f(a)+f(b)+f(c)\geq 3f(s),\quad s=\frac{a+b+c}{3}=1,$$

where

$$f(u)=\ln(1-u+u^4),\quad u\in[0,3].$$

From

$$f'(u) = \frac{4u^3-1}{1-u+u^4},$$

it follows that f is decreasing on $[0, s_0 1]$ and increasing on $[s_0, 4]$, where $s_0 = 1/\sqrt[3]{4}$. Also, f is convex for $u \in [s_0, 1]$ because

$$f''(u) = \frac{-4u^6 - 4u^3 + 12u^2 - 1}{(1-u+u^4)^2} \geq \frac{-4u^2 - 4u^2 + 12u^2 - 1}{(1-u+u^4)^2} = \frac{4u^2-1}{(1-u+u^4)^2} > 0.$$

According to LPCF Theorem, we only need to show that $f(x) + 2f(y) \geq 3f(1)$ for all $x, y \geq 0$ such that $x + 2y = 3$. Using Remark 2, it suffices to prove that $H(x, y) \geq 0$, where

$$H(x,y) = \frac{f'(x) - f'(y)}{x-y}.$$

We have

$$\begin{aligned}
H(x,y) &= \frac{(x+y)(x-y)^2 - 1 + 4(x^2+y^2+xy) - 2xy(x+y) - 4x^3y^3}{(1-x+x^4)(1-y+y^4)} \\
&\geq \frac{-1 + 4(x^2+y^2+xy) - 2xy(x+y) - 4x^3y^3}{(1-x+x^4)(1-y+y^4)} \\
&= \frac{-1 + 4(x+y)^2 - 2xy(x+y) - 4xy - 4x^3y^3}{(1-x+x^4)(1-y+y^4)}.
\end{aligned}$$

Thus, we only need to show that

$$-1 + 4(x+y)^2 - 2xy(x+y) - 4xy - 4x^3y^3 \geq 0.$$

From $3 = x + 2y \geq 2\sqrt{2xy}$ and $(1-x)(1-y) \leq 0$, we get

$$xy \leq \frac{9}{8}, \quad x + y \geq 1 + xy.$$

Therefore,

$$\begin{aligned}
&-1 + 4(x+y)^2 - 2xy(x+y) - 4xy - 4x^3y^3 \geq \\
&\geq -1 + 2(x+y)[2(x+y) - xy] - 4xy - 4x^3y^3 \\
&\geq -1 + 2(1+xy)[2(1+xy) - xy] - 4xy - 4x^3y^3 \\
&= 3 + 2xy + 2x^2y^2 - 4x^3y^3 \geq 3 + 2xy + 2x^2y^2 - 5x^2y^2 \\
&= 3 + 2xy - 3x^2y^2 \geq 3 + 2xy - 4xy = 3 - 2xy > 0.
\end{aligned}$$

The proof is completed. In accordance with Remark 3, the equality holds for $a = b = c = 1$.

□

P 3.14. *If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then*

$$(1-a+a^3)(1-b+b^3)(1-c+c^3)(1-d+d^3) \ge 1.$$

(Vasile Cîrtoaje, 2012)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d) \ge 4f(s), \quad s=\frac{a+b+c+d}{4}=1,$$

where

$$f(u)=\ln(1-u+u^3), \quad u \in [0,4].$$

From

$$f'(u)=\frac{3u^2-1}{1-u+u^3},$$

it follows that f is decreasing on $[0,s_0]$ and increasing on $[s_0,4]$, where $s_0=1/\sqrt{3}4$. Also, f is convex for $u \in [s_0,1]$ because

$$f''(u)=\frac{-3u^4+6u-1}{(1-u+u^3)^2} \ge \frac{-3u+6u-1}{(1-u+u^3)^2}=\frac{3u-1}{(1-u+u^3)^2}>0.$$

According to LPCF Theorem, we only need to show that $f(x)+3f(y) \ge 4f(1)$ for all $x,y \ge 0$ such that $x+3y=4$. Using Remark 2, it suffices to prove that $H(x,y) \ge 0$, where

$$H(x,y)=\frac{f'(x)-f'(y)}{x-y}.$$

We have

$$H(x,y)=\frac{(x-y)^2+3(x+y)-1-3x^2y^2}{(1-x+x^3)(1-y+y^3)} \ge \frac{3(x+y)-1-3x^2y^2}{(1-x+x^3)(1-y+y^3)}.$$

Thus, we only need to show that

$$3(x+y)-1-3x^2y^2 \ge 0.$$

From $4=x+3y \ge 2\sqrt{3xy}$ and $(1-x)(1-y) \le 0$, we get

$$xy \le \frac{4}{3}, \quad x+y \ge 1+xy.$$

Therefore,

$$3(x+y)-1-3x^2y^2 \ge 3(1+xy)-1-3x^2y^2 \ge 3(1+xy)-1-4xy=2-xy>0.$$

The proof is completed. The equality holds for $a=b=c=d=1$.

□

P 3.15. *If $a_1, a_2, \ldots, a_{10}$ are real numbers such that $a_1 + a_2 + \cdots + a_{10} = 10$, then*

$$(1 - a_1 + a_1^2)(1 - a_2 + a_2^2) \cdots (1 - a_{10} + a_{10}^2) \ge 1.$$

(Vasile Cîrtoaje, 2006)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_{10}) \ge 10f(s), \quad s = \frac{a_1 + a_2 + \cdots + a_{10}}{10} = 1$$

where

$$f(u) = \ln(1 - u + u^2), \quad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{2u - 1}{1 - u + u^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where $s_0 = 1/2$. Also, from

$$f''(u) = \frac{1 + 2u(1 - u)}{(1 - u + u^2)^2},$$

it follows that $f''(u) > 0$ for $u \in [s_0, 1]$, hence f is convex on $[s_0, 1]$. According to LPCF Theorem, we only need to show that $f(x) + 9f(y) \ge 10f(1)$ for all real x, y such that $x + 9y = 10$. Using Remark 2, it suffices to prove that $H(x, y) \ge 0$, where

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$

Since

$$H(x, y) = \frac{1 + x + y - 2xy}{(1 - x + x^2)(1 - y + y^2)}$$

and

$$1 + x + y - 2xy = 18y^2 - 8y + 1 = 2y^2 + (4y - 1)^2 > 0,$$

the conclusion follows. The equality holds for $a_1 = a_2 = \cdots = a_{10} = 1$.

Remark. By replacing $a_1, a_2, \ldots, a_{10}$ respectively with $1 - a_1, 1 - a_2, \ldots, 1 - a_{10}$, we get the following statement.

- *If $a_1, a_2, \ldots, a_{10}$ are real numbers such that $a_1 + a_2 + \cdots + a_{10} = 0$, then*

$$(1 - a_1 + a_1^2)(1 - a_2 + a_2^2) \cdots (1 - a_{10} + a_{10}^2) \ge 1,$$

with equality for $a_1 = a_2 = \cdots = a_n = 0$.

□

P 3.16. *If a,b,c,d,e are nonzero real numbers such that $a+b+c+d+e=5$, then*

$$5\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}+\frac{1}{e^2}\right)+45\ge 14\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right).$$

(Vasile Cîrtoaje, 2013)

Solution. Write the desired inequality as

$$f(a)+f(b)+f(c)+f(d)+f(e)\ge 5f(s),\quad s=\frac{a+b+c+d+e}{5}=1,$$

where

$$f(u)=\frac{5}{u^2}-\frac{14}{u}+9,\quad u\in\mathbb{I}=\mathbb{R}\setminus\{0\}.$$

From

$$f'(u)=\frac{2(7u-5)}{u^3},$$

it follows that f is increasing on $(-\infty,0)\cup[s_0,\infty)$ and decreasing on $(0,s_0]$, where

$$s_0=\frac{5}{7}.$$

Since

$$\lim_{u\to-\infty}f(u)=9$$

and $f(s_0)<f(1)=0$, we have

$$\min_{u\in\mathbb{I}}f(u)=f(s_0).$$

From

$$f''(u)=\frac{2(15-14u)}{u^4},$$

it follows that f is convex on $[s_0,1]$. By LPCF Theorem, Remark 4 and Remark 1, it suffices to show that $h(x,y)\ge 0$ for all $x,y\in\mathbb{I}$ which satisfy $x+4y=5$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y},\quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

Indeed, we have

$$g(u)=\frac{9}{u}-\frac{5}{u^2},$$

$$h(x,y)=\frac{5x+5y-9xy}{x^2y^2}=\frac{(6y-5)^2}{x^2y^2}\ge 0.$$

In accordance with Remark 3, the equality holds for $a=b=c=d=e=1$, and also for $a=\frac{5}{3}$ and $b=c=d=e=\frac{5}{6}$ (or any cyclic permutation).

□

P 3.17. *If a,b,c are positive real numbers such that $abc=1$, then*

$$\frac{7-6a}{2+a^2}+\frac{7-6b}{2+b^2}+\frac{7-6c}{2+c^2}\ge 1.$$

(Vasile Cîrtoaje, 2008)

Solution. Using the substitution

$$a=e^x,\ b=e^y,\ c=e^z,$$

we need to show that

$$f(x)+f(y)+f(z)\ge 3f(s),\quad s=\frac{x+y+z}{3}=0,$$

where

$$f(u)=\frac{7-6e^u}{2+e^{2u}},\quad u\in\mathbb{R}.$$

From

$$f'(u)=\frac{2(3e^u+2)(e^u-3)}{(2+e^{2u})^2},$$

it follows that f is decreasing on $(-\infty,s_0]$ and increasing on $[s_0,\infty)$, where $s_0=\ln 3$. Also, we have

$$f''(u)=\frac{2t\cdot h(t)}{(2+t^2)^3},\quad t=e^u,$$

where

$$h(t)=-3t^4+14t^3+36t^2-28t-12.$$

We will show that $h(t)>0$ for $t\in[1,3]$, hence f is convex on $[0,s_0]$. We have

$$h(t)=3(t^2-1)(9-t^2)+14t^3+6t^2-28t+15$$

$$=3(t^2-1)(9-t^2)+14t^2(t-1)+14(t-1)^2+6t^2+1>0.$$

By RPCF Theorem, we only need to prove that $f(x)+2f(y)\ge 3f(0)$ for all real x,y such that $x+2y=0$. That is, to show that the original inequality holds for $b=c:=t$ and $a=1/t^2$, where $t>0$. Write this inequality as

$$\frac{t^2(7t^2-6)}{2t^4+1}+\frac{2(7-6t)}{2+t^2}\ge 1,$$

$$(t-1)^2(t-2)^2(5t^2+6t+3)\ge 0.$$

Since the last inequality is true, the proof is completed. In accordance with Remark 3, the equality holds for $a=b=c=1$, and also for $a=1/4$ and $b=c=2$ (or any cyclic permutation).

□

P 3.18. *If a,b,c are positive real numbers such that $abc=1$, then*

$$\frac{1}{a+5bc}+\frac{1}{b+5ca}+\frac{1}{c+5ab}\le\frac{1}{2}.$$

(Vasile Cîrtoaje, 2008)

Solution. Write the inequality as

$$\frac{a}{a^2+5}+\frac{b}{b^2+5}+\frac{c}{c^2+5}\le\frac{1}{2}.$$

Using the substitution

$$a=e^x,\ b=e^y,\ c=e^z,$$

we need to show that

$$f(x)+g(y)+g(z)\ge 3f(s),\quad s=\frac{x+y+z}{3}=0,$$

where

$$f(u)=\frac{-e^u}{e^{2u}+5},\ \ u\in\mathbb{R}.$$

From

$$f'(u)=\frac{e^u(e^{2u}-5)}{(e^{2u}+5)^2},$$

it follows that f is decreasing on $(-\infty,s_0]$ and increasing on $[s_0,\infty)$, where $s_0=\dfrac{1}{2}\ln 5$. Also, from

$$f''(u)=\frac{e^u(-e^{4u}+30e^{2u}-25)}{(e^{2u}+5)^3},$$

it follows that f is convex on $[0,s_0]$, because $0\le u\le s_0$ involves $1\le e^u\le\sqrt{5}$, hence

$$-e^{4u}+30e^{2u}-25\ge e^{2u}(5-e^{2u})+25(e^{2u}-1)>0.$$

By RPCF Theorem, we only need to prove the original inequality for $b=c:=t$ and $a=1/t^2$, where $t>0$. Write this inequality as

$$\frac{t^2}{5t^4+1}+\frac{2t}{t^2+5}\le\frac{1}{2},$$

$$(t-1)^2(5t^4-10t^3-2t^2+6t+5)\ge 0,$$

$$(t-1)^2[5(t-1)^4+2t(5t^2-16t+13)]\ge 0.$$

Since the last inequality is true, the proof is completed. The equality holds for $a=b=c=1$.

□

P 3.19. *If a,b,c are positive real numbers such that $abc=1$, then*

$$\frac{1}{4-3a+4a^2}+\frac{1}{4-3b+4b^2}+\frac{1}{4-3c+4c^2}\le\frac{3}{5}.$$

(Vasile Cirtoaje, 2008)

Solution. Let

$$a=e^x,\quad b=e^y,\quad c=e^z.$$

We need to show that

$$f(x)+f(y)+f(z)\ge 3f(s),\quad s=\frac{x+y+z}{3}=0,$$

where

$$f(u)=\frac{-1}{4-3e^u+4e^{2u}},\quad u\in\mathbb{R}.$$

From

$$f'(u)=\frac{e^u(8e^u-3)}{(4-3e^u+4e^{2u})^2},$$

it follows that f is decreasing on $(-\infty,s_0]$ and increasing on $[s_0,\infty)$, where

$$s_0=\ln\frac{3}{8}<0.$$

We claim that f is convex on $[s_0,0]$. Since

$$f''(u)=\frac{e^u(-64e^{3u}+36e^{2u}+55e^u-12)}{(4-3e^u+4e^{2u})^3},$$

we need to show that

$$-64x^3+36x^2+55x-12\ge 0,$$

where $x=e^u\in[3/8,1]$. Indeed,

$$-64x^3+36x^2+55x-12>-72x^3+36x^2+48x-12=12(1-x)(6x^2+3x-1)\ge 0.$$

By LPCF Theorem, we only need to prove the original inequality for $b=c:=t$ and $a=1/t^2$, where $t>0$. Write this inequality as follows:

$$\frac{t^4}{4t^4-3t^2+4}+\frac{2}{4-3t+4t^2}\le\frac{3}{5},$$

$$28t^6-21t^5-48t^4+27t^3+42t^2-36t+8\ge 0,$$

$$(t-1)^2(28t^4+35t^3-6t^2-20t+8)\ge 0.$$

It suffices to show that

$$7(4t^4+5t^3-t^2-3t+1)\ge 0.$$

Indeed,

$$4t^4+5t^3-t^2-3t+1=t^2(2t-1)^2+9t^3-2t^2-3t+1$$

and

$$9t^3-2t^2-3t+1=t(3t-1)^2+(2t-1)^2>0.$$

The equality holds for $a=b=c=1$.

Remark. Since

$$\frac{1}{4-3a+4a^2}\geq\frac{1}{4-3a+4a^2+(1-a)^2}=\frac{1}{5(1-a+a^2)},$$

we get the following known inequality

$$\frac{1}{1-a+a^2}+\frac{1}{1-b+b^2}+\frac{1}{1-c+c^2}\leq 3.$$

□

P 3.20. *If a,b,c are positive real numbers such that $abc=1$, then*

$$\frac{1}{(3a+1)(3a^2-5a+3)}+\frac{1}{(3b+1)(3b^2-5b+3)}+\frac{1}{(3c+1)(3c^2-5c+3)}\leq\frac{3}{4}.$$

Solution. Let

$$a=e^x,\quad b=e^y,\quad c=e^z.$$

We need to show that

$$f(x)+f(y)+f(z)\geq 3f(s),\qquad s=\frac{x+y+z}{3}=0,$$

where

$$f(u)=\frac{-1}{(3e^u+1)(3e^{2u}-5e^u+3)},\quad u\in\mathbb{R}.$$

From

$$f'(u)=\frac{(3e^u-2)(9e^u-2)}{(3e^u+1)^2(3e^{2u}-5e^u+3)^2},$$

it follows that f is increasing on $(-\infty,s_1]\cup[s_0,\infty)$ and decreasing on $[s_1,s_0]$, where

$$s_1=\ln 2-\ln 9,\quad s_0=\ln 2-\ln 3,\qquad s_1<s_0<0.$$

Since

$$\lim_{u\to-\infty}f(u)=f(s_0)=-1/3,$$

we get

$$\min_{u\in\mathbb{R}} f(u) = f(s_0).$$

We claim that f is convex on $[s_0, 0]$. We have

$$f''(u) = \frac{t\cdot h(t)}{(3t+1)^3(3t^2-5t+3)^3},$$

where

$$t = e^u, \quad h(t) = -729t^5 + 1188t^4 - 648t^3 + 387t^2 - 160t + 12.$$

Since the polynomial $h(t)$ has the real roots

$$t_1 \approx 0.0933, \quad t_2 \approx 0.5072, \quad t_3 \approx 1.11008,$$

it follows that $h(t) > 0$ for $t \in [2/3, 1]$, hence f is convex for $u \in [s_0, 0]$. By LPCF Theorem, we only need to prove the original inequality for $b = c := t$ and $a = 1/t^2$, where $t \in (0, 1]$. Write this inequality as follows:

$$\frac{t^6}{(t^2+3)(3t^4-5t^2+3)} + \frac{2}{(3t+1)(3t^2-5t+3)} \le \frac{3}{4}.$$

Since

$$t^2 + 3 \ge 2(t+1)$$

and

$$3t^4 - 5t^2 + 3 \ge t(3t^2 - 5t + 3),$$

it suffices to prove that

$$\frac{t^5}{2(t+1)(3t^2-5t+3)} + \frac{2}{(3t+1)(3t^2-5t+3)} \le \frac{3}{4}.$$

This is equivalent to the obvious inequality

$$(1-t)^2(1 + 15t + 5t^2 - 14t^3 - 6t^4) \ge 0.$$

The equality holds for $a = b = c = 1$.

□

P 3.21. *Let $a_1, a_2, \cdots, a_n$ ($n \ge 3$) be positive real numbers such that $a_1a_2\cdots a_n = 1$. If $p, q \ge 0$ such that $p + 4q \ge n - 1$, then*

$$\frac{1-a_1}{1+pa_1+qa_1^2} + \frac{1-a_2}{1+pa_2+qa_2^2} + \cdots + \frac{1-a_n}{1+pa_n+qa_n^2} \ge 0.$$

(Vasile Cîrtoaje, 2008)

Solution. For $q=0$, we get a known inequality (see Remark 2 from the proof of P 1.51). Consider further that $q>0$. Using the substitutions $a_i=e^{x_i}$ for $i=1,2,\ldots,n$, we need to show that

$$f(x_1)+f(x_2)+\cdots+f(x_n)\geq nf(s), \quad s=\frac{x_1+x_2+\cdots+x_n}{n}=0,$$

where

$$f(u)=\frac{1-e^u}{1+pe^u+qe^{2u}}, \quad u\in\mathbb{R}.$$

From

$$f'(t)=\frac{e^u(qe^{2u}-2qe^u-p-1)}{(1+pe^u+qe^{2u})^2},$$

it follows that f is decreasing on $(-\infty,s_0]$ and increasing on $[s_0,\infty)$, where

$$s_0=\ln r_0>0, \quad r_0=1+\sqrt{1+\frac{p+1}{q}}.$$

Also, we have

$$f''(u)=\frac{t\cdot h(t)}{(1+pt+qt^2)^3},$$

where

$$h(t)=-q^2t^4+q(p+4q)t^3+3q(p+2)t^2+(p-4q+p^2)t-p-1, \quad t=e^u.$$

We will show that $h(t)\geq 0$ for $t\in[1,r_0]$, hence f is convex on $[0,s_0]$. We have

$$h'(t)=-4q^2t^3+3q(p+4q)t^2+6q(p+2)t+p-4q+p^2,$$

$$h''(t)=6q[-2qt^2+(p+4q)t+p+2].$$

Since

$$h''(t)=6q[2(-qt^2+2qt+p+1)+p(t-1)]\geq 12q(-qt^2+2qt+p+1)\geq 0,$$

$h'(t)$ is increasing,

$$h'(t)\geq h'(1)=p^2+9pq+8q^2+p+8q>0,$$

h is increasing, hence

$$h(t)\geq h(1)=p^2+4pq+3q^2+2q-1=(p+2q)^2-(q-1)^2$$
$$=(p+q+1)(p+3q-1).$$

Therefore, f is convex on $[0,s_0]$ if $p+3q-1\geq 0$. Indeed,

$$p+3q-1\geq p+3q-\frac{p+4q}{n-1}=\frac{p+2q}{2}>0.$$

By RPCF Theorem, we only need to prove the original inequality for $a_2 = \cdots = a_n := t$ and $a_1 = 1/t^{n-1}$, where $t \geq 1$. Write this inequality as

$$\frac{t^{n-1}(t^{n-1}-1)}{t^{2n-2}+pt^{n-1}+q} + \frac{(n-1)(1-t)}{1+pt+qt^2} \geq 0,$$

or

$$pA + qB \geq C,$$

where

$$A = t^{n-1}(t^n - nt + n - 1),$$
$$B = t^{2n} - t^{n+1} - (n-1)(t-1),$$
$$C = t^{n-1}[(n-1)t^n - nt^{n-1} + 1].$$

Since $p+4q \geq n-1$ and $C \geq 0$ (by the AM-GM inequality applied to n positive numbers), it suffices to show that

$$pA + qB \geq \frac{(p+4q)C}{n-1},$$

which is equivalent to

$$p[(n-1)A - C] + q[(n-1)B - 4C] \geq 0.$$

Clearly, this is true if $(n-1)A - C \geq 0$ and $(n-1)B - 4C \geq 0$ for $t \geq 1$. By the AM-GM inequality, we have

$$(n-1)A - C = nt^{n-1}[t^{n-1} - (n-1)t + n - 2] \geq 0.$$

For $n = 3$ and $n = 4$.... Consider further that $n \geq 5$. Since

$$t - 1 \leq t^{n-1}(t-1),$$

we have

$$\begin{aligned} B &\geq t^{2n} - t^{n+1} - (n-1)t^{n-1}(t-1) \\ &= t^{n-1}[t^{n+1} - t^2 - (n-1)t + n - 1] \\ &\geq t^{n-1}\left[t^{n+1} - t^2 - (n-1)\frac{t^2+1}{2} + n - 1\right] \\ &= \frac{1}{2}t^{n-1}[2t^{n+1} - (n+1)t^2 + n - 1] \end{aligned}$$

hence

$$\begin{aligned} 2(n-1)B - 8C &\geq (n-1)t^{n-1}[2t^{n+1} - (n+1)t^2 + n - 1] - 8C \\ &= (n-1)t^{n-1}h(t), \end{aligned}$$

where

$$h(t) = 2t^{n+1} - 8t^n + \frac{8nt^{n-1} - 8}{n-1} - (n+1)t^2 + n - 1.$$

We have

$$h'(t) = 2th_1(t), \quad h_1(t) = (n+1)t^{n-1} - 4nt^{n-2} + 4nt^{n-3} - n - 1,$$

$$h_1'(t) = t^{n-4}g(t), \quad g(t) = (n^2-1)t^2 + 4n(n-3) - 4n(n-2)t,$$

where

$$g(t) \ge 2\sqrt{(n^2-1)t^2 \cdot 4n(n-3)} - 4n(n-2)t$$
$$= 4\sqrt{n}\left[\sqrt{(n^2-1)(n-3)} - (n-2)\sqrt{n}\right]t.$$

Since

$$(n^2-1)(n-3) - n(n-2)^2 = n(n-5) + 3 > 0,$$

we have $g(t) > 0$, $h_1'(t) > 0$, $h_1(t)$ is increasing for $t \ge 1$, $h_1(t) \ge h_1(1) = 0$, $h'(t) \ge 0$, $h(t)$ is increasing for $t \ge 1$, $h(t) \ge h(1) = 0$, hence $(n-1)B - 4C \ge 0$. Thus, the proof is completed. In accordance with Remark 3, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark 1. For $p = 0$ and $q = 1$, we get the following inequality (*Vasile Cîrtoaje*, 2006):

$$\frac{1-a}{1+a^2} + \frac{1-b}{1+b^2} + \frac{1-c}{1+c^2} + \frac{1-d}{1+d^2} + \frac{1-e}{1+e^2} \ge 0,$$

where a,b,c,d,e are positive real numbers such that $abcde = 1$. Replacing a,b,c,d,e by $1/a, 1/b, 1/c, 1/d, 1/e$, we get the inequality

$$\frac{1+a}{1+a^2} + \frac{1+b}{1+b^2} + \frac{1+c}{1+c^2} + \frac{1+d}{1+d^2} + \frac{1+e}{1+e^2} \le 5,$$

where a,b,c,d,e are positive real numbers such that $abcde = 1$.

Notice that the inequality

$$\frac{1-a_1}{1+a_1^2} + \frac{1-a_2}{1+a_2^2} + \frac{1-a_3}{1+a_3^2} + \frac{1-a_4}{1+a_4^2} + \frac{1-a_5}{1+a_5^2} + \frac{1-a_6}{1+a_6^2} \ge 0$$

is not true for all positive numbers $a_1, a_2, a_3, a_4, a_5, a_6$ satisfying $a_1a_2a_3a_4a_5a_6 = 1$. Indeed, for $a_2 = a_3 = a_4 = a_5 = a_6 = 2$, the inequality becomes

$$\frac{1-a_1}{1+a_1^2} - 1 \ge 0,$$

which is false for $a_1 > 0$.

□

P 3.22. *If a,b,c are positive real numbers such that $abc=1$, then*

$$\frac{1-a}{17+4a+6a^2}+\frac{1-b}{17+4b+6b^2}+\frac{1-c}{17+4c+6c^2}\ge 0.$$

(Vasile Cîrtoaje, 2008)

Solution. Using the substitution

$$a=e^x,\ b=e^y,\ c=e^z,$$

we need to show that

$$f(x)+g(y)+g(z)\ge 3f(s),\quad s=\frac{x+y+z}{3}=0,$$

where

$$f(u)=\frac{1-e^u}{1+pe^u+qe^{2u}},\ \ u\in\mathbb{R},$$

with

$$p=\frac{4}{17},\qquad q=\frac{6}{17}.$$

As we have shown in the proof of the preceding P 3.21, f is decreasing on $(-\infty,s_0]$ and increasing on $[s_0,\infty)$, where

$$s_0=\ln r_0>0,\quad r_0=1+\sqrt{1+\frac{p+1}{q}}=1+\sqrt{\frac{9}{2}}.$$

In addition, f is convex on $[0,s_0]$ if $p+3q-1\ge 0$. Indeed,

$$p+3q-1=\frac{5}{17}>0.$$

By RPCF Theorem, we only need to prove the original inequality for $b=c:=t$ and $a=1/t^2$, where $t>0$. Write this inequality as follows:

$$\frac{t^2(t^2-1)}{t^4+pt^2+q}+\frac{2(1-t)}{1+pt+qt^2}\ge 0,$$

$$pA+qB\ge C,$$

where

$$A=t^2(t-1)^2(t+2),$$

$$B=(t-1)^2(t^4+2t^3+2t^2+2t+2),$$

$$C=t^2(t-1)^2(2t+1).$$

Indeed, we have

$$pA+qB-C=\frac{3(t-1)^2(t-2)^2(2t^2+2t+1)}{17}\geq 0.$$

In accordance with Remark 3, the equality holds for $a=b=c=1$, and also for $a=1/4$ and $b=c=2$ (or any cyclic permutation).

□

P 3.23. *If $a_1,a_2,\cdots,a_8$ are positive real numbers such that $a_1a_2\cdots a_8=1$, then*

$$\frac{1-a_1}{(1+a_1)^2}+\frac{1-a_2}{(1+a_2)^2}+\cdots+\frac{1-a_8}{(1+a_8)^2}\geq 0.$$

(Vasile Cîrtoaje, 2006)

Solution. Using the substitutions $a_i=e^{x_i}$ for $i=1,2,\ldots,8$, we need to show that

$$f(x_1)+f(x_2)+\cdots+f(x_8)\geq 8f(s),\quad s=\frac{x_1+x_2+\cdots+x_8}{8}=0,$$

where

$$f(u)=\frac{1-e^u}{(1+e^u)^2},\quad u\in\mathbb{R}.$$

From

$$f'(t)=\frac{e^u(e^u-3)}{(1+e^u)^3},$$

it follows that f is decreasing on $(-\infty,s_0]$ and increasing on $[s_0,\infty)$, where

$$s_0=\ln 3>1.$$

Also, we have

$$f''(u)=\frac{e^u(8e^u-e^{2u}-3)}{(1+e^u)^4}.$$

We will show that $8e^u-e^{2u}-3>0$ for $u\in[0,s_0]$, hence f is convex on $[0,s_0]$. Indeed,

$$8e^u-e^{2u}-3\geq 8e^u-3e^u-3=5e^u-3>5-3>0.$$

By RPCF Theorem, we only need to prove the original inequality for $a_2=\cdots=a_8:=t$ and $a=1/t^7$, where $t\geq 1$. For the nontrivial case $t>1$, write this inequality as follows:

$$\frac{t^7(t^7-1)}{(t^7+1)^2}\geq\frac{7(t-1)}{(t+1)^2}.$$

$$\frac{t^7(t^7-1)(t+1)^2}{(t-1)(t^7+1)^2} \ge 7,$$

$$\frac{t^7(t^6+t^5+t^4+t^3+t^2+t+1)}{(t^6-t^5+t^4-t^3+t^2-t+1)^2} \ge 7.$$

Since

$$t^6-t^5+t^4-t^3+t^2-t+1 = t^4(t^2-t+1)-(t-1)(t^2+1) > t^4(t^2-t+1),$$

it suffices to show that

$$\frac{t^6+t^5+t^4+t^3+t^2+t+1}{t(t^2-t+1)^2} \ge 7,$$

which is equivalent to the obvious inequality $(t-1)^6 \ge 0$. Thus, the proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_8 = 1$.

Remark. The inequality

$$\frac{1-a_1}{(1+a_1)^2} + \frac{1-a_2}{(1+a_2)^2} + \cdots + \frac{1-a_9}{(1+a_9)^2} \ge 0$$

is not true for all positive numbers $a_1, a_2, \cdots, a_9$ satisfying $a_1a_2\cdots a_9 = 1$. Indeed, for $a_2 = a_3 = \cdots = a_9 = 3$, the inequality becomes

$$\frac{1-a_1}{(1+a_1)^2} - 1 \ge 0,$$

which is false for $a_1 > 0$.

□

P 3.24. *Let a, b, c be positive real numbers such that $abc = 1$. If $k \in \left[\frac{-13}{3\sqrt{3}}, \frac{13}{3\sqrt{3}}\right]$, then*

$$\frac{a+k}{a^2+1} + \frac{b+k}{b^2+1} + \frac{c+k}{c^2+1} \le \frac{3(1+k)}{2}.$$

(Vasile Cîrtoaje, 2012)

Solution. The desired inequality is equivalent to

$$\sum \frac{(a-1)^2}{a^2+1} \ge k\left(\sum \frac{2}{a^2+1} - 3\right).$$

Thus, it suffices to prove this inequality for only $|k| = \dfrac{13}{3\sqrt{3}}$. On the other hand, replacing a, b, c by $1/a, 1/b, 1/c$, the inequality becomes

$$\sum \frac{(a-1)^2}{a^2+1} \ge k\left(3 - \sum \frac{2}{a^2+1}\right).$$

Thus, we only need to prove the desired inequality for

$$k = \frac{13}{3\sqrt{3}}.$$

Using the substitution

$$a = e^x,\ b = e^y,\ c = e^z,$$

we need to show that

$$f(x) + g(y) + g(z) \ge 3f(s), \quad s = \frac{x+y+z}{3} = 0,$$

where

$$f(u) = \frac{-e^u - k}{e^{2u}+1},\ \ u \in \mathbb{R}.$$

From

$$f'(t) = \frac{e^{2u} + 2ke^u - 1}{(e^{2u}+1)^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where

$$s_0 = \ln r_0 < 0, \quad r_0 = \frac{1}{3\sqrt{3}}.$$

Also, we have

$$f''(u) = \frac{t \cdot h(t)}{(1+t^2)^3},$$

where

$$h(t) = -t^4 - 4kt^3 + 6t^2 + 4kt - 1, \quad t = e^u.$$

We will show that $h(t) > 0$ for $t \in [r_0, 1]$, hence f is convex on $[s_0, 0]$. Indeed, since

$$4kt = \frac{52t}{3\sqrt{3}} \ge \frac{52}{27} > 1,$$

we have

$$h(t) = -t^4 + 6t^2 - 1 + 4kt(1-t^2) \ge -t^4 + 6t^2 - 1 + (1-t^2) = t^2(5-t^2) > 0.$$

By LPCF Theorem, we only need to prove the original inequality for $b = c := t$ and $a = 1/t^2$, where $t > 0$. Write this inequality as

$$\frac{t^2(kt^2+1)}{t^4+1} + \frac{2(t+k)}{t^2+1} \le \frac{3(1+k)}{2},$$

$$3t^6 - 4t^5 + t^4 + t^2 - 4t + 3 - k(1-t^2)^3 \ge 0,$$

$$(t-1)^2[(3+k)t^4 + 2(1+k)t^3 + 2t^2 + 2(1-k)t + 3 - k] \ge 0$$

$$(t-1)^2(t-2+\sqrt{3})^2[(27+13\sqrt{3})t^2 + 24(2+\sqrt{3})t + 33 + 17\sqrt{3}] \ge 0.$$

The last inequality is clearly true. Thus, the proof is completed. The equality holds for $a = b = c = 1$. If $k = \frac{13}{3\sqrt{3}}$, then the equality holds also for $a = 7+4\sqrt{3}$ and $b = c = 2-\sqrt{3}$ (or any cyclic permutation). If $k = \frac{-13}{3\sqrt{3}}$, then the equality holds also for $a = 7-4\sqrt{3}$ and $b = c = 2+\sqrt{3}$ (or any cyclic permutation).

□

P 3.25. *If a, b, c are positive real numbers and $0 < k \le 2+2\sqrt{2}$, then*

$$\frac{a^3}{ka^2+bc} + \frac{b^3}{kb^2+ca} + \frac{c^3}{kc^2+ab} \ge \frac{a+b+c}{k+1}.$$

(Vasile Cîrtoaje, 2011)

Solution. For $k < 2+2\sqrt{2}$, the proof is similar to the one of the main case $k = 2+2\sqrt{2}$. For this reason, we consider further only the case where

$$k = 2+2\sqrt{2}.$$

Due to homogeneity, we may assume that $abc = 1$. On this hypothesis,

$$\sum \frac{a^3}{ka^2+bc} - \frac{1}{k+1}\sum a = \sum\left(\frac{a^4}{ka^3+1} - \frac{a}{k+1}\right) = \frac{1}{k+1}\sum \frac{a^4-a}{ka^3+1}.$$

Thus, we can write the inequality as

$$\frac{a^4-a}{ka^3+1} + \frac{b^4-b}{kb^3+1} + \frac{c^4-c}{kc^3+1} \ge 0.$$

Using the substitution

$$a = e^x,\ b = e^y,\ c = e^z,$$

we need to show that

$$f(x)+g(y)+g(z)\geq 3f(s), \quad s=\frac{x+y+z}{3}=0,$$

where

$$f(u)=\frac{e^{4u}-e^{u}}{ke^{3u}+1}, \;\; u\in\mathbb{R}.$$

From

$$f'(t)=\frac{ke^{6u}+2(k+2)e^{3u}-1}{(ke^{3u}+1)^2},$$

it follows that f is decreasing on $(-\infty,s_0]$ and increasing on $[s_0,\infty)$, where

$$s_0=\ln r_0<0, \quad r_0=\sqrt[3]{\frac{-k-2+\sqrt{(k+1)(k+4)}}{k}}\approx 0.4149.$$

Also, we have

$$f''(u)=\frac{t\cdot h(t)}{(kt^3+1)^3},$$

where

$$h(t)=k^2t^9-k(4k+1)t^6+(13k+16)t^3-1, \quad t=e^u.$$

We will show that $h(t)>0$ for $t\in[r_0,1]$, hence f is convex on $[s_0,0]$. Indeed, we have $h(t)\geq 0$ for $t\in[t_1,t_2]$, where $t_1\approx 0.2345$ and $t_2\approx 1.02$. Since $[r_0,1]\subset[t_1,t_2]$, the conclusion follows. By LPCF Theorem, we only need to prove the original inequality for $b=c$. Due to homogeneity, we may consider that $b=c=1$. Thus, we need to show that

$$\frac{a^3}{ka^2+1}+\frac{2}{a+k}\geq\frac{a+2}{k+1},$$

which is equivalent to the obvious inequality

$$(a-1)^2(a-\sqrt{2})^2\geq 0.$$

Thus, the proof is completed. In accordance with Remark 3, the equality holds for $a=b=c$. If $k=2+2\sqrt{2}$, then the equality holds also for $\frac{a}{\sqrt{2}}=b=c$ (or any cyclic permutation).

□

P 3.26. *If $a_1,a_2,\ldots,a_n$ $(n\geq 3)$ are real numbers such that*

$$a_1,a_2,\ldots,a_n\geq\frac{-1}{n-2}, \quad a_1+a_2+\cdots+a_n=n,$$

then

$$\frac{a_1^2}{a_1^2-a_1+1}+\frac{a_2^2}{a_2^2-a_2+1}+\cdots+\frac{a_n^2}{a_n^2-a_n+1}\le n.$$

(Vasile Cirtoaje, 2012)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s),\quad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=\frac{1-u}{u^2-u+1},\quad u\in\mathbb{I}=\left[\frac{-1}{n-2},\frac{n^2-n-1}{n-2}\right].$$

From

$$f'(u)=\frac{u(u-2)}{(u^2-u+1)^2},\quad f''(u)=\frac{2(3u^2-u^3-1)}{(u^2-u+1)^3}=\frac{2u^2(2-u)+2(u^2-1)}{(u^2-u+1)^3},$$

it follow that f is convex on $[1,s_0]$, decreasing for $1\le u\le s_0$ and increasing for $u\ge s_0$, where $s_0=2\in\mathbb{I}$. Since f is not decreasing on $\mathbb{I}_{u\le s_0}$, we can not apply RPCF Theorem in its original form. However, according to Remark 5, we may replace this condition with the condition $ns-(n-1)s_0\le\inf\mathbb{I}$. Indeed, we have

$$ns-(n-1)s_0=n-2(n-1)=-n+2\le\frac{-1}{n-2}=\inf\mathbb{I}.$$

So, it suffices to show that $f(x)+(n-1)f(y)\ge nf(1)$ for all $x,y\in\mathbb{I}$ such that $x\le 1\le y$ and $x+(n-1)y=n$. According to Remark 1, we only need to show that $h(x,y)\ge 0$, where

$$g(u)=\frac{f(u)-f(1)}{u-1},\quad h(x,y)=\frac{g(x)-g(y)}{x-y}.$$

We have

$$g(u)=\frac{-1}{u^2-u+1},$$

$$h(x,y)=\frac{x+y-1}{(x^2-x+1)(y^2-y+1)}=\frac{(n-2)x+1}{(n-1)(x^2-x+1)(y^2-y+1)}\ge 0.$$

The proof is completed. The equality holds for $a_1=a_2=\cdots=a_n=1$, and also for $a_1=\dfrac{-1}{n-2}$ and $a_2=a_3=\cdots=a_n=\dfrac{n-1}{n-2}$ (or any cyclic permutation).

□

P 3.27. *If $a_1,a_2,\ldots,a_n$ $(n\ge 3)$ are nonzero real numbers such that*

$$a_1,a_2,\ldots,a_n\ge \frac{-n}{n-2},\quad a_1+a_2+\cdots+a_n=n,$$

then

$$\frac{1}{a_1^2}+\frac{1}{a_2^2}+\cdots+\frac{1}{a_n^2}\ge \frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}.$$

(Vasile Cirtoaje, 2012)

Solution. According to P 2.25-(a) in Volume 1, the inequality is true for $n=3$. Assume further that $n\ge 4$ and write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\ge nf(s),\quad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=\frac{1}{u^2}-\frac{1}{u},\quad u\in\mathbb{I}=\left[\frac{-n}{n-2},\frac{n(2n-3)}{n-2}\right]\setminus\{0\}.$$

From

$$f'(u)=\frac{u-2}{u^3},\quad f''(u)=\frac{2(3-u)}{u^4},$$

it follow that f is convex on $[1,3]$, decreasing for $1\le u\le s_0$ and increasing for $u\ge s_0$, where $s_0=2\in\mathbb{I}$. We see that f is not decreasing on $\mathbb{I}_{u\le s_0}$. Therefore, according to Remark 4 and Remark 5, we may replace this condition in RPCF Theorem with the condition $ns-(n-1)s_0\le \inf\mathbb{I}$. Indeed, for $n\ge 4$, we have

$$ns-(n-1)s_0=n-2(n-1)=-n+2\le\frac{-n}{n-2}=\inf\mathbb{I}.$$

So, it suffices to show that $f(x)+(n-1)f(y)\ge nf(1)$ for all $x,y\in\mathbb{I}$ such that $x\le 1\le y$ and $x+(n-1)y=n$. According to Remark 1, we only need to show that $h(x,y)\ge 0$, where

$$g(u)=\frac{f(u)-f(1)}{u-1},\quad h(x,y)=\frac{g(x)-g(y)}{x-y}.$$

We have

$$g(u)=\frac{-1}{u^2},\quad h(x,y)=\frac{x+y}{x^2y^2}=\frac{(n-2)x+n}{(n-1)x^2y^2}\ge 0.$$

The proof is completed. In accordance with Remark 3, the equality holds for $a_1=a_2=\cdots=a_n=1$, and also for $a_1=\dfrac{-n}{n-2}$ and $a_2=a_3=\cdots=a_n=\dfrac{n}{n-2}$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization.

- *Let* $a_1, a_2, \cdots, a_n \geq \frac{-n}{n-2}$ *such that* $a_1 + a_2 + \cdots + a_n = n$. *If* $n \geq 3$ *and* $k \geq 0$, *then*

$$\frac{1-a_1}{k+a_1^2} + \frac{1-a_2}{k+a_2^2} + \cdots + \frac{1-a_n}{k+a_n^2} \geq 0,$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, *and also for* $a_1 = \frac{-n}{n-2}$ *and* $a_2 = a_3 = \cdots = a_n = \frac{n}{n-2}$ *(or any cyclic permutation).*

□

P 3.28. *If* $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ *are real numbers such that*

$$a_1, a_2, \ldots, a_n \geq \frac{-(3n-2)}{n-2}, \quad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{1-a_1}{(1+a_1)^2} + \frac{1-a_2}{(1+a_2)^2} + \cdots + \frac{1-a_n}{(1+a_n)^2} \geq 0.$$

(Vasile Cîrtoaje, 2014)

Solution. According to P 2.25-(b) in Volume 1, the inequality is true for $n = 3$. Assume further that $n \geq 4$ and write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf(s), \quad s = \frac{a_1 + a_2 + \cdots + a_n}{n} = 1,$$

where

$$f(u) = \frac{1-u}{(1+u)^2}, \quad u \in \mathbb{I} = \left[\frac{-(3n-2)}{n-2}, \frac{4n^2-7n+2}{n-2}\right] \setminus \{-1\}.$$

From

$$f'(u) = \frac{u-3}{(u+1)^3}, \quad f''(u) = \frac{2(5-u)}{(u+1)^4},$$

it follow that f is convex on $[1,5]$, decreasing for $1 \leq u \leq s_0$ and increasing for $u \geq s_0$, where $s_0 = 3 \in \mathbb{I}$. We see that f is not decreasing on $\mathbb{I}_{u \leq s_0}$. Therefore, according to Remark 4 and Remark 5, we may apply RPCF Theorem by replacing this condition with the sufficient condition $ns - (n-1)s_0 \leq \inf \mathbb{I}$. Indeed, for $n \geq 4$, we have

$$ns - (n-1)s_0 = n - 3(n-1) = -2n + 3 \leq \frac{-(3n-2)}{n-2} = \inf \mathbb{I}.$$

Thus, it suffices to show that $f(x) + (n-1)f(y) \geq nf(1)$ for all $x, y \in \mathbb{I}$ such that $x \leq 1 \leq y$ and $x + (n-1)y = n$. According to Remark 1, we only need to show that $h(x,y) \geq 0$, where

$$g(u) = \frac{f(u) - f(1)}{u-1}, \quad h(x,y) = \frac{g(x) - g(y)}{x-y}.$$

We have

$$g(u)=\frac{-1}{(u+1)^2}, \quad h(x,y)=\frac{x+y+2}{(1+x)^2(1+y)^2}=\frac{(n-2)x+3n-2}{(n-1)(1+x)^2(1+y)^2}\geq 0.$$

The proof is completed. In accordance with Remark 3, the equality holds for $a_1=a_2=\cdots=a_n=1$, and also for $a_1=\frac{-(3n-2)}{n-2}$ and $a_2=a_3=\cdots=a_n=\frac{n+2}{n-2}$ (or any cyclic permutation).

□

P 3.29. *Let $a_1,a_2,\ldots,a_n$ be nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$. If $n\geq 3$ and $k\geq 2-\frac{2}{n}$, then*

$$\frac{1-a_1}{(1-ka_1)^2}+\frac{1-a_2}{(1-ka_2)^2}+\cdots+\frac{1-a_n}{(1-ka_n)^2}\geq 0.$$

(Vasile Cîrtoaje, 2012)

Solution. According to P 3.96 in Volume 1, the inequality is true for $n=3$. Assume further that $n\geq 4$ and write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n)\geq nf(s), \quad s=\frac{a_1+a_2+\cdots+a_n}{n}=1,$$

where

$$f(u)=\frac{1-u}{(1-ku)^2}, \quad u\in\mathbb{I}=[0,3]\setminus\{1/k\}.$$

From

$$f'(u)=\frac{-ku+2k-1}{(1-ku)^3}, \quad f''(u)=\frac{2k(-ku+3k-2)}{(1-ku)^4},$$

it follow that f is convex on $[1,s_0]$, increasing on $[0,1/k)\cup[s_0,3]$ and decreasing on $(1/k,s_0]$, where

$$s_0=2-1/k\geq\frac{3n-4}{2(n-1)}>1.$$

Since f is not decreasing on $\mathbb{I}_{u\leq s_0}$, we can not apply RPCF Theorem in its original form. However, according to Remark 5, we may replace this condition with the condition $s_1\leq\inf\mathbb{I}$, where $s_1=ns-(n-1)s_0$. Indeed, we have

$$s_1\leq n-(n-1)\cdot\frac{3n-4}{2(n-1)}=\frac{4-n}{2}\leq 0=\inf\mathbb{I}.$$

So, it suffices to show that $f(x)+(n-1)f(y) \geq nf(1)$ for all $x, y \in \mathbb{I}$ such that $x \leq 1 \leq y$ and $x+(n-1)y = n$. According to Remark 1, we only need to show that $h(x,y) \geq 0$, where

$$g(u) = \frac{f(u)-f(1)}{u-1}, \quad h(x,y) = \frac{g(x)-g(y)}{x-y}.$$

Since

$$g(u) = \frac{-1}{(1-ku)^2},$$

$$h(x,y) = \frac{k[k(x+y)-2]}{(1-kx)^2(1-ky)^2},$$

we need to show that $k(x+y)-2 \geq 0$. Indeed,

$$k(x+y) = \frac{k[(n-2)x+n]}{n-1} \geq \frac{kn}{n-1} \geq 2.$$

The proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = 2-\frac{2}{n}$, then the equality also holds for $a_1 = 0$ and $a_2 = a_3 = \cdots = a_n = \frac{n}{n-1}$ (or any cyclic permutation).

□

Chapter 4

Partially Convex Function Method for Ordered Variables

4.1 Theoretical Basis

Right Partially Convex Function Theorem for Ordered Variables (*Vasile Cirtoaje*, 2012). *Let f be a function defined on a real interval $\mathbb{I}$ and convex on $[s, s_0]$, where $s, s_0 \in \mathbb{I}$, $s < s_0$. In addition, $f(u)$ is decreasing on $\mathbb{I}_{u \le s_0}$ and*

$$\min_{u \in \mathbb{I}} f(u) = f(s_0).$$

The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ such that $a_1 + a_2 + \cdots + a_n = ns$ and at least $n - m$ of $a_1, a_2, \ldots, a_n$ are smaller than or equal to s if and only if

$$f(x) + mf(y) \ge (1 + m)f(s)$$

for all $x, y \in \mathbb{I}$ such that $x \le s \le y$ and $x + my = (1 + m)s$.

Proof. The necessity is obvious. By Lemma from section 3, to prove the sufficiency, it suffices to consider that $a_1, a_2, \ldots, a_n \in \mathbb{J}$, where

$$\mathbb{J} = \mathbb{I}_{u \le s_0}.$$

Because $f(u)$ is convex on $\mathbb{J}_{u \ge s}$, the desired inequality follows from HCF-OV Theorem applied to the interval $\mathbb{J}$.

Similarly, we can prove Left Partially Convex Function Theorem for Ordered Variables (*Vasile Cirtoaje*, 2012).

Left Partially Convex Function Theorem for Ordered Variables. *Let f be a function defined on a real interval $\mathbb{I}$ and convex on $[s_0, s]$, where $s_0, s \in \mathbb{I}$, $s_0 < s$. In addition, $f(u)$ is increasing on $\mathbb{I}_{u \ge s_0}$ and satisfies*

$$\min_{u \in \mathbb{I}} = f(s_0).$$

The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \cdots, a_n \in \mathbb{I}$ such that $a_1 + a_2 + \cdots + a_n = ns$ and at least $n - m$ of $a_1, a_2, \ldots, a_n$ are greater than or equal to s if and only if

$$f(x) + mf(y) \ge (1 + m)f(s)$$

for all $x, y \in \mathbb{I}$ such that $x \ge s \ge y$ and $x + my = (1 + m)s$.

Remark 1. Let us denote

$$g(u) = \frac{f(u) - f(s)}{u - s}, \quad h(x, y) = \frac{g(x) - g(y)}{x - y}.$$

We may replace the hypothesis condition in RPCF-OV Theorem and LPCF-OV Theorem, namely

$$f(x) + mf(y) \ge (1 + m)f(s),$$

by the condition

$$h(x, y) \ge 0 \;\; \textit{for all} \;\; x, y \in \mathbb{I} \;\; \textit{such that} \;\; x + my = (1 + m)s.$$

Remark 2. Assume that f is differentiable on $\mathbb{I}$, and let

$$H(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$

Then, the desired inequality of Jensen's type in RPCF-OV Theorem and LPCF-OV Theorem holds true by replacing the hypothesis

$$f(x) + mf(y) \ge (1 + m)f(s)$$

with the more restrictive condition

$$H(x, y) \ge 0 \;\; \textit{for all} \;\; x, y \in \mathbb{I} \;\; \textit{such that} \;\; x + my = (1 + m)s.$$

Remark 3. RPCF-OV Theorem is a generalization of LPCF Theorem, because the last theorem can be obtained from the first theorem for $m = n - 1$. Similarly, LPCF-OV Theorem is a generalization of LPCF Theorem.

Remark 4. If

$$a_1 \geq \cdots \geq a_m \geq s \geq a_{m+1} \geq \cdots \geq a_n,$$

then at least $n-m$ of $a_1, a_2, \ldots, a_n$ are smaller than or equal to s. Also, if

$$a_1 \geq \cdots \geq a_{n-m} \geq s \geq a_{n-m+1} \geq \cdots \geq a_n,$$

then at least $n-m$ of $a_1, a_2, \ldots, a_n$ are greater than or equal to s. Thus, from RPCF-OV Theorem and LPCF-OV Theorem, we get the following corollaries.

RPCF-OV Corollary. *Let f be a function defined on a real interval $\mathbb{I}$ and convex on $[s, s_0]$, where $s, s_0 \in \mathbb{I}$, $s < s_0$. In addition, $f(u)$ is decreasing on $\mathbb{I}_{u \leq s_0}$ and*

$$\min_{u \in \mathbb{I}} f(u) = f(s_0).$$

The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying

$$a_1 \geq \cdots \geq a_m \geq s \geq a_{m+1} \geq \cdots \geq a_n, \qquad a_1 + a_2 + \cdots + a_n = ns,$$

if and only if

$$f(x) + mf(y) \geq (1+m)f(s)$$

for all $x, y \in \mathbb{I}$ such that $x \leq s \leq y$ and $x + my = (1+m)s$.

LPCF-OV Corollary. *Let f be a function defined on a real interval $\mathbb{I}$ and convex on $[s_0, s]$, where $s_0, s \in \mathbb{I}$, $s_0 < s$. In addition, $f(u)$ is increasing on $\mathbb{I}_{u \geq s_0}$ and satisfies*

$$\min_{u \in \mathbb{I}} f(u) = f(s_0).$$

The inequality

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying

$$a_1 \leq \cdots \leq a_m \leq s \leq a_{m+1} \leq \cdots \leq a_n, \qquad a_1 + a_2 + \cdots + a_n = ns,$$

if and only if

$$f(x) + mf(y) \geq (1+m)f(s)$$

for all $x, y \in \mathbb{I}$ such that $x \geq s \geq y$ and $x + my = (1+m)s$.

Remark 5. The inequality in RPCF-OV Corollary becomes an equality for

$$a_1 = a_2 = \cdots = a_n = s$$

and also for

$$a_1 = \cdots = a_m = y, \quad a_{m+1} = \cdots = a_{n-1} = s, \quad a_n = x \quad (x < y),$$

where $x, y \in \mathbb{I}$ satisfy the equations

$$x + my = (1+m)s, \quad f(x) + mf(y) = (1+m)f(s).$$

For $x \neq y$, these equations are equivalent to

$$x + my = (1+m)s, \quad h(x,y) = 0.$$

Remark 6. The inequality in LPCF-OV Corollary becomes an equality for

$$a_1 = a_2 = \cdots = a_n = s$$

and also for

$$a_1 = x, \quad a_2 = \cdots = a_{n-m} = s, \quad a_{n-m+1} = \cdots = a_n = y \quad (x > y),$$

where $x, y \in \mathbb{I}$ satisfy the equations

$$x + my = (1+m)s, \quad f(x) + mf(y) = (1+m)f(s).$$

For $x \neq y$, these equations are equivalent to

$$x + my = (1+m)s, \quad h(x,y) = 0.$$

Remark 7. RPCF-OV Theorem and RPCF-OV Corollar are also valid in the case when $\mathbb{I} = [a,b] \setminus \{u_0\}$ or $\mathbb{I} = (a,b) \setminus \{u_0\}$, where a, b, u_0 are real numbers such that

$$a < s < s_0 < u_0 < b.$$

Similarly, LPCF Theorem is also valid in the case when $\mathbb{I} = [a,b]\setminus\{u_0\}$ or $\mathbb{I} = (a,b)\setminus\{u_0\}$, where a, b, u_0 are real numbers such that

$$a < u_0 < s_0 < s < b.$$

4.2 Applications

4.1. If a,b,c,d are real numbers such that

$$a\le 1\le b\le c\le d,\qquad a+b+c+d=4,$$

then

$$\frac{a}{3a^2+1}+\frac{b}{3b^2+1}+\frac{c}{3c^2+1}+\frac{d}{3d^2+1}\le 1.$$

4.2. If a,b,c,d are real numbers such that

$$a\ge b\ge 1\ge c\ge d,\qquad a+b+c+d=4,$$

then

$$\frac{16a-5}{32a^2+1}+\frac{16b-5}{32b^2+1}+\frac{16c-5}{32c^2+1}+\frac{16d-5}{32d^2+1}\le \frac{4}{3}.$$

4.3. If a,b,c,d,e are real numbers such that

$$a\ge b\ge 1\ge c\ge d\ge e,\qquad a+b+c+d+e=5,$$

then

$$\frac{18a-5}{12a^2+1}+\frac{18b-5}{12b^2+1}+\frac{18c-5}{12c^2+1}+\frac{18d-5}{12d^2+1}+\frac{18e-5}{12e^2+1}\le 5.$$

4.4. If a,b,c,d,e are real numbers such that

$$a\ge b\ge 1\ge c\ge d\ge e,\qquad a+b+c+d+e=5,$$

then

$$\frac{a(a-1)}{3a^2+4}+\frac{b(b-1)}{3b^2+4}+\frac{c(c-1)}{3c^2+4}+\frac{d(d-1)}{3d^2+4}+\frac{e(e-1)}{3e^2+4}\ge 0.$$

4.5. Let $a_1,a_2,\dots,a_{2n}\ne -k$ be real numbers such that

$$a_1\ge\cdots\ge a_n\ge 1\ge a_{n+1}\ge\cdots\ge a_{2n},\qquad a_1+a_2+\cdots+a_{2n}=2n.$$

If $k\ge \dfrac{n+1}{2\sqrt{n}}$, then

$$\frac{a_1(a_1-1)}{(a_1+k)^2}+\frac{a_2(a_2-1)}{(a_2+k)^2}+\cdots+\frac{a_{2n}(a_{2n}-1)}{(a_{2n}+k)^2}\ge 0.$$

4.6. Let $a_1, a_2, \ldots, a_{2n} \neq -k$ be real numbers such that

$$a_1 \geq \cdots \geq a_n \geq 1 \geq a_{n+1} \geq \cdots \geq a_{2n}, \qquad a_1 + a_2 + \cdots + a_{2n} = 2n.$$

If $k \geq 1 + \dfrac{n+1}{\sqrt{n}}$, then

$$\frac{a_1^2 - 1}{(a_1 + k)^2} + \frac{a_2^2 - 1}{(a_2 + k)^2} + \cdots + \frac{a_{2n}^2 - 1}{(a_{2n} + k)^2} \geq 0.$$

4.7. If $a_1, a_2, \cdots, a_n$ are positive real numbers such that

$$a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$a_1^{2/a_1} + a_2^{2/a_2} + \cdots + a_n^{2/a_n} \leq n.$$

4.8. If $a_1, a_2, \cdots, a_{11}$ are real numbers such that

$$a_1 \geq a_2 \geq 1 \geq a_3 \geq \cdots \geq a_{11}, \qquad a_1 + a_2 + \cdots + a_{11} = 11,$$

then

$$(1 - a_1 + a_1^2)(1 - a_2 + a_2^2) \cdots (1 - a_{11} + a_{11}^2) \geq 1.$$

4.9. If $a_1, a_2, \cdots, a_8$ are nonzero real numbers such that

$$a_1 \geq a_2 \geq a_3 \geq a_4 \geq 1 \geq a_5 \geq a_6 \geq a_7 \geq a_8, \qquad a_1 + a_2 + \cdots + a_8 = 8,$$

then

$$5\left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_8^2}\right) + 72 \geq 14\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_8}\right).$$

4.10. If a, b, c, d are positive real numbers such that

$$a \geq b \geq 1 \geq c \geq d, \qquad abcd = 1,$$

then

$$\frac{7 - 6a}{2 + a^2} + \frac{7 - 6b}{2 + b^2} + \frac{7 - 6c}{2 + c^2} + \frac{7 - 6d}{2 + d^2} \geq \frac{4}{3}.$$

4.11. If a, b, c are positive real numbers such that

$$a \ge 1 \ge b \ge c, \qquad abc = 1,$$

then

$$\frac{7-4a}{2+a^2} + \frac{7-4b}{2+b^2} + \frac{7-4c}{2+c^2} \ge 3.$$

4.12. Let $a_1, a_2, \cdots, a_n$ be positive real numbers such that

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1 a_2 \cdots a_n = 1.$$

If $p, q \ge 0$ such that $p + 3q \ge 1$, then

$$\frac{1-a_1}{1+pa_1+qa_1^2} + \frac{1-a_2}{1+pa_2+qa_2^2} + \cdots + \frac{1-a_n}{1+pa_n+qa_n^2} \ge 0.$$

4.3 Solutions

P 4.1. *If a, b, c, d are real numbers such that*

$$a \le 1 \le b \le c \le d, \qquad a+b+c+d=4,$$

then

$$\frac{a}{3a^2+1}+\frac{b}{3b^2+1}+\frac{c}{3c^2+1}+\frac{d}{3d^2+1} \le 1.$$

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d) \ge 4f(s), \qquad s=\frac{a+b+c+d}{4}=1,$$

where

$$f(u)=\frac{-u}{3u^2+1}, \qquad u \in \mathbb{R}.$$

From

$$f'(u)=\frac{3u^2-1}{(3u^2+1)^2},$$

it follows that f is increasing on $(-\infty,-s_0]\cup[s_0,\infty)$ and decreasing on $[-s_0,s_0]$, where $s_0=1/\sqrt{3}$. Since

$$\lim_{u\to-\infty} f(u)=0$$

and $f(s_0)<0$, it follows that

$$\min_{u\in\mathbb{R}} f(u)=f(s_0).$$

From

$$f''(u)=\frac{18u(1-u^2)}{(3u^2+1)^3},$$

it follows that f is convex on $[0,1]$, hence on $[s_0,1]$. Therefore, we apply LPCF-OV Theorem or LPCF-OV Corollary for $n=4$ and $m=1$. We only need to show that $f(x)+f(y)\ge 2f(1)$ for all real x,y such that $x+y=2$. Using Remark 1, it suffices to prove that $h(x,y)\ge 0$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y}, \quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

Indeed, we have

$$g(u)=\frac{3u-1}{4(3u^2+1)},$$

$$h(x,y)=\frac{3(1+x+y-3xy)}{4(3x^2+1)(3y^2+1)}=\frac{9(1-xy)}{4(3x^2+1)(3y^2+1)}\ge 0,$$

since

$$4(1-xy)=(x+y)^2-4xy=(x-y)^2\geq 0.$$

Thus, the proof is completed. The equality holds for $a=b=c=d=1$.

Remark. Similarly, we can prove the following generalization.

- *If $a_1,a_2,\dots,a_n$ are real numbers such that*

$$a_1\leq 1\leq a_2\leq\cdots\leq a_n,\qquad a_1+a_2+\cdots+a_n=n,$$

then

$$\frac{a_1}{3a_1^2+1}+\frac{a_2}{3a_2^2+1}+\cdots+\frac{a_n}{3a_n^2+1}\leq\frac{n}{4},$$

with equality for $a_1=a_2=\cdots=a_n=1$.

□

P 4.2. *If a,b,c,d are real numbers such that*

$$a\geq b\geq 1\geq c\geq d,\qquad a+b+c+d=4,$$

then

$$\frac{16a-5}{32a^2+1}+\frac{16b-5}{32b^2+1}+\frac{16c-5}{32c^2+1}+\frac{16d-5}{32d^2+1}\leq\frac{4}{3}.$$

(Vasile Cîrtoaje, 2012)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)\geq 4f(s),\qquad s=\frac{a+b+c+d}{4}=1,$$

where

$$f(u)=\frac{5-16u}{32u^2+1},\qquad u\in\mathbb{R}.$$

As shown in the proof of P 3.1, f is convex on $[s_0,1]$, increasing for $u\geq s_0$ and

$$\min_{u\in\mathbb{R}}f(u)=f(s_0),$$

where

$$s_0=\frac{5+\sqrt{33}}{16}\approx 0.6715.$$

Therefore, we may apply LPCF-OV Theorem or LPCF-OV Corollary for $n=4$ and $m=2$. We only need to show that $f(x)+2f(y)\geq 3f(1)$ for all real x,y such that $x+2y=3$. Using Remark 1, it suffices to prove that $h(x,y)\geq 0$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y},\quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

Indeed, we have

$$g(u) = \frac{32(2u-1)}{3(32u^2+1)},$$

$$h(x,y) = \frac{64(1+16x+16y-32xy)}{3(32x^2+1)(32y^2+1)} = \frac{64(4x-5)^2}{3(32x^2+1)(32y^2+1)} \geq 0.$$

Thus, the proof is completed. From $x+2y=3$ and $h(x,y)=0$, we get $x=5/4$ and, $y=7/8$. Therefore, in accordance with Remark 6, the equality holds for $a=b=c=d=1$, and also for $a=5/4$, $b=1$ and $c=d=7/8$.

Remark. Similarly, we can prove the following generalization.

- *If $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ are real numbers such that*

$$a_1 \geq \cdots \geq a_{n-2} \geq 1 \geq a_{n-1} \geq a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{16a_1-5}{32a_1^2+1} + \frac{16a_2-5}{32a_2^2+1} + \cdots + \frac{16a_n-5}{32a_n^2+1} \leq \frac{n}{3},$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = 5/4$, $a_2 = \cdots = a_{n-2} = 1$ and $a_{n-1} = a_n = 7/8$.

□

P 4.3. *If a,b,c,d,e are real numbers such that*

$$a \geq b \geq 1 \geq c \geq d \geq e, \qquad a+b+c+d+e=5,$$

then

$$\frac{18a-5}{12a^2+1} + \frac{18b-5}{12b^2+1} + \frac{18c-5}{12c^2+1} + \frac{18d-5}{12d^2+1} + \frac{18e-5}{12e^2+1} \leq 5.$$

(Vasile Cîrtoaje, 2012)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)+f(e) \geq 5f(s), \qquad s = \frac{a+b+c+d+e}{5} = 1,$$

where

$$f(u) = \frac{5-18u}{12u^2+1}, \qquad u \in \mathbb{R}.$$

As shown in the proof of P 3.2, f is convex on $[s_0, 1]$, increasing for $u \geq s_0$ and

$$\min_{u\in\mathbb{R}} f(u) = f(s_0),$$

where

$$s_0 = \frac{5+\sqrt{52}}{18} \approx 0.678.$$

Therefore, we may apply LPCF-OV Theorem or LPCF-OV Corollary for $n = 5$ and $m = 3$. We only need to show that $f(x) + 3f(y) \geq 4f(1)$ for all real x, y such that $x + 3y = 4$. Using Remark 1, it suffices to prove that $h(x, y) \geq 0$, where

$$h(x,y) = \frac{g(x)-g(y)}{x-y}, \quad g(u) = \frac{f(u)-f(1)}{u-1}.$$

Indeed, we have

$$g(u) = \frac{6(2u-1)}{12u^2+1},$$

$$h(x,y) = \frac{12(1+6x+6y-12xy)}{(12x^2+1)(12y^2+1)} = \frac{12(2x-3)^2}{(12x^2+1)(12y^2+1)} \geq 0.$$

Thus, the proof is completed. From $x + 3y = 4$ and $h(x, y) = 0$, we get $x = 3/2$ and, $y = 5/6$. Therefore, in accordance with Remark 6, the equality holds for $a = b = c = d = e = 1$, and also for $a = 3/2$, $b = 1$ and $c = d = e = 5/6$.

Remark. Similarly, we can prove the following generalization.

- *If $a_1, a_2, \ldots, a_n$ $(n \geq 4)$ are real numbers such that*

$$a_1 \geq \cdots \geq a_{n-3} \geq 1 \geq a_{n-2} \geq a_{n-1} \geq a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{18a_1-5}{12a_1^2+1} + \frac{18a_2-5}{12a_2^2+1} + \cdots + \frac{18a_n-5}{12a_n^2+1} \leq n,$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = 3/2$, $a_2 = \cdots = a_{n-3} = 1$ and $a_{n-2} = a_{n-1} = a_n = 5/6$.

□

P 4.4. *If a, b, c, d, e are real numbers such that*

$$a \geq b \geq 1 \geq c \geq d \geq e, \qquad a+b+c+d+e = 5,$$

then

$$\frac{a(a-1)}{3a^2+4} + \frac{b(b-1)}{3b^2+4} + \frac{c(c-1)}{3c^2+4} + \frac{d(d-1)}{3d^2+4} + \frac{e(e-1)}{3e^2+4} \geq 0.$$

(Vasile Cîrtoaje, 2012)

Solution. Write the inequality as

$$f(a)+f(b)+f(c)+f(d)+f(e)\ge 5f(s),\quad s=\frac{a+b+c+d+e}{5}=1,$$

where

$$f(u)=\frac{u^2-u}{3u^2+4},\quad u\in\mathbb{R}.$$

As shown in the proof of P 3.5, f is convex on $[s_0,1]$, increasing for $u\ge s_0$ and

$$\min_{u\in\mathbb{R}} f(u)=f(s_0),$$

where

$$s_0=\frac{-4+2\sqrt{7}}{3}\approx 0.43.$$

Therefore, we may apply LPCF-OV Theorem or LPCF-OV Corollary for $n=5$ and $m=3$. We only need to show that $f(x)+3f(y)\ge 4f(1)$ for all real x,y such that $x+3y=4$. Using Remark 1, it suffices to prove that $h(x,y)\ge 0$, where

$$h(x,y)=\frac{g(x)-g(y)}{x-y},\quad g(u)=\frac{f(u)-f(1)}{u-1}.$$

Indeed, we have

$$g(u)=\frac{u)}{3u^2+4},$$

$$h(x,y)=\frac{4-3xy}{(3x^2+4)(3y^2+4)}=\frac{(x-2)^2}{(12x^2+1)(12y^2+1)}\ge 0.$$

Thus, the proof is completed. From $x+3y=4$ and $h(x,y)=0$, we get $x=2$ and, $y=2/3$. Therefore, in accordance with Remark 6, the equality holds for

$$a=b=c=d=e=1,$$

and also for

$$a=2,\ b=1,\ c=d=e=2/3.$$

Remark. Similarly, we can prove the following generalizations.

- *If $a_1,a_2,\ldots,a_n$ $(n\ge 4)$ are real numbers such that*

$$a_1\ge\cdots\ge a_{n-3}\ge 1\ge a_{n-2}\ge a_{n-1}\ge a_n,\qquad a_1+a_2+\cdots+a_n=n,$$

then

$$\frac{a_1(a_1-1)}{3a_1^2+4}+\frac{a_2(a_2-1)}{3a_2^2+4}+\cdots+\frac{a_n(a_n-1)}{3a_n^2+4}\ge 0,$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$*, and also for* $a_1 = 2$*,* $a_2 = \cdots = a_{n-3} = 1$ *and* $a_{n-2} = a_{n-1} = a_n = 2/3$.

- *If* $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ *are real numbers such that*

$$a_1 \ge a_2 \ge 1 \ge a_3 \ge \cdots \ge a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{a_1(a_1-1)}{4(n-2)a_1^2 + (n-1)^2} + \frac{a_2(a_2-1)}{4(n-2)a_2^2 + (n-1)^2} + \cdots + \frac{a_n(a_n-1)}{4(n-2)a_n^2 + (n-1)^2} \ge 0,$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$*, and also for* $a_1 = \dfrac{n-1}{2}$*,* $a_2 = 1$ *and* $a_3 = \cdots = a_n = \dfrac{n-1}{2(n-2)}$.

□

P 4.5. *Let* $a_1, a_2, \ldots, a_{2n} \ne -k$ *be real numbers such that*

$$a_1 \ge \cdots \ge a_n \ge 1 \ge a_{n+1} \ge \cdots \ge a_{2n}, \qquad a_1 + a_2 + \cdots + a_{2n} = 2n.$$

If $k \ge \dfrac{n+1}{2\sqrt{n}}$*, then*

$$\frac{a_1(a_1-1)}{(a_1+k)^2} + \frac{a_2(a_2-1)}{(a_2+k)^2} + \cdots + \frac{a_{2n}(a_{2n}-1)}{(a_{2n}+k)^2} \ge 0.$$

(Vasile Cîrtoaje, 2012)

Solution. Write the inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_{2n}) \ge 2nf(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_{2n}}{2n} = 1,$$

where

$$f(u) = \frac{u(u-1)}{(u+k)^2}, \quad u \in \mathbb{I} = \mathbb{R} \setminus \{-k\}.$$

As shown in the proof of P 3.6, f is convex on $[s_0, 1]$, increasing for $u \ge s_0$ and

$$\min_{u \in \mathbb{I}} f(u) = f(s_0),$$

where

$$s_0 = \frac{k}{2k+1} < 1.$$

Therefore, we may apply LPCF-OV Theorem or LPCF-OV Corollary for $2n$ real numbers and $m = n$. We only need to show that $f(x)+nf(y) \ge (n+1)f(1)$ for all real x, y such that $x + ny = n+1$. Using Remark 1, it suffices to prove that $h(x, y) \ge 0$, where

$$h(x,y) = \frac{g(x)-g(y)}{x-y}, \quad g(u) = \frac{f(u)-f(1)}{u-1}.$$

Indeed, we have

$$g(u) = \frac{u}{(u+k)^2},$$

$$h(x,y) = \frac{k^2-xy}{(x+k)^2(y+k)^2} \ge 0,$$

because

$$k^2 - xy \ge \frac{(n+1)^2}{4n} - xy = \frac{[2ny-n-1]^2}{4n} \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = \dfrac{n+1}{2\sqrt{n}}$, then the equality holds also for $a_1 = \dfrac{n+1}{2}$, $a_2 = \cdots = a_n = 1$ and $a_{n+1} = \cdots = a_{2n} = \dfrac{n+1}{2n}$.

□

P 4.6. *Let $a_1, a_2, \ldots, a_{2n} \neq -k$ be real numbers such that*

$$a_1 \ge \cdots \ge a_n \ge 1 \ge a_{n+1} \ge \cdots \ge a_{2n}, \qquad a_1 + a_2 + \cdots + a_{2n} = 2n.$$

If $k \ge 1 + \dfrac{n+1}{\sqrt{n}}$, then

$$\frac{a_1^2-1}{(a_1+k)^2} + \frac{a_2^2-1}{(a_2+k)^2} + \cdots + \frac{a_{2n}^2-1}{(a_{2n}+k)^2} \ge 0.$$

(Vasile Cîrtoaje, 2012)

Solution. Write the inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_{2n}) \ge 2nf(s), \quad s = \frac{a_1+a_2+\cdots+a_{2n}}{2n} = 1,$$

where

$$f(u) = \frac{u^2-1}{(u+k)^2}, \quad u \in \mathbb{I} = \mathbb{R} \setminus \{-k\}.$$

As shown in the proof of P 3.7, f is convex on $[s_0, 1]$, increasing for $u \ge s_0$ and

$$\min_{u \in \mathbb{I}} f(u) = f(s_0),$$

where
$$s_0 = \frac{-1}{k} \in (-1, 0).$$
Therefore, we may apply LPCF-OV Theorem or LPCF-OV Corollary for $2n$ real numbers and $m = n$. We only need to show that $f(x)+nf(y) \ge (n+1)f(1)$ for all real x, y such that $x + ny = n+1$. Using Remark 1, it suffices to prove that $h(x, y) \ge 0$, where

$$h(x, y) = \frac{g(x)-g(y)}{x-y}, \quad g(u) = \frac{f(u)-f(1)}{u-1}.$$

Indeed, we have
$$g(u) = \frac{u+1}{(u+k)^2},$$
$$h(x, y) = \frac{(k-1)^2-1-x-y-xy}{(x+k)^2(y+k)^2} \ge 0,$$

because

$$(k-1)^2-1-x-y-xy \ge \frac{(n+1)^2}{n} - 1-x-y-xy = \frac{(ny-1)^2}{n} \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$. If $k = 1 + \dfrac{n+1}{\sqrt{n}}$, then the equality holds also for $a_1 = n$, $a_2 = \cdots = a_n = 1$ and $a_{n+1} = \cdots = a_{2n} = \dfrac{1}{n}$.

□

P 4.7. *If $a_1, a_2, \cdots, a_n$ are positive real numbers such that*

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then
$$a_1^{2/a_1} + a_2^{2/a_2} + \cdots + a_n^{2/a_n} \le n.$$

(Vasile Cîrtoaje, 2012)

Solution. Rewrite the desired inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_n) \ge nf(s), \quad s = \frac{a_1+a_2+\cdots+a_n}{n} = 1,$$

where
$$f(u) = -u^{2/u}, \quad u \in \mathbb{I} = (0, n).$$

We have
$$f'(u) = 2u^{\frac{2}{u}-2}(\ln u - 1),$$

$$f''(u) = 2u^{\frac{2}{u}-4}[u + 2(1 - \ln u)(u - 1 + \ln u)].$$

From the expression of f', it follows that f is decreasing on $(0, s_0]$ and increasing on $[s_0, n)$, where $s_0 = e$. In addition, we claim that f is convex on $[1, s_0]$. Indeed, since $1 - \ln u \geq 0$ and

$$u - 1 + \ln u \geq u - 1 \geq 0,$$

we have $f'' > 0$ for $u \in [1, s_0]$. Therefore, we may apply RPCF-OV Theorem or RPCF-OV Corollary for $m = 1$. We only need to show that $f(x) + f(y) \geq 2f(1)$ for all $x, y > 0$ such that $x + y = 2$. The inequality $f(x) + f(y) \geq 2f(1)$ is equivalent to

$$x^{2/x} + y^{2/y} \leq 2,$$

which is just the inequality in P 3.27 from Volume 2. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 4.8. *If $a_1, a_2, \cdots, a_{11}$ are real numbers such that*

$$a_1 \geq a_2 \geq 1 \geq a_3 \geq \cdots \geq a_{11}, \qquad a_1 + a_2 + \cdots + a_{11} = 11,$$

then

$$(1 - a_1 + a_1^2)(1 - a_2 + a_2^2) \cdots (1 - a_{11} + a_{11}^2) \geq 1.$$

(Vasile Cîrtoaje, 2012)

Solution. Rewrite the desired inequality as

$$f(a_1) + f(a_2) + \cdots + f(a_{11}) \geq 11f(s), \qquad s = \frac{a_1 + a_2 + \cdots + a_{11}}{11} = 1,$$

where

$$f(u) = \ln(1 - u + u^2), \qquad u \in \mathbb{R}.$$

From

$$f'(u) = \frac{2u - 1}{1 - u + u^2},$$

it follows that f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where $s_0 = 1/2$. Also, from

$$f''(u) = \frac{1 + 2u(1 - u)}{(1 - u + u^2)^2},$$

it follows that $f''(u) > 0$ for $u \in [s_0, 1]$, hence f is convex on $[s_0, 1]$. Therefore, we may apply LPCF-OV Theorem or LPCF-OV Corollary for $n = 11$ and $m = 9$. We only need to show that $f(x)+9f(y) \ge 9f(1)$ for all real x, y such that $x+9y=10$. Using Remark 2, it suffices to prove that $H(x,y) \ge 0$, where

$$H(x,y) = \frac{f'(x)-f'(y)}{x-y}.$$

Since

$$H(x,y) = \frac{1+x+y-2xy}{(1-x+x^2)(1-y+y^2)}$$

and

$$1+x+y-2xy = 18y^2-8y+1 = 2y^2+(4y-1)^2 > 0,$$

the conclusion follows. The equality holds for $a_1 = a_2 = \cdots = a_{11} = 1$.

Remark. By replacing $a_1, a_2, \ldots, a_{11}$ respectively with $1-a_1, 1-a_2, \ldots, 1-a_{11}$, we get the following statement.

- *If $a_1, a_2, \ldots, a_{11}$ are real numbers such that*

$$a_1 \le a_2 \le 0 \le a_3 \le \cdots \le a_{11}, \qquad a_1+a_2+\cdots+a_{11} = 0,$$

then

$$(1-a_1+a_1^2)(1-a_2+a_2^2)\cdots(1-a_{11}+a_{11}^2) \ge 1,$$

with equality for $a_1 = a_2 = \cdots = a_n = 0$.

□

P 4.9. *If $a_1, a_2, \cdots, a_8$ are nonzero real numbers such that*

$$a_1 \ge a_2 \ge a_3 \ge a_4 \ge 1 \ge a_5 \ge a_6 \ge a_7 \ge a_8, \qquad a_1+a_2+\cdots+a_8 = 8,$$

then

$$5\left(\frac{1}{a_1^2}+\frac{1}{a_2^2}+\cdots+\frac{1}{a_8^2}\right)+72 \ge 14\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_8}\right).$$

(Vasile Cîrtoaje, 2012)

Solution. Write the desired inequality as

$$f(a_1)+f(a_2)+\cdots+f(a_8) \ge 8f(s), \quad s = \frac{a_1+a_2+\cdots+a_8}{8} = 1,$$

where

$$f(u) = \frac{5}{u^2} - \frac{14}{u} + 9, \quad u \in \mathbb{I} = \mathbb{R} \setminus \{0\}.$$

As shown in the proof of P 3.16, f is convex on $[s_0, 1]$, increasing for $u \ge s_0$ and

$$\min_{u\in\mathbb{I}} f(u) = f(s_0),$$

where

$$s_0 = \frac{5}{7}.$$

Taking into account Remark 7, we may apply LPCF-OV Theorem or LPCF-OV Corollary for $n = 8$ and $m = 4$. We only need to show that $f(x) + 4f(y) \ge 5f(1)$ for all $x, y \in \mathbb{I}$ such that $x + 4y = 5$. Using Remark 1, it suffices to prove that $h(x, y) \ge 0$, where

$$h(x,y) = \frac{g(x) - g(y)}{x - y}, \quad g(u) = \frac{f(u) - f(1)}{u - 1}.$$

Indeed, we have

$$g(u) = g(u) = \frac{9}{u} - \frac{5}{u^2},$$

$$h(x,y) = \frac{5x + 5y - 9xy}{x^2 y^2} = \frac{(6y-5)^2}{x^2 y^2} \ge 0.$$

In accordance with Remark 6, the equality holds for $a_1 = a_2 = \cdots = a_8 = 1$, and also for $a_1 = \frac{5}{3}$, $a_2 = a_3 = a_4 = 1$ and $a_5 = a_6 = a_7 = a_8 = \frac{5}{6}$ (or any cyclic permutation).

□

P 4.10. *If a, b, c, d are positive real numbers such that*

$$a \ge b \ge 1 \ge c \ge d, \qquad abcd = 1,$$

then

$$\frac{7-6a}{2+a^2} + \frac{7-6b}{2+b^2} + \frac{7-6c}{2+c^2} + \frac{7-6d}{2+d^2} \ge \frac{4}{3}.$$

(Vasile Cîrtoaje, 2012)

***Solution*.** Using the substitution

$$a = e^x,\ b = e^y,\ c = e^z,\ d = e^w$$

we need to show that

$$f(x) + f(y) + f(z) + f(w) \ge 4f(s),$$

where

$$x \ge y \ge 0 \ge z \ge w, \quad s = \frac{x+y+z+w}{4} = 0,$$

$$f(u) = \frac{7 - 6e^u}{2 + e^{2u}}, \quad u \in \mathbb{R}.$$

As shown in the proof of P 3.17, f is convex on $[0, s_0]$, is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$, where $s_0 = \ln 3$. Therefore, we may apply RPCF-OV Theorem or

RPCF-OV Corollary for $n=4$ and $m=2$. We only need to show that $f(x)+2f(y)\geq 3f(0)$ for all real x,y such that $x+2y=0$. That is, to prove that

$$\frac{7-6a}{2+a^2}+\frac{2(7-6b)}{2+b^2}\geq 1$$

for all $a,b>0$ such that $ab^2=1$. This is equivalent to

$$(b-1)^2(b-2)^2(5b^2+6bt+3)\geq 0,$$

which is clearly true. In accordance with Remark 5, the equality holds for $a=b=c=d=1$, and also for $a=b=2$, $c=1$ and $d=1/4$.

□

P 4.11. *If a,b,c are positive real numbers such that*

$$a\geq 1\geq b\geq c,\qquad abc=1,$$

then

$$\frac{7-4a}{2+a^2}+\frac{7-4b}{2+b^2}+\frac{7-4c}{2+c^2}\geq 3.$$

(Vasile Cîrtoaje, 2012)

Solution. Using the substitution

$$a=e^x,\ b=e^y,\ c=e^z,$$

we need to show that

$$f(x)+f(y)+f(z)\geq 3f(s),$$

where

$$x\geq 1\geq y\geq z,\qquad s=\frac{x+y+z}{3}=0,$$

$$f(u)=\frac{7-4e^u}{2+e^{2u}},\quad u\in\mathbb{R}.$$

From

$$f'(u)=\frac{2e^u(2e^u+1)(e^u-4)}{(2+e^{2u})^2},$$

it follows that f is decreasing on $(-\infty,s_0]$ and increasing on $[s_0,\infty)$, where $s_0=\ln 4$. Also, we have

$$f''(u)=\frac{4t\cdot h(t)}{(2+t^2)^3},\qquad t=e^u,$$

where

$$h(t)=-t^4+7t^3+12t^2-14t-4.$$

We will show that $h(t) \ge 0$ for $t \in [1,4]$, hence f is convex on $[0, s_0]$. Indeed,

$$h(t) = (t-1)[t^2(-t+6) + 18t + 4] \ge 0.$$

Therefore, we may apply RPCF-OV Theorem or RPCF-OV Corollary for $n = 3$ and $m = 1$. We only need to show that $f(x) + f(y) \ge 2f(0)$ for all real x, y such that $x + y = 0$. That is, to prove that

$$\frac{7-4a}{2+a^2} + \frac{7-4b}{2+b^2} \ge 2$$

for all $a, b > 0$ such that $ab = 1$. This is equivalent to

$$(a-1)^4 \ge 0.$$

The equality holds for $a = b = c = 1$.

□

P 4.12. *Let $a_1, a_2, \cdots, a_n$ be positive real numbers such that*

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1 a_2 \cdots a_n = 1.$$

If $p, q \ge 0$ such that $p + 3q \ge 1$, then

$$\frac{1-a_1}{1+pa_1+qa_1^2} + \frac{1-a_2}{1+pa_2+qa_2^2} + \cdots + \frac{1-a_n}{1+pa_n+qa_n^2} \ge 0.$$

(Vasile Cîrtoaje, 2012)

Solution. For $q = 0$, we need to show that $p \ge 1$ involves

$$\frac{1-a_1}{1+pa_1} + \frac{1-a_2}{1+pa_2} + \cdots + \frac{1-a_n}{1+pa_n} \ge 0.$$

This is just the inequality from P 2.19. Consider next that $q > 0$. Using the substitutions $a_i = e^{x_i}$ for $i = 1, 2, \ldots, n$, we need to show that

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf(s),$$

where

$$x_1 \ge 1 \ge x_2 \ge \cdots \ge x_n, \qquad s = \frac{x_1 + x_2 + \cdots + x_n}{n} = 0,$$

$$f(u) = \frac{1-e^u}{1+pe^u+qe^{2u}}, \quad u \in \mathbb{R}.$$

As shown in the proof of P 3.21, f is convex on $[0, s_0]$ if $p + 3q - 1 \ge 0$, where

$$s_0 = \ln r_0 > 0, \qquad r_0 = 1 + \sqrt{1 + \frac{p+1}{q}}.$$

Clearly, the condition $p+3q-1 \ge 0$ is satisfied by hypothesis. In addition, f is decreasing on $(-\infty, s_0]$ and increasing on $[s_0, \infty)$. Therefore, we may apply RPCF-OV Theorem or RPCF-OV Corollary for $m = 1$. We only need to show that $f(x) + f(y) \ge 2f(0)$ for all real x, y such that $x + y = 0$. That is, to prove that

$$\frac{1-a}{1+pa+qa^2} + \frac{1-b}{1+pb+qb^2} \ge 0$$

for all $a, b > 0$ such that $ab = 1$. This is equivalent to

$$(a-1)^2[(p-1)a + q(a^2+a+1)] \ge 0,$$

which is true because

$$(p-1)a + q(a^2+a+1) \ge (p-1)a + q(3a) = (p+3q-1)a \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

Chapter 5

Bibliography